

Random sets of isomorphism of linear operators on Hilbert space

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Abstract: This note deals with a problem of the probabilistic Ramsey theory in functional analysis. Given a linear operator T on a Hilbert space with an orthogonal basis, we define the *isomorphic structure* $\Sigma(T)$ as the family of all subsets of the basis so that T restricted to their span is a nice isomorphism. Our main result is a dimension-free optimal estimate of the size of $\Sigma(T)$. It improves and extends in several ways the principle of restricted invertibility due to Bourgain and Tzafriri. With an appropriate notion of randomness, we obtain a randomized principle of restricted invertibility.

1. Introduction

1.1. Randomized Ramsey-type problems

Finding a nice structure in a big unstructured object is a recurrent theme in mathematics. This direction of thought is often called Ramsey theory, although Ramsey theory was originally only associated with combinatorics. One celebrated example is Van der Waerden’s theorem: for any partition of the integers into two sets, one of these sets contains arbitrary long arithmetic progressions.

Ramsey theory meets probability theory when one asks about the quality of *most* sub-structures of a given structure. Can one improve the quality of a structure by passing to its *random* sub-structure? (a random subgraph, for example). A remarkable example of the randomized Ramsey theory is Dvoretzky’s theorem in geometric functional analysis in the form of V. Milman (see [4], 4.2). One of its corollaries states that, for any n -dimensional finite-dimensional Banach space, a random $O(\log n)$ -dimensional subspace (with respect to some natural measure) is well isomorphic to a Hilbert space.

1.2. The isomorphism structure of a linear operator

In this note we are trying to find a nice structure in an arbitrary bounded linear operator on a separable Hilbert space. Let T be a bounded linear operator on a Hilbert space H with an orthonormal basis $(e_i)_{i \in \mathbb{N}}$. We naturally think of T as being nice if it is a nice isomorphism on H . However, this situation is rather rare; instead, T may be a nice isomorphism on the subspace spanned by some subsets of the basis. So, instead of being a “global” isomorphism, T may be a “local” isomorphism when restricted to certain subspaces of H . A central question is then – how many such subspaces are there? Let us call these subspaces an isomorphism structure of T :

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Definition 1.1. Let T be a bounded linear operator on a Hilbert space H , and $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis of H . Let $0 < \varepsilon < 1$. A set σ of \mathbb{N} is called a *set of ε -isomorphism* of T if the equivalence

$$(1) \quad (1 - \varepsilon) \sum_{i \in \sigma} \|a_i T e_i\|^2 \leq \left\| \sum_{i \in \sigma} a_i T e_i \right\|^2 \leq (1 + \varepsilon) \sum_{i \in \sigma} \|a_i T e_i\|^2$$

holds for every choice of scalars $(a_i)_{i \in \sigma}$. The *ε -isomorphism structure* $\Sigma(T, \varepsilon)$ consists of all such sets σ .

How big is the isomorphism structure? From the probabilistic point of view, we can ask for the probability that a random subset of (a finite interval of) the basis is the set of isomorphism. Unfortunately, this probability is in general exponentially small. For example, if T acts as $T e_i = e_{\lceil (i+1)/2 \rceil}$, then every set of isomorphism contains no pairs of the form $\{2i - 1, 2i\}$. Hence a random subset of a finite interval is unlikely to be a set of isomorphism of T . However, an appropriate notion of randomness yields a clean optimal bound on the size of the isomorphic structure. This is the main result of this note, which extends in several ways the Bourgain-Tzafriri’s principle of the restricted invertibility [1], as we will see shortly.

Theorem 1.2. *Let T be a norm-one linear operator on a Hilbert space H , and let $0 < \varepsilon < 1$. Then there exists a probability measure ν on the isomorphism structure $\Sigma(T, \varepsilon)$, such that*

$$(2) \quad \nu\{\sigma \in \Sigma(T, \varepsilon) \mid i \in \sigma\} \geq c\varepsilon^2 \|T e_i\|^2 \quad \text{for all } i.$$

Here and thereafter c, C, c_1, \dots denote positive absolute constants.

Theorem 1.2 gives a lower bound on the average of the characteristic functions of the sets of the isomorphism. Indeed, the left hand side in (2) clearly equals $\int_{\Sigma(T, \varepsilon)} \chi_\sigma(i) d\nu(\sigma)$. Thus, in absence of “true” randomness in the isomorphic structure $\Sigma(T, \varepsilon)$, we can still measure the size of $\Sigma(T, \varepsilon)$ by bounding below the *average of the characteristic functions* of its sets. It might be that considering this weak type of randomness might help in other problems, in which the usual, strong randomness, fails.

1.3. Principle of restricted invertibility

One important consequence of Theorem 1.2 is that *there always exists a big set of isomorphism of T* . This extends and strengthens a well known result due to Bourgain and Tzafriri, known under the name of the principle of restricted invertibility [1]. We will show how to find a big set of isomorphism; its size can be measured with respect to an arbitrary measure μ on \mathbb{N} . For the rest of the paper, we denote the measure of the singletons $\mu(\{i\})$ by μ_i . Summing over i with weights μ_i in (2) and using Theorem 1.2, we obtain

$$(3) \quad \begin{aligned} \int_{\Sigma(T, \varepsilon)} \mu(\sigma) d\nu(\sigma) &= \sum_i \mu_i \int_{\Sigma(T, \varepsilon)} \chi_\sigma(i) d\nu(\sigma) \\ &= \sum_i \mu_i \nu\{\sigma \in \Sigma(T, \varepsilon) \mid i \in \sigma\} \geq c\varepsilon^2 \sum_i \mu_i \|T e_i\|^2. \end{aligned}$$

Replacing the integral in the left hand side of (4) by the maximum shows that there exists a big set of isomorphism:

Corollary 1.3. *Let T be a norm-one linear operator on a Hilbert space H , and let μ be a measure on \mathbb{N} . Then, for every $0 < \varepsilon < 1$, there exists a set of ε -isomorphism σ of T such that*

$$(4) \quad \mu(\sigma) \geq c\varepsilon^2 \sum_i \mu_i \|Te_i\|^2.$$

Earlier, Bourgain and Tzafriri [1] proved a weaker form of Corollary 1.3 with only the *lower* bound in the definition (1) of the set of isomorphism, for a uniform measure μ on an interval, under an additional assumption on the uniform lower bound on $\|Te_i\|$, and for some fixed ε .

Theorem 1.4 (Bourgain-Tzafriri's principle of restricted invertibility).

Let T be a linear operator on an n -dimensional Hilbert space H with an orthonormal basis (e_i) . Assume that $\|Te_i\| = 1$ for all i . Then there exists a subset σ of $\{1, \dots, n\}$ such that $|\sigma| \geq cn/\|T\|^2$ and

$$\|Tf\| \geq c\|f\|$$

for all $f \in \text{span}(e_i)_{i \in \sigma}$.

This important result has found applications in Banach space theory and harmonic analysis. Corollary 1.3 immediately yields a stronger result, which is dimension-free and which yields an almost isometry:

Corollary 1.5. *Let T be a linear operator on a Hilbert space H with an orthonormal basis (e_i) . Assume that $\|Te_i\| = 1$ for all i . Let μ be a measure on \mathbb{N} . Then, for every $0 < \varepsilon < 1$, there exists a subset σ of \mathbb{N} such that $\mu(\sigma) \geq c\varepsilon^2/\|T\|^2$ and such that*

$$(5) \quad (1 - \varepsilon)\|f\| \leq \|Tf\| \leq (1 + \varepsilon)\|f\|$$

for all $f \in \text{span}(e_i)_{i \in \sigma}$.

Szarek [5] proved a weaker form of Corollary 1.3 with only the *upper* bound in the definition (1) of the set of isomorphism, and with some fixed ε .

For the counting measure on \mathbb{N} , Corollary 1.3 was proved in [7]. In this case, bound (4) reads as

$$(6) \quad |\sigma| \geq c\varepsilon^2 \|T\|_{\text{HS}}^2,$$

where $\|T\|_{\text{HS}}$ denotes the Hilbert-Schmidt norm of T . (If T is not a Hilbert-Schmidt operator, then an infinite σ exists).

2. Proof of Theorem 1.2

Corollary 1.3 is a consequence of two suppression results due to Szarek [5] and Bourgain-Tzafriri [2]. We will then deduce Theorem 1.2 from Corollary 1.3 by a simple separation argument from [2].

To prove Corollary 1.3, we can assume by a straightforward approximation that our Hilbert space H is finite dimensional. We can thus identify H with the n -dimensional Euclidean space ℓ_2^n , and identify the basis $(e_i)_{i=1}^n$ of H with the canonical basis of ℓ_2^n . Given a subset σ of $\{1, \dots, n\}$ (or of \mathbb{N}), by ℓ_2^σ we denote the subspace of ℓ_2^n (of ℓ_2 respectively) spanned by $(e_i)_{i \in \sigma}$. The orthogonal projection onto ℓ_2^σ is denoted by Q_σ .

With a motivation different from ours, Szarek proved in ([5], Lemma 4) the following suppression result for operators in ℓ_2^n .

Theorem 2.1 (Szarek). *Let T be a norm-one linear operator on ℓ_2^n . Let $\lambda_1, \dots, \lambda_n$, $\sum_{i=1}^n \lambda_i = 1$, be positive weights. Then there exists a subset σ of $\{1, \dots, n\}$ such that*

$$(7) \quad \sum_{i \in \sigma} \lambda_i \|Te_i\|^{-2} \geq c$$

and such that the inequality

$$\left\| \sum_{i \in \sigma} a_i Te_i \right\|^2 \leq C \sum_{i \in \sigma} \|a_i Te_i\|^2$$

holds for every choice of scalars $(a_i)_{i \in \sigma}$.

Remark 2.2. Inequality (7) for a probability measure λ on $\{1, \dots, n\}$ is equivalent to the inequality

$$(8) \quad \mu(\sigma) \geq c \sum_i \mu_i \|Te_i\|^2$$

for a positive measure μ on $\{1, \dots, n\}$.

Indeed, (7) implies (8) with

$$\lambda_i = \frac{\mu_i \|Te_i\|^2}{\sum_i \mu_i \|Te_i\|^2}.$$

Conversely, (8) implies (7) with $\mu_i = \lambda_i \|Te_i\|^{-2}$.

Theorem 2.1 and Remark 2.2 yield a weaker version of Corollary 1.3 – with only the upper bound in the definition (1) of the set of isomorphism, and with some fixed ε .

To prove Corollary 1.3 in full strength, we will use the following suppression analog of Theorem 1.2 due to Bourgain and Tzafriri [2].

Theorem 2.3 (Bourgain-Tzafriri). *Let S be a linear operator on ℓ_2 whose matrix relative to the unit vector basis has zero diagonal. For a $\delta > 0$, denote by $\Sigma'(S, \delta)$ the family of all subsets σ of \mathbb{N} such that $\|Q_\sigma S Q_\sigma\| \leq \delta \|S\|$. Then there exists a probability measure ν' on $\Sigma'(S, \delta)$ such that*

$$(9) \quad \nu' \{ \sigma \in \Sigma'(S, \delta) \mid i \in \sigma \} \geq c \delta^2 \quad \text{for all } i.$$

Proof of Corollary 1.3. We define a linear operator T_1 on $H = \ell_2^n$ as

$$T_1 e_i = Te_i / \|Te_i\|, \quad i = 1, \dots, n.$$

Theorem 2.1 and the remark below it yield the existence of a subset σ of $\{1, \dots, n\}$ whose measure satisfies (8) and such that the inequality

$$\|T_1 f\| \leq C \|f\|$$

holds for all $f \in \text{span}(e_i)_{i \in \sigma}$. In other words, the operator

$$T_2 = T_1 Q_\sigma$$

satisfies

$$(10) \quad \|T_2\| \leq C.$$

We will apply Theorem 2.3 for the operator S on ℓ_2^σ defined as

$$(11) \quad S = T_2^* T_2 - I \quad \text{and with} \quad \delta = \varepsilon / \|S\|.$$

Indeed, S has zero diagonal:

$$\langle S e_i, e_i \rangle = \|T_2 e_i\|^2 - 1 = \|T_1 e_i\|^2 - 1 = 0 \quad \text{for all } i \in \sigma.$$

Also, S has nicely bounded norm by (10):

$$\|S\| \leq \|T_2\|^2 + 1 \leq C^2 + 1,$$

which yields a lower bound on δ :

$$(12) \quad \delta \geq \varepsilon / (C^2 + 1).$$

So, Theorem 2.3 yields a family $\Sigma'(S, \delta)$ of subsets of σ and a measure ν' on this family. It follows as before that $\Sigma'(S, \delta)$ must contain a big set, because

$$\begin{aligned} \int_{\Sigma'(S, \delta)} \mu(\sigma') \, d\nu'(\sigma') &= \sum_{i \in \sigma} \mu_i \int_{\Sigma'(S, \delta)} \chi_{\sigma'}(i) \, d\nu'(\sigma') \\ &= \sum_{i \in \sigma} \mu_i \nu' \{ \sigma' \in \Sigma'(S, \delta) \mid i \in \sigma' \} \\ &\geq \sum_{i \in \sigma} \mu_i \cdot c \delta^2 \geq c' \varepsilon^2 \mu(\sigma) \end{aligned}$$

where the last inequality follows from (12) with $c' = c(C^2 + 1)^{-2}$. Thus there exists a set $\sigma' \in \Sigma'(S, \delta)$ such that by (8) we have

$$\mu(\sigma') \geq c' \varepsilon^2 \mu(\sigma) \geq c'' \varepsilon^2 \sum_{i=1}^n \mu_i \|T e_i\|^2,$$

so with the measure as required in (4).

It remains to check that σ' is a set of ε -isomorphism of T . Consider an $f \in \text{span}(e_i)_{i \in \sigma'}$, $\|f\| = 1$. By the suppression estimate in Theorem 2.3 and by our choice of S and δ made in (11), we have

$$\begin{aligned} \varepsilon = \delta \|S\| &\geq |\langle Q_{\sigma'} S Q_{\sigma'} f, f \rangle| \\ &= |\langle S f, f \rangle| \quad \text{because } Q_{\sigma'} f = f \\ &= \| \|T_2 f\|^2 - \|f\|^2 \| \quad \text{by the definition of } S \\ &= \| \|T_1 f\|^2 - 1 \| \quad \text{because } Q_{\sigma'} f = Q_{\sigma} f = f \text{ as } \sigma' \subset \sigma. \end{aligned}$$

It follows by homogeneity that

$$(1 - \varepsilon) \|f\|^2 \leq \|T_1 f\|^2 \leq (1 + \varepsilon) \|f\|^2 \quad \text{for all } f \in \text{span}(e_i)_{i \in \sigma'}.$$

By the definition of T_1 , this means that σ' is a set of ε -isomorphism of T . This completes the proof. \square

Proof of Theorem 1.2. We deduce Theorem 1.2 from Corollary 1.3 by a separation argument, which is a minor adaptation of the proof of Corollary 1.4 in [2].

We first note that, by Remark 2.2, an equivalent form of the consequence of Corollary 1.3 is the following. For every probability measure λ on \mathbb{N} , there exists a set $\sigma \in \Sigma(T, \varepsilon)$ such that

$$(13) \quad \sum_{i \in \sigma} \lambda_i \|Te_i\|^{-2} \geq c\varepsilon^2.$$

We consider the space of continuous functions $C(\Sigma(T, \varepsilon))$ on the isomorphism structure $\Sigma(T, \varepsilon)$, which is compact in its natural topology (of pointwise convergence of the indicators of the sets $\sigma \in \Sigma(T, \varepsilon)$). For each $i \in \mathbb{N}$, define a function $\pi_i \in C(\Sigma(T, \varepsilon))$ by setting

$$\pi_i(\sigma) = \chi_\sigma(i) \|Te_i\|^{-2}, \quad \sigma \in \Sigma(T, \varepsilon).$$

Let \mathcal{C} be the convex hull of the set of functions $\{\pi_i, i \in \mathbb{N}\}$. Every $\pi \in \mathcal{C}$ can be expressed a convex combination $\pi = \sum_i \lambda_i \pi_i$. By Corollary 1.3 in the form (13), there exists a set $\sigma \in \Sigma(T, \varepsilon)$ such that $\pi(\sigma) \geq c\varepsilon^2$. Thus $\|\pi\|_{C(\Sigma(T, \varepsilon))} \geq c\varepsilon^2$. We conclude by the Hahn-Banach theorem that there exists a probability measure $\nu \in C(\Sigma(T, \varepsilon))^*$ such that

$$\nu(\pi) = \int_{\Sigma(T, \varepsilon)} \pi(\sigma) d\nu(\sigma) \geq c\varepsilon^2 \quad \text{for all } \pi \in \mathcal{C}.$$

Applying this estimate for $\pi = \pi_i$, we obtain

$$\int_{\Sigma(T, \varepsilon)} \chi_\sigma(i) d\nu(\sigma) \geq c\varepsilon^2 \|Te_i\|^2,$$

which is exactly the conclusion of the theorem. □

Remark 2.4. The proof of Theorem 1.2 given above is a combination of previously known tools – two suppression results due to [5] and [2] and a separation argument from [2]. The new point was to realize that the suppression result of Szarek [5], developed with a different purpose in mind, gives a sharp estimate when combined with the results of [2]. To find a set of the isomorphism as in (1), one needs to reduce the norm of the operator with [5] *before* applying restricted invertibility principles from [2].

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