POWER CONCAVITY OF SOLUTIONS OF DIRICHLET PROBLEMS

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This talk consists of two items. The first is a simplified version of a concavity theorem. The second is an indication of how the result might be extended to a certain class of Dirichlet problems for degenerate quasilinear equations.

Let Ω be a bounded convex domain in $\mathbb{R}^n$ with $n \geqslant 2$. It was recently shown ([2]), under some complicated conditions on the positive function $b\colon \Omega \times (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, that if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a solution to the problem

$$\Delta u + b(x,u,Du) = 0 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

then u is power concave. That is, $u(x)^\alpha$ is a concave function of x in $\overline{\Omega}$ for some $\alpha > 0$. The conditions stated for b were inequalities for various polynomials of derivatives of b. These conditions were difficult to interpret, but have now been considerably simplified, as shown in case (7) of the following table, which summarises power concavity results for various categories of function b. For any category C of function b, let

$$\alpha_0(C) = \inf \{\alpha \in \mathbb{R};\ (u(x)^\alpha - 1)/\alpha \text{ is concave in } \Omega\}$$

where the infimum is taken over all b in C and all bounded convex Ω, and $(u(x)^\alpha-1)/\alpha$ is understood to mean $\log(u(x))$ when $\alpha = 0$. [Note: the above set of α is an interval, since if $(u^\alpha-1)/\alpha$ is concave, then $(u^\beta-1)/\beta$ is concave for all $\beta \leqslant \alpha$.]

Category of $b(x,u,Du)$	Best concavity power α_0	
(1) $b \equiv k > 0$	$\alpha_0 = \frac{1}{2}$	
(2) $b = \lambda_0 u,\ \lambda_0 > 0$	$\alpha_0 = 0$	
(3) $b = \lambda u^\gamma,\ 0 < \gamma < 1,\ \lambda > 0$	$\alpha_0 \geqslant (1-\gamma)/2$	
(4) $b = f(x)$, f^β concave, $\beta \geqslant 1$	$\alpha_0 = \beta/(1+2\beta)$	
(5) $b = f(x)u^\gamma$, f^β concave, $\beta \geqslant 1,\ 0 < \gamma < 1$	$\alpha_0 \geqslant (1-\gamma)\beta/(1+2\beta)$	
(6) $b = h(u),\ h > 0,\ 0 < \alpha \leqslant 1$, $u^{\alpha-1}h(u)$ is decreasing with respect to u, and $u^{(3\alpha-1)/\alpha}h(u^{1/\alpha})$ is concave with respect to u	$\alpha_0 \geqslant \alpha$	
(7) $b = b(x,u,Du),\ b > 0,\ 0 < \alpha \leqslant 1$, $u^{\alpha-1}b(x,u,Du)$ is decreasing with respect to u, and $u^{(3\alpha-1)/\alpha}b(x,u^{1/\alpha},Du)$ is jointly concave with respect to the variable (x,u) in $\Omega\times(0,\infty)$	$\alpha_0 \geqslant \alpha$	

The concavity powers indicated by a badger have been proved only for Ω satisfying an interior cone condition. The last result in the table includes the other results as special limiting cases. Note that none of the above results give $\alpha_0 > \frac{1}{2}$. Indeed, if Δu is bounded and $\partial\Omega$ has a cone-like point of small enough aperture, then comparison of u with the torsion function for Ω (case (1)), shows that $\alpha_0 \leqslant \frac{1}{2}$.

Now that a wide range of concavity results has been obtained for the Laplacian operator, the question of generalisations to other operators arises. The second item of this talk is a semi-conjectural outline of a possible short proof (three pages) of a recent result of Huisken ([1]), namely that a convex surface in $\mathbb{R}^n$ contracting with a velocity proportional to its mean curvature will contract to a point. The "proof" proposed here uses a Dirichlet problem formulation rather than evolution equations for the metric of the surface. If u(x) denotes the time taken by the surface to move from its starting position on the boundary $\partial\Omega$ of a convex set Ω to x, then it is readily shown that u satisfies

$$L_1 u + 1 = 0 \quad \text{in } \{x \in \Omega;\ Du \neq 0\}$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where for $q \in \mathbb{R}$, L_q denotes the operator $L_q u = \Delta u + (q-2)u_{nn}$, and $u_{nn}(x)$ denotes the second derivative of u in a direction normal to the level surface of u passing through x; that is, $u_{nn} = u_{ij}u_i u_j |\nabla u|^{-2}$. For $q = 2$, $L_q = \Delta$, and so $\alpha_0 = \frac{1}{2}$. But when $q \neq 2$, L_q is degenerate in the sense that at a stationary point of u the discontinuity of $L_q u$ can not be removed unless the Hessian of u converges to a multiple of the identity matrix (so that in some sense the level curves of u converge to spheres near the stationary point of u).

If, however, $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is known to satisfy the above Dirichlet problem for $q \geqslant 1$, then the same methods used to obtain case (7) in the above table show that $\alpha_0 \geqslant (q-1)/q$. Thus if solutions exist for all $q > 1$ with a uniform bound, then a limit function will exist as $q \to 1$ and the logarithm of this function will be concave. This would then imply Huisken's result. It remains only to show the existence of suitable solutions for $q > 1$, a task which may or may not be aided by the apriori power concavity of such solutions.

The attraction of this attempt to duplicate Huisken's result by a different method lies principally in the hope that the technique may provide existence proofs for classes of Dirichlet problem generalising the above example.

REFERENCES

[1] G. Huisken, "Flow by mean curvature of convex surfaces into spheres", Research Report CMA-R10-84, Aust. Nat. Univ., Canberra (1984).

[2] A. U. Kennington, "An improved convexity maximum principle and some applications", thesis, Univ. of Adelaide (Feb. 1984).

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