2. PROJECTION OPERATORS

A projection operator allows us to decompose a Banach space X as well as a commuting bounded operator T on X. In this way, we are able to concentrate only on a 'part' of X, or of T. These projection operators will often occur in the spectral theory as well as in various approximation procedures that we shall study.

A complex Banach space X is said to be <u>decomposed by a pair (Y,Z) of its closed subspaces</u> if $X = Y + Z$ and $Y \cap Z = \{0\}$. In this case, we write

$$X = Y \oplus Z .$$

This happens if and only if every $x \in X$ can be written in a unique way as $y + z$ with $Y \in Y$ and $z \in Z$; if we let $Px = y$, then P is a linear map from X to X and satisfies $P^2 = P$, i.e., P is a <u>projection</u>. Also, the set $\{(x,Px) : x \in X\}$ is closed in $X \times X$. This can be seen as follows. Let $x_n \to x$ and $Px_n \to y$. Since $Px_n \in Y$ and Y is closed, we see that $y \in Y$. Also, $x_n - Px_n \in Z$ and Z is closed, so that $x - y \in Z$. Since $x = y + (x-y)$ with $y \in Y$ and $x - y \in Z$, we have $Px = y$. This shows that P is a *closed* operator; the *closed graph theorem* tells us that P is, in fact, continuous ([L], 10.3). This operator P is called the <u>projection from</u> X <u>on</u> Y <u>along</u> Z.

On the other hand, starting with a projection operator $P \in BL(X)$ we obtain a decomposition of X as follows: Let $Y = R(P)$ and $Z = Z(P)$. Since P is continuous, Z is closed; also, since $Y = Z(I-P)$, where $I - P$ is continuous, we see that Y is closed. Moreover, for every $x \in X$, we have $x = Px + (x-Px)$, so that $X = Y + Z$. Clearly, $x \in Y \cap Z$ implies $x = Px = 0$. Thus,

$$X = R(P) \oplus Z(P) .$$

It is worthwhile to note that for $P \in BL(X)$, we have

(2.1) $$\text{either } P = 0 \text{ or } \|P\| \geq 1 .$$

This follows easily from $\|P\| = \|P^2\| \leq \|P\|^2$.

Let X be decomposed by (Y,Z) and consider $T \in BL(X)$. We say that T <u>is decomposed by</u> (Y,Z) if $T(Y) \subset Y$ and $T(Z) \subset Z$, i.e., if Y and Z are <u>invariant subspaces for</u> T.

In this case, if we let $T_Y = T|_Y : Y \to Y$ and $T_Z = T|_Z : Z \to Z$, then for

$$x = y + z , \; y \in Y , \; z \in Z ,$$

we have

$$Tx = T_Y y + T_Z z .$$

This allows us to write

$$T = T_Y \oplus T_Z .$$

We now give a criterion for T to be decomposed by (Y,Z).

PROPOSITION 2.1 Let $X = Y \oplus Z$ and P be the projection on Y along Z. Then $T \in BL(X)$ is decomposed by (Y,Z) if and only if $PT = TP$, i.e., T and P commute. In this case, we have

$$T_Y = PTP|_Y \quad \text{and} \quad T_Z = (I-P)T(I-P)|_Z .$$

Proof $PT = TP$ if and only if $PTx = TPx$ for all $x \in X$ if and only if $PTy + PTz = TPy + TPz$ for all $y \in Y$ and $z \in Z$, i.e., $PTy + PTz = Ty$ for all $y \in Y$ and $z \in Z$ if and only if $PTy = Ty$

and $PTz = 0$ for all $y \in Y$ and $z \in Z$ (upon applying P to both sides). This happens if and only if $Ty \in Y$ and $Tz \in Z$ for all $y \in Y$ and $z \in Z$, i.e., $T(Y) \subset Y$ and $T(Z) \subset Z$. The rest is easy. //

Let us now relate the results of this section to the adjoint considerations of Section 1.

PROPOSITION 2.2 (a) Let $X = Y \oplus Z$. Then

$$X^* = Z^\perp \oplus Y^\perp .$$

If P is the projection on Y along Z, then P^* is the projection on $Z^\perp$ along $Y^\perp$; thus,

$$R(P^*) = Z^\perp \quad \text{and} \quad Z(P^*) = Y^\perp . \tag{2.2}$$

The linear map $F : Z^\perp \to Y^*$ given by

$$Fy^* = y^*|_Y$$

is one to one and onto.

(b) Let $T \in BL(X)$ and $T = T_Y \oplus T_Z$. Then

$$T^* = (T^*)_{Z^\perp} \oplus (T^*)_{Y^\perp} .$$

The map $(T^*)_{Z^\perp}$ can be identified with $(T_Y)^*$ as a linear map via the map F, i.e., the following diagram commutes

$$\begin{tikzcd} Z^\perp \arrow[r, "F"] \arrow[d, "(T^*)_{Z^\perp}"'] & Y^* \arrow[d, "(T_Y)^*"] \\ Z^\perp \arrow[r, "F"] & Y^* \end{tikzcd}$$

Proof (a) Since P is a projection, we have

$$(P^*)^2 = (P^2)^* = P^* ,$$

Also, by 1.3(c),

$$Z(P^*) = R(P)^\perp = Y^\perp ,$$
$$R(P^*) = Z(I-P^*) = R(I-P)^\perp = Z^\perp .$$

Hence P^* is a projection from X^* on $Z^\perp$ along $Y^\perp$. Thus,

$$X^* = Z^\perp \oplus Y^\perp .$$

Now, let $y^* \in Z^\perp$. If $Fy^* = y^*|_Y = 0$, then $\langle y^*,x\rangle = 0$ for all $x \in Z \cup Y$, i.e., $y^* = 0$. This shows that the map F is one to one. Next, for $w^* \in Y^*$, define $y^* \in X^*$ by

$$\langle y^*,x\rangle = \langle w^*,Px\rangle , \quad x \in X .$$

Then $y^* \in Z^\perp$ and $Fy^* = w^*$. Thus, the map F is onto.

(b) Since T is decomposed by (Y,Z) , we see by Proposition 2.1 that $TP = PT$. Hence

$$T^*P^* = (PT)^* = (TP)^* = P^*T^* ,$$

so that T^* is decomposed by $R(P^*) = Z^\perp$ and $Z(P^*) = Y^\perp$.

Lastly, for $y^* \in Z^\perp$ and $y \in Y$, we have

$$\begin{aligned} \langle (T_Y)^*Fy^*,y\rangle &= \langle Fy^*,T_Yy\rangle \\ &= \langle y^*,Ty\rangle \\ &= \langle T^*y^*,y\rangle \\ &= \langle (T^*)_{Z^\perp}y^*,y\rangle \\ &= \langle F(T^*)_{Z^\perp}y^*,y\rangle . \end{aligned}$$

This shows that we can identify $(T^*)_{Z^\perp}$ with $(T_Y)^*$ via the map F . //

The result in part (a) of the above proposition can be illustrated as follows.

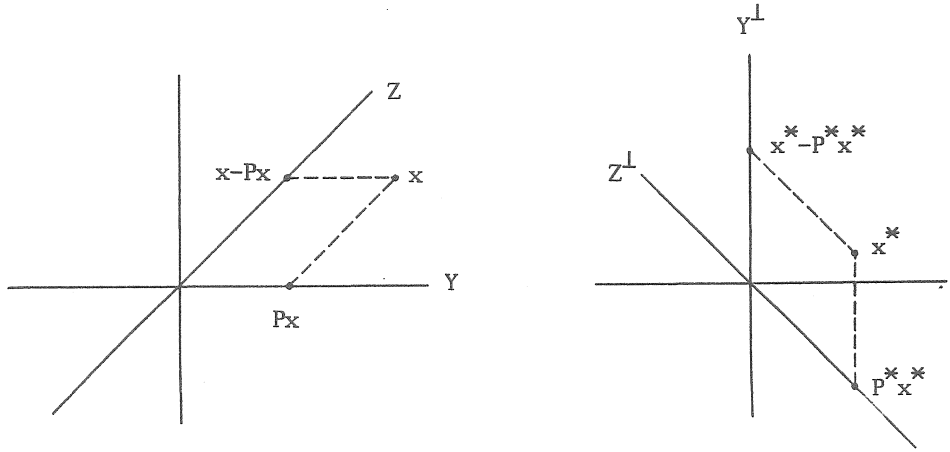

Figure 2.1

So far we have not said anything about the existence of a decomposition of X . Indeed, given a closed subspace Y of X , there may not exist any closed subspace Z of X such that $X = Y \oplus Z$. Such is the case if X is the space of all complex-valued bounded functions on $[a,b]$ and $Y = C([a,b])$. (See [F].) However, if Y is a finite dimensional subpace of a Banach space X , then there exist many closed subspaces Z of X such that $X = Y \oplus Z$, as we shall see in the next section. We now show that if X is a Hilbert space, then every closed subspace Y of X can be 'complemented', and that too in a canonical manner.

PROPOSITION 2.3 Let X be a Hilbert space and Y be a closed subspace of X . Then

$$X = Y \oplus Y^\perp .$$

The projection P on Y along $Y^{\perp}$ satisfies

$$P = 0 \quad \text{or} \quad \|P\| = 1 \ , \quad \text{and}$$

$$\langle Px,x\rangle \geq 0 \quad \text{for all} \quad x \in X \ .$$

In particular, P is self-adjoint. Conversely, if a projection $P \in BL(X)$ is normal, then

$$R(P)^{\perp} = Z(P) \ .$$

Proof Let $x \in X$ and $d = \text{dist}(x,Y)$. Find $y_n \in Y$ such that

$$\|x-y_n\| \to d \quad \text{as} \quad n \to \infty \ .$$

By the *parallelogram law*,

$$2\|x-y_n\|^2 + 2\|x-y_m\|^2 = \|2x-y_n-y_m\|^2 + \|y_n-y_m\|^2 \ .$$

Now,

$$2d \leq 2\|x-(y_n+y_m)/2\| = \|2x-y_n-y_m\| \leq \|x-y_n\| + \|x-y_m\| \ ,$$

which tends to $2d$ as $n,m \to \infty$. Hence $\|y_n-y_m\|^2 \to 0$ as $n,m \to \infty$, i.e., (y_n) is a Cauchy sequence in Y . Let $y_n \to y \in Y$, since Y is closed. We show that $x - y \in Y^{\perp}$. Let $y_0 \in Y$ with $\|y_0\| = 1$. Since

$$x - y = [(x-y) - \langle x-y,y_0\rangle y_0] + \langle x-y,y_0\rangle y_0 \ ,$$

the *Pythagoras theorem* shows that

$$\|x-y\|^2 = \|(x-y) - \langle x-y,y_0\rangle y_0\|^2 + |\langle x-y,y_0\rangle|^2 \ .$$

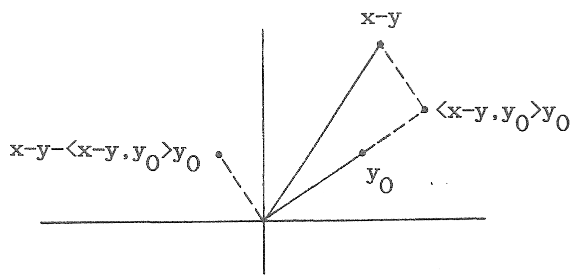

Figure 2.2

On the other hand, since the element $y + \langle x-y, y_0\rangle y_0$ belongs to Y,

$$\|x-y\| = d \leq \|(x-y) - \langle x-y, y_0\rangle y_0\| .$$

Hence $\langle x-y, y_0\rangle = 0$, i.e., $x - y$ is orthogonal to y_0. Since y_0 is an arbitrary element of norm 1 in Y, we see that $x - y \in Y^{\perp}$.

Thus, $x = y + (x-y)$ with $y \in Y$ and $x - y \in Y^{\perp}$. Since $Y \cap Y^{\perp} = \{0\}$, we have $X = Y \oplus Y^{\perp}$.

Let P be the projection on Y along $Y^{\perp}$. Then for all $x \in X$,

$$\begin{aligned}\langle Px, x\rangle &= \langle Px, Px\rangle + \langle Px, x-Px\rangle \\ &= \langle Px, Px\rangle \geq 0 .\end{aligned}$$

This implies, in particular, that $\langle Px, x\rangle$ is real for all $x \in X$. Hence by (1.8), P is self-adjoint. Also, for $x \in X$, the *Pythagoras theorem* shows that

$$\|x\|^2 = \|Px + (x-Px)\|^2 = \|Px\|^2 + \|x-Px\|^2$$

since $\langle Px, x-Px\rangle = 0$. Thus, $\|Px\|^2 \leq \|x\|^2$, i.e., $\|P\| \leq 1$. But we always have $P = 0$ or $\|P\| \geq 1$ for any projection P. Hence in the present case, $P = 0$ or $\|P\| = 1$.

Lastly, let $P \in BL(X)$ be a normal projection. Then by (1.8), $Z(P) = Z(P^*)$. But by Proposition 1.3(c), $Z(P^*) = R(P)^{\perp}$. Hence $Z(P) = R(P)^{\perp}$. //

The projection on a closed subspace Y of a Hilbert space X along its orthogonal complement $Y^\perp$ is called the <u>orthogonal projection on</u> Y. Thus, a projection $P \in BL(X)$ is orthogonal if and only if $Z(P) = R(P)^\perp$.

Before we conclude this section, we introduce the concept of the gap between two closed subspaces of a Banach space X and relate it to projections on them.

Let Y and $\tilde{Y}$ be closed subspaces of X. If $Y = \{0\}$, let $\delta(Y,\tilde{Y}) = 0$, and otherwise let

$$\delta(Y,\tilde{Y}) = \sup\{\mathrm{dist}(y,\tilde{Y}) : y \in Y ,\ \|y\| = 1\} .$$

Thus, $\delta(Y,\tilde{Y})$ is the smallest number δ such that

$$\mathrm{dist}(y,\tilde{Y}) \leq \delta\|y\| \quad \text{for all} \quad y \in Y .$$

It is clear that $0 \leq \delta(Y,\tilde{Y}) \leq 1$ and $\delta(Y,\tilde{Y}) = 0$ if and only if $Y \subset \tilde{Y}$. We note that $\delta(Y,\tilde{Y})$ can be zero even when $Y \neq \tilde{Y}$, and may not equal $\delta(\tilde{Y},Y)$. To mend these matters, define the <u>gap</u> between Y and $\tilde{Y}$ by

$$\hat{\delta}(Y,\tilde{Y}) = \max\{\delta(Y,\tilde{Y}) ,\ \delta(\tilde{Y},Y)\} . \tag{2.3}$$

Then $\hat{\delta}(Y,\tilde{Y}) = 0$ if and only if $Y = \tilde{Y}$ and $\delta(Y,\tilde{Y}) = \delta(\tilde{Y},Y)$.

Let P and $\tilde{P}$ be projections onto Y and $\tilde{Y}$, respectively. Then it follows that

$$\begin{aligned} &\delta(Y,\tilde{Y}) \leq \|(P-\tilde{P})P\| , \\ &\hat{\delta}(Y,\tilde{Y}) \leq \max\{\|(P-\tilde{P})P\| ,\ \|(\tilde{P}-P)\tilde{P}\|\} . \end{aligned} \tag{2.4}$$

In case X is a Hilbert space and P as well as $\tilde{P}$ are *orthogonal* projections, then it can be easily seen that

$$\delta(Y,\tilde{Y}) = \|(P-\tilde{P})P\|_2 \ ,$$

(2.5)

$$\hat{\delta}(Y,\tilde{Y}) = \max\{\|(P-\tilde{P})P\|_2 \ , \ \|(\tilde{P}-P)\tilde{P}\|_2\} \ .$$

In fact, Kato has proved that

(2.6) $$\hat{\delta}(Y,\tilde{Y}) = \|P-\tilde{P}\|_2 \leq \|Q-\tilde{Q}\|_2 \ ,$$

where Q and $\tilde{Q}$ are any projections on Y and $\tilde{Y}$ respectively ([K], Problem 6.33, Theorems 6.34 and 6.35). In particular, $\hat{\delta}$ is a metric on the set of all closed subspaces of a Hilbert space.

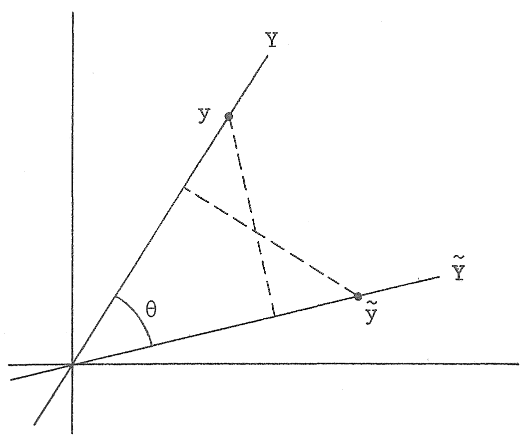

Figure 2.3

$\hat{\delta}(Y,\tilde{Y})$ can be interpreted to be the sine of 'the acute angle between Y and $\tilde{Y}$'. (See [GV], p.22.)

Problems

2.1 If P is a projection, then $I - P$ is a projection on $Z(P)$ along $R(P)$; if P is orthogonal then so is $I - P$.

2.2 Let $Y_1,\dots,Y_n$ be closed subspaces of X . Then $X = Y_1 + \dots + Y_n$ and $Y_i \cap Y_j = \{0\}$ for $i \neq j$ (i.e., $X = Y_1 \oplus \dots \oplus Y_n$) if and only if there are projections $P_1,\dots,P_n$ such that $R(P_i) = Y_i$, $P_iP_j = 0$ if $i \neq j$, and $I = P_1 + \dots + P_n$. Let $T \in BL(X)$. Then $T = T_{Y_1} \oplus \dots \oplus T_{Y_n}$ if and only if $TP_i = P_iT$ for $i = 1,\dots,n$.

2.3 Let P and Q be projections. (i) $P + Q$ is a projection if and only if the <u>Jordan product</u> $P \circ Q \equiv (PQ+QP)/2 = 0$, and then $PQ = 0$. (ii) For P and Q orthogonal, $PQ = 0$ if and only if $P \circ Q = 0$. (iii) $(P-Q)^2 + (I-P-Q)^2 = I$ and $(P-Q)^2$ commutes with both P and Q.

2.4 Let X be a Hilbert space and Y a closed subspace of X. If (u_α) is an orthonormal basis of Y, then the orthogonal projection on Y is given by

$$Px = \sum_\alpha \langle x, u_\alpha \rangle u_\alpha , \quad x \in X .$$

(Cf. 22.6 of [L].) For $x \in X$, Px is the best approximation to x from Y, i.e., $\|x-Px\| = \mathrm{dist}(x,Y)$. (Cf. 23.2 of [L].)

2.5 The map F of Proposition 2.2 which sends $y^* \in Z^\perp$ to $y^*|_Y$ need not be an isometry of $Z^\perp$ onto Y^*.

2.6 For $Y \subset X$, let $S_Y = \{y \in Y : \|y\| = 1\}$. Let Y and $\tilde{Y}$ be closed subspaces of X. Define

$$(2.5) \qquad d(Y,\tilde{Y}) = \begin{cases} 0 , & \text{if } Y = \{0\} \\ \sup\{\mathrm{dist}(y,S_{\tilde{Y}}) : y \in S_Y\} , & \text{if } Y \neq \{0\} \neq \tilde{Y} \\ 2 , & \text{if } Y \neq \{0\} ,\ \tilde{Y} = \{0\} . \end{cases}$$

Then $d(Y,Z) \leq d(Y,\tilde{Y}) + d(\tilde{Y},Z)$ for a closed subspace Z of X. Let

$$(2.6) \qquad \hat{d}(Y,\tilde{Y}) = \max\{d(Y,\tilde{Y}) ,\ d(\tilde{Y},Y)\} .$$

Then $\hat{d}$ is a metric on the closed subspaces of a Banach space X.

2.7 Let $X = \mathbb{C}^n$ and $q_1,\ldots,q_k$ form an orthonormal basis of a closed subspace Y of X. Let Q denote the $n \times k$ matrix whose j-th column is q_j. Then $Q^H Q = I_k$, and the orthogonal projection on Y is given by QQ^H.