MINIMISING CURVATURE — A HIGHER DIMENSIONAL ANALOGUE OF THE PLATEAU PROBLEM

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The classical problem at the heart of contemporary geometric measure theory is the Plateau problem. One is given a smooth compact $(k-1)$-dimensional manifold ("boundary") B in $\mathbb{R}^n$ and one asks whether there is a k dimensional object M with boundary ∂M equal to B and having least k-dimensional volume among all such objects.

In order to make the above problem precise we need to clarify in particular the following notions: k dimensional object; k dimensional volume; boundary. In order to solve the problem by means of the usual variational approach (i.e. take a minimising sequence, extract a convergent subsequence, and show the limit has the required properties) our class of k dimensional objects must carry a topology which gives the required compactness and lower semi continuity properties. In a landmark paper [FF], Federer and Fleming introduced the class of k-dimensional integer multiplicity currents in $\mathbb{R}^n$, proved the appropriate compactness property (very difficult) and semi continuity property, and solved the Plateau problem in this context (see the references [F] and [S] for details and further references).

In order to fix our ideas we remark that we can represent a k dimensional integer multiplicity current T in $\mathbb{R}^n$ as a triple $\underline{t}(M,\theta,\xi)$ where M is a countably k-rectifiable subset of $\mathbb{R}^n$, θ is a $\mathcal{H}^k$ measurable summable non-negative integer valued function defined over M, and ξ is a $\mathcal{H}^k$ measurable function defined over M which assigns to $\mathcal{H}^k$ a.e. $x \in M$ one of the two possible orientations of the approximate tangent plane to M at x . In other words, T is the current corresponding to the set M with orientation ξ and multiplicity θ . (Using the earlier conventions in [FF] or [F], T is called a rectifiable current.)

After proving the existence of a minimizer in the context of integer multiplicity currents one is left with further regularity questions. Is the minimizer an analytic manifold or are there necessarily singularities, and is the boundary taken in a regular way? It turns out that in general a minimizing current can have singularities on a "small" set, but that outside this set the current corresponds to an analytic manifold possibly with multiplicity (see [A], [F], [S],[AF]).

The previous considerations generalise from $\mathbb{R}^n$ to an arbitrary ambient Riemannian manifold N . Recently this has been applied to the study of the geometry of N , see [S, §7] for a discussion.

Rather than considering further the Plateau problem, I would like to discuss here an analogous problem which turns out to be quite interesting. Instead of prescribing ∂M and seeking to minimise $\int_M 1$ (where M is a manifold or more generally an integer multiplicity current), we seek to minimise $\int_M F(A)$ where $A = A(M)$ is the second fundamental form of M and F is an appropriate convex function. In terms of the number of differentiations performed in the integrand this is the next natural geometrically invariant problem. There is no notion involving first derivatives which is invariant under rotations; the second fundamental form involves "second derivatives" and encodes all the local curvature information of M as imbedded in $\mathbb{R}^n$.

The following discussion is an outline of the ideas involved in particular in [H1], [H2].

As in the Plateau problem we need an appropriate class $\mathcal{C}$ of objects with which to work. $\mathcal{C}$ should include the class of smooth manifolds and $\mathcal{C}$ should, as before, have appropriate compactness and semicontinuity properties.

For the rest of this discussion I assume the reader has some basic familiarity with the notions of currents and varifolds, as discussed for example in [S].

We first need a notion of boundary, and so $\mathcal{C}$ should presumably be a subset of the set of integral currents. On the other hand current convergence is very weak and seems unlikely to be usefully related to curvature (which involves "second derivatives" if we are working in a local graphical setting). It would appear that on these grounds we need to consider something like varifold convergence (which involves "first derivative" or "tangent plane" information), but then the problem is that varifolds do not have an appropriate notion of boundary. We are thus led to consider the class of oriented integer multiplicity varifolds, which we will presently discuss. For a further discussion along the above lines see [H1, §1].

Henceforth we assume $U \subset \mathbb{R}^n$, U open.

Because of the previous considerations we will be interested in the class $\mathcal{IV}_k^o(U)$ of finite mass oriented integer multiplicity k dimensional varifolds in U. Any varifold $V \in \mathcal{IV}_k^o(U)$ can be written naturally in the form

$$\underline{\underline{v}}(E,\xi,\theta_1) + \underline{\underline{v}}(E,-\xi,\theta_2)$$

where E is a countably k-rectifiable $\mathcal{H}^k$-measurable set, ξ is a $\mathcal{H}^k$-measurable function on E which assigns to $\mathcal{H}^k$ a.e. $x \in E$ one of the two possible orientations of the approximate tangent plane to E at x, and θ_1 and θ_2 are $\mathcal{H}^k$-measurable and $\mathcal{H}^k$-summable non-negative integer valued functions over E. We think of V as being the set E with orientation ξ taken with multiplicity θ_1 and orientation $-\xi$ taken with multiplicity θ_2.

In a natural way we associate to V the integer multiplicity current $\underline{\underline{t}}(V)$ obtained by cancellation, i.e.

$$\underline{\underline{t}}(V) = \underline{\underline{t}}(E,\xi,\theta_1-\theta_2) .$$

The boundary ∂V of V is defined to be the boundary of $\underline{\underline{t}}(V)$ in the usual current sense, and we write

$$\partial V = \partial(\underline{\underline{t}}(V)) .$$

We also associate to each $V \in IV_k^o(U)$ an (unoriented) integer multiplicity current $\underline{\underline{q}}(V) \in IV_k(U)$, the class of (unoriented) integer multiplicity currents, where $\underline{\underline{q}}(V)$ is obtained by ignoring orientation, i.e.

$$\underline{\underline{q}}(V) = \underline{\underline{v}}(E,\theta_1+\theta_2) .$$

The first variation $\|\delta V\|$ of V is obtained by taking the first variation of $q(V)$, i.e.

$$\|\delta V\| = \|\delta(q(V))\| .$$

We put a topology on $IV_k^o(U)$ by regarding $IV_k^o(U)$ as a subset of the set of Radon measures on

$$\underline{\underline{G}}_k^o(U) = U \times \underline{\underline{G}}^o(k,n) ,$$

where $\underline{\underline{G}}^o(k,n)$ is the set of oriented k dimensional subspaces of $\underline{\underline{\mathbb{R}}}^n$ with the usual topology. For each $\phi \in C_c^o(\underline{\underline{G}}_k^o(U))$ and $V = \underline{\underline{v}}(E,\theta_1,\xi) + \underline{\underline{v}}(E,\theta_2,-\xi)$ we define

$$V(\phi) = \int_E [\theta_1\phi(\xi) + \theta_2\phi(-\xi)]\, dH^k .$$

Oriented varifold convergence will mean convergence in the sense of Radon measures.

We have a compactness theorem for $IV_k^o(U)$, analogous to the usual compactness theorems for integer multiplicity currents and integer multiplicity varifolds. Although our theorem does not seem to follow directly from these earlier results, the proof combines the techniques of these earlier proofs.

1. THEOREM [H1, 4.6.1]: *Let* $U = \bigcup_{i=1}^{\infty} A_i$ *where the* A_i *are open. Then for any sequence* $\{M_i\}_{i=1}^{\infty}$ *of non-negative constants the following is sequentially compact with respect to oriented varifold convergence:*

$$\{V \in IV_k^o(U) : (\mu_V + \|\delta V\| + \mu_{\partial V})(A_i) \le M_i \quad \text{for all} \quad i\} .$$

(μ_V *is the measure in* U *obtained by projecting the measure* V *on* $\underline{\underline{G}}_k^o(U) = U \times \underline{\underline{G}}^o(k,n)$ *onto* U *, and similarly for* $\mu_{\partial V}$ *).*

We next define the appropriate notion of generalised second fundamental form for certain (unoriented) $V \in IV_k(U)$. The generalised second fundamental form for $V \in IV_k^o(U)$ is then defined to be that of $\underline{\underline{q}}(V) \in IV_k(U)$.

First observe that $\underline{\underline{G}}(k,n) \subset \mathbb{R}^{n^2}$, where the imbedding is obtained by identifying $P \in \underline{\underline{G}}(k,n)$ with orthogonal projection of $\mathbb{R}^n$ onto $P \subset \mathbb{R}^n$, and thence with a matrix $[P_{ij}]_{1 \le i,j \le n}$. Consider those $\phi = \phi(x,P) \in C^1(U \times \mathbb{R}^{n^2})$ which have compact support in the x variables. Define

$$D_j \phi \ , \qquad D_{jk}^* \phi$$

to be the functions obtained by partial differentiation with respect to the x_j and P_{jk} variables respectively. Finally let us recall that just as we can regard $IV_k^o(U)$ as a subset of the set of Radon measures on $\underline{\underline{G}}_k^o(U)$, so we can regard $IV_k(U)$ as a subset of the set of Radon measures on $\underline{\underline{G}}_k(U)$, and so in particular if $V \in IV_k(U)$ and ϕ is as above, then $\int \phi(x,P)\, dV(x,P)$ is well defined.

We now can make the appropriate definition.

2. Definition: With the above conventions we say that $V \in IV_k(U)$ has second fundamental form $A = A(V) = [A_{ijh}]_{1 \le i,j,h \le n}$ in U if

(i) $A_{ijh} \in L^1(V; U)$ for $1 \le i,j,h \le n$,

(ii) $0 = \int [P_{ij} D_j \phi + A_{ijh} D^*_{jh} \phi + A_{jij} \phi] \, dV$

for $1 \le i \le n$ and summing j and h from 1 to n . The class of all $V \in IV_k(U)$ having second fundamental form in this sense is denoted by $CV_k(U)$ and any such V is called a curvature varifold.

We say V is an oriented curvature varifold in U if $V \in IV^o_k(U)$ and $\underline{\underline{q}}(V) \in CV_k(U)$, and we denote the set of such varifolds by $CV^o_k(U)$.

The definition is motivated by a calculation [H1, §5.2] which shows that the above holds if V is the varifold $\underline{\underline{v}}(M,1)$ where M is a C^2 embedded submanifold of U with $\partial M \cap U = \phi$ and if $A_{ijh}(x) = (\text{grad}^M P_{jh}(x))_i$. Notice that $[\text{grad}^M P_{jh}(x)]_{1 \le j,h \le n}$ is equivalent to the usual notion of second fundamental form of M at x .

One easily checks that $A(V)$, if it exists, is V a.e. unique. It is worth remarking that $A(V)$ is defined over $\underline{\underline{G}}_k(U)$, and not over U as is the case for generalised mean curvature in the sense of Allard [A].

We next define a notion of convergence in $CV_k(U)$ (called *weak* curvature convergence in [H1]).

3. Definition: Suppose $\{V_i\}_{i=1}^{\infty}$, $V \in CV_k(U)$ and $V_k \to V$ in the sense of varifolds. Then we say V_k converges to V in the *curvature sense* and write

$$V_k \xrightarrow{C} V$$

if moreover

$$V_k \llcorner A(V_k) \to V \llcorner A(V)$$

in the sense of vector-valued measures in U .

We next have the following theorem.

4. THEOREM: *Suppose* $\{V_i\}_{i=1}^{\infty} \subset CV_k(U)$, $V \in IV_k(U)$, $V_i \to V$ *in the varifold sense, and* $\int |A(V_i)|^p dV_i$ *is bounded from above uniformly in* i *for some* $p > 1$. *Then*

$$V \in CV_k(U) ,$$

$$V_i \xrightarrow{C} V ,$$

and

$$\int |A(V)|^p dV \le \liminf_i \int |A(V_i)|^p dV_i .$$

Proof (see [H1, §5] for more details): Compactness and lower semi continuity for vector-valued measures imply on passing to a subsequence that $V_i \llcorner A(V_i) \to V \llcorner f$ for some f such that $\int |f|^p dV \le \liminf \int |A(V_i)|^p dV_i$ (cf. [H1, 4.4.3(b)]). The definition of curvature convergence, the *linearity* of A in 2(ii) and the V almost uniqueness of $A(V)$ imply that $A(V)$ exists and $F = A(V)$ V a.e. Hence $V \in CV_k(U)$. Since the above holds for some subsequence of any subsequence of $\{V_i\}$, we see that $V_i \xrightarrow{C} V$ without passing to a subsequence. □

In [H1, §5] the above is generalised to convex functions of A other than $\xi \mapsto |\xi|^p$ and the relationship between (weak) curvature convergence and a notion of strong curvature convergence is also considered.

The following result now allows us to solve, in certain cases, the problem of minimising curvature within the context of (oriented) curvature varifolds.

5. THEOREM: *Suppose* B *is a smooth compact oriented* $k-1$ *dimensional manifold without boundary imbedded in* $\mathbb{R}^n$. *Let* $U = \mathbb{R}^n \sim \operatorname{spt} B$ *and let*

$$E = \{V \in CV_k^o(\mathbb{R}^n) : \partial V = B , \ \|\partial V\| \llcorner \operatorname{spt} B \le \mu_B\} .$$

and assume $E \ne \phi$.

Then if $1 < p < m$ or if $p = m$ and $\int_U |A(\underline{\underline{g}}(V))|^p d\underline{\underline{g}}(V) \le \underline{\underline{\gamma}}$ (for some absolute constant $\underline{\underline{\gamma}} = \underline{\underline{\gamma}}(k)$), there exists $V^* \in \mathcal{E}$ such that

$$\int |A(\underline{\underline{g}}(V^*))|^p d\underline{\underline{g}}(V^*) = \inf\{\int |A(\underline{\underline{g}}(V))|^p d\underline{\underline{g}}(V) : V \in \mathcal{E}\} .$$

Proof (see [H1, 6.-7 for details): One first uses the isoperimetric inequality for varifolds to obtain a uniform mass bound on some minimising sequence $\{V_i\}$. A first variation bound is simultaneously obtained basically by using the fact that the generalised second fundamental form dominates the generalised mean curvature. The compactness theorem 1 for oriented integral varifolds implies some subsequence of $\{V_i\}$ converges in the oriented varifold sense to V^* , say. From the previous theorem it follows $V^* \in CV_k^o(U)$ and $\int |A(\underline{\underline{g}}(V^*)|^p d\underline{\underline{g}}(V) = \inf\{\int |A(\underline{\underline{g}}(V))|^p d\underline{\underline{g}}(V) : V \in \mathcal{E}\}$. It remains only to show that $V \in \mathcal{E}$, in particular that $\|V\| \llcorner \operatorname{spt} B \le \mu_B$ — this argument uses in particular that $p > 1$ (see the proof of [H1, 6.2]). □

Interesting questions now include the interior and boundary regularity of minimisers. Although it is not relevant to the preceding theorem, we do have the following complete regularity theorem in case $\int |A(V)|^p dV < \infty$ for some $p > k$, $V \in IV_k(\mathbb{R}^n)$. It is interesting to compare this with the partial regularity results which are due to Allard [A] and hold in case V has generalised mean curvature in $L^p(\mu_V)$.

6. THEOREM [H2]: *Suppose* $V \in CV_k(\mathbb{R}^n)$ *and* $\int |A(V)|^p dV < \infty$ *for some* $p > k$. *Then for each* $x \in \operatorname{spt} \mu_V$ *there exists* $r > 0$, *a positive integer* Q , *positive integers* $m_1, \ldots, m_Q$, *subspaces* $P_1, \ldots, P_Q \in \underline{\underline{G}}(k,n)$, *and varifolds* $V_1, \ldots, V_Q \in CV(\underline{\underline{B}}_r(x))$ *such that*

(i) $V \llcorner \underline{\underline{B}}_r(x) = \sum_{i=1}^{Q} V_i$,

(ii) V_i *corresponds in* $\underline{\underline{B}}_r(x)$ *to the graph of a* $C^{1,1-k/p}$ m_i

valued function defined over P_i *, for each* i *,*

(iii) V_i *has approximate and classical tangent plane at* x *given by* P_i *taken with multiplicity* m_i . □

The above theorem is the main result in [H2], see also that paper for the relevant definitions. The theorem could be regarded as a Sobolev type result in a non-parametric setting. The multiple valued functions in (ii) can occur in a non-trivial way. For example if V is the varifold corresponding to the algebraic variety $\{w^2 = z^3\} \subset \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^2 \times \mathbb{R}^2$ then $A(V) \in L^p(V)$ for any $p < 4$ (this is easily checked by using an appropriate cut-off function around 0 in Definition 2). From the previous theorem it follows V corresponds to a $C^{1,\alpha}$ 2-valued function defined over $\mathbb{C} \cong \mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}$ for any $\alpha < \frac{1}{2}$ (of course, one can here take $\alpha = \frac{1}{2}$).

The proof of Theorem 6 is quite long but a key ingredient is the following monotonicity type result for functions depending on tangent plane direction.

PROPOSITION [H2]: *Suppose* $V \in CV_k(B_R)$ *where* $p > k$ *, and* $\int_{B_R} |A(V)|^p dV \le \Gamma^p R^{k-p} < \infty$. *Suppose also that* $\psi \in C^1(\mathbb{R}^{n^2})$ *and that for all* $P \in \mathbb{R}^{n^2}$ *we have* $0 \le \psi(P) \le 1$ *and* $\left(\sum_{j,h=1}^{n} (D^*_{jh}\psi(P))^2\right)^{\frac{1}{2}} \le \lambda\psi(P)$. *Then*

$$\left(\frac{\int_{B_\sigma} \psi dV}{\sigma^n}\right)^{1/p} \le \left(\frac{\int_{B_\rho} \psi dV}{\rho^n}\right)^{1/p} + (1+\lambda)\frac{\Gamma}{p-k}\left(\left(\frac{\rho}{R}\right)^{1-k/p} - \left(\frac{\sigma}{R}\right)^{1-k/p}\right) .$$

whenever $0 < \sigma < \rho < R$.

The proof follows the argument used to prove analogous results for functions $\psi = \psi(x)$. The hypothesis concerning D^*_{jh} is used to dominate the term $A_{ijh} D^*_{jh}$ in Definition 2.

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