MINIMUM PROBLEMS FOR NONCONVEX INTEGRALS

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1. INTRODUCTION

Let us consider an integral of the Calculus of Variations of the following type :

$$(1.1) \qquad F(u;\Omega) = \int_\Omega f(x,u(x),Du(x))dx \ ,$$

where Ω is a bounded open set in $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^m$ is a function belonging to $W^{1,p}(\Omega;\mathbb{R}^m)$, $p>1$ and $f(x,u,\xi)$ is a Carathéodory function, i.e. measurable with respect to x, continuous in (u,ξ). The direct method to get the existence of minima for the Dirichlet problem

$$(P) \qquad \text{Inf}\ \{F(u;\Omega) \ : \ u-u_0 \in W_0^{1,p}(\Omega;\ \mathbb{R}^m)\} \ ,$$

where u_0 is a fixed function in $W^{1,p}$, is based on the sequential lower semicontinuity of F (s.l.s.c.) in the weak topology of $W^{1,p}$.

If $m=1$, it is well known (see [7],[8],[10]) that the l.s.c. of F is equivalent, under very general growth assumptions on f, to the condition that the integrand is a convex function of the variable ξ. But if $m>1$, convexity is no longer a necessary condition. To see this, let us consider a continuous function $f : \mathbb{R}^{mn} \to \mathbb{R}$ such that the functional $\int_\Omega f(Du(x))dx$ is weakly* s.l.s.c. on $W^{1,\infty}(\Omega;\mathbb{R}^m)$. Let Q be a fixed cube containing Ω . If we fix $\xi \in \mathbb{R}^{mn}$, $z(x) \in C_0^1(\Omega;\mathbb{R}^m)$, then, thinking of z as a C_0^1 function defined on Q , we may extend it by periodicity to all $\mathbb{R}^n$. Let us still denote this extension by z . Then, if $u_h(x) = \xi\cdot x+2^{-h}z(2^h x)$, $u_h(x) \to \xi\cdot x$ weakly* and, by the l.s.c. of the integral of f , we get :

$$f(\xi)(\text{meas }\Omega) \leq \liminf_h \int_\Omega f(\xi+(Dz)(2^h x))dx).$$

Since $f(\xi+(Dz)(2^h x))$ converges to $(\text{meas }Q)^{-1}\int_Q f(\xi+Dz(x)dx$ in $\sigma(L^\infty,L^1)$, from the above inequality we deduce that

$$f(\xi)(\text{meas }\Omega) \leq \int_\Omega f(\xi+Dz(x)dx \tag{1.2}$$

for any $\xi \in \mathbb{R}^{mn}$ and any $z \in C_0^1(\Omega;\mathbb{R}^m)$. We shall call *quasi-convex* a function verifying the condition (1.2). If $m=1$, (1.2) is equivalent to Jensen's inequality, and so quasi-convexity reduces to the usual convexity. But if $m>1$, (1,2) is a more general condition as one can see, for instance, in the simple case $m=n$ and $f(\xi) = |\det\xi|$. A study of the properties of quasi-convex functions is contained in [12], [13], and [3]. Here we just recall the following result ([2]) :

<u>THEOREM 1.1</u> - *Let us suppose* $f(x,u,\xi) : \Omega\times\mathbb{R}^m\times\mathbb{R}^{mn}$ *is a Carathéodory function verifying*

$$0 \leq f(x,u,\xi) \leq a(x)+C(|u|^p+|\xi|^p) \qquad p\geq 1 ,$$

where $a(x) \in L^1_{loc}(\Omega)$, $a(x)\geq 0$, $C>0$. *Then* $F(u;\Omega)$ *is weakly s.l.s.c. in* $W^{1,p}$ *if and only if for a.e.* $x\in\Omega$ *and any* $u \in \mathbb{R}^m$ *the function* $\xi \to f(x,u,\xi)$ *is quasi-convex.*

In this talk we shall be concerned with problems of the type $(\mathcal{P})$ in which the integrand f is not quasi-convex. So, by the above theorem, the integral is not l.s.c. and the problem in general will lack a solution. However we shall see that the relaxation methods introduced by Ekeland and Temam ([8]) in the case $m=1$ can be extended also to integrals depending on vector-valued functions. What they prove in the scalar case (see also [10]) is that if one considers the so called 'relaxed problem'

$$(\mathcal{PR}) \qquad \operatorname{Inf}\left\{\int_\Omega f^{**}(x,u(x),Du(x))\,dx \;:\; u-u_0 \in W_0^{1,p}(\Omega)\right\} \;,$$

where for any fixed x and u, $f^{**}(x,u,\cdot)$ is the convex envelope of the function $f(x,u,\cdot)$, then $\operatorname{Inf}(\mathcal{P}) = \operatorname{Inf}(\mathcal{PR})$ and, if f verifies the usual growth assumptions, $(\mathcal{PR})$ has a solution. Moreover its solutions are limit points in the weak topology of $W^{1,p}(\Omega)$ of the minimizing sequences of the problem $(\mathcal{P})$.

In the case $m>1$, one can still define a relaxed problem by replacing $F(u;\Omega)$ with the integral of $\bar{f}(x,u,\xi)$, where now for any x and u fixed $\xi \to \bar{f}(x,u,\xi)$ is the greatest quasi-convex function less than or equal to $\xi \to f(x,u,\xi)$. We shall see that with such a definition one can prove essentially the same results which hold in the scalar case.

Because of the fact that quasi-convexity is defined by an integral condition, one cannot expect that the formula which represents $\bar{f}$ should have the same simple geometrical character as the formula representing the convex envelope f^{**} of the function f with respect to ξ. But it is interesting to note that in some special cases one can explicitly say how $\bar{f}$ is obtained from f.

The proofs given in this talk are essentially, with some minor changes and simplifications, the ones given in [2] and [1]. However, similar results of relaxation have been also given by Dacorogna in [4], [5] and [6], but his proofs are based on completely different techniques.

2. MAIN RESULTS

Although most of the results given here can be extended to the case in which f is a function depending on (x,u,ξ), verifying some kind of uniform continuity in u with respect to ξ, for simplicity we shall restrict to the case in which f does not depend on u.
So we shall assume $f(x,\xi) : \mathbb{R}^n \times \mathbb{R}^{m.n} \to \mathbb{R}$ to be a Carathéodory function, Ω a bounded open set. We shall say that Ω is *regular* if $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\Omega)$ (for instance if Ω has the segment property) and we shall put

$$F(u;\Omega) = \int_\Omega f(x,Du(x))dx \quad ,$$

where $u : \Omega \to \mathbb{R}^m$ is any function for which the integral on the right (possibly $= +\infty$) has sense. Let us denote by $\bar{F}_p(u;\Omega)$ the greatest functional less than or equal to $F(u;\Omega)$ and which is weakly s.l.s.c. in $W^{1,p}(\Omega;\mathbb{R}^m)$. The following result gives a representation of $\bar{F}_p$.

<u>THEOREM 2.1</u> - *If* $f(x,\xi)$ *is a Carathêodory function verifying*

$$(2.1) \qquad 0 \leq f(x,\xi) \leq a(x)+C|\xi|^p \quad ,$$

where $a(x) \in L^1_{loc}(\mathbb{R}^n)$, $a(x) \geq 0$, $C > 0$, $p \geq 1$, *then there exists a Caratheodory function* $\bar{f}(x,\xi)$ *such that for any* Ω *regular and any* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$

$$\bar{F}_p(u;\Omega) = \int_\Omega \bar{f}(x,Du(x))dx \ .$$

Moreover for a.e. $x \in \mathbb{R}^n$ *the function* $\xi \to \bar{f}(x,\xi)$ *is the greatest quasi-convex function less than or equal to* $\xi \to f(x,\xi)$.

In the scalar case this characterization becomes (see [8],[10]) $\bar{f}(x,\xi) = f^{**}(x,\xi)$, since if $m=1$ quasi-convexity is equivalent to convexity. The above result shows also that in order to represent $\bar{f}$ it is sufficient to consider the case in which f is just a function of ξ . In this case, denoting by Y the unit cube $(0,1)^n$ we can prove the following.

THEOREM 2.2 - *If* $f : \mathbb{R}^{mn} \to \mathbb{R}$ *is a continuous function, then the quasi-convex envelope of* f *is given by*

$$\bar{f}(\xi) = \mathrm{Inf}\ \{\liminf_h \int_Y f(Du_h(x))dx \ : \ u_h \in C^1(\bar{Y};\mathbb{R}^m),\ u_h = \xi \cdot x \ \textit{on} \ \partial Y$$
$$Du_h(x) \to \xi \ \textit{in} \ \sigma(L^\infty,L^1)\}$$

Although in general this formula is not very easy to handle, it may be used to obtain a sharper characterization of $\bar{f}$ in particular cases. Let us regard now the vector $\xi \in \mathbb{R}^{mn}$ as an $m \times n$ matrix and denote by $X(\xi)$ the vector whose components are the subdeterminants of ξ of highest order . Let $N(n,m)$ denote the dimension of $X(\xi)$. For instance, if $n=m$ $X(\xi) = \det\xi$ and $N(n,m) = 1$, and if $m = n+1$ $N(n,n+1) = n+1$ and so on. Then the following result holds (see [1]) :

THEOREM 2.3 - *Let us suppose* $m \geq n$ *and* $f(x,\xi) = \phi(x,X(\xi))$, *where* $\phi(x,X) : \mathbb{R}^n \times \mathbb{R}^{N(n,m)} \to \mathbb{R}$ *is a Carathéodory function such that*

$$0 \leq \phi(x,X) \leq g(x,|X|) \tag{2.2}$$

and $g : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ *is a Carathéodory function such that for any* $t \geq 0, g(\cdot,t) \in L^1_{loc}(\mathbb{R}^n)$ *and for any* $x \in \mathbb{R}^n$ *a.e.* $g(x,\cdot)$ *is a non-decreasing function, then there exists another Carethéodory function* $\psi : \mathbb{R}^n \times \mathbb{R}^{N(n,m)} \to \mathbb{R}$, *still verifying* (2.2) *such that for any regular* Ω *and any* $u \in W^{1,n}(\Omega;\mathbb{R}^m)$

$$\bar{F}_n(u;\Omega) = \int_\Omega \psi(x,X(Du(x)))dx .$$

Moreover, if $m = n$ *or* $m = n+1$ $\psi(x,X) = \phi^{**}(x,X)$.

From Theorem 2.1 one can prove the following relaxation result :

<u>THEOREM 2.4</u> - *Let us suppose* f *is a Carathéodory function such that*

$$-a(x)+|\xi|^p \leq f(x,\xi) \leq a(x)+C|\xi|^p \tag{2.3}$$

where $a(x) \in L^1_{loc}(\mathbb{R}^n)$, $a(x) \geq 0$, $C \geq 1$ *and* $p > 1$. *Let us fix an open regular set* Ω *and* $u_0 \in W^{1,p}(\Omega;\mathbb{R}^m)$ *and consider the following problems :*

(*P*) $$\operatorname{Inf}\left\{\int_\Omega f(x,Du(x))dx : u-u_0 \in W_0^{1,p}(\Omega;\mathbb{R}^m)\right\}$$

(*PR*) $$\operatorname{Inf}\left\{\int_\Omega \bar{f}(x,Du(x))dx : u-u_0 \in W_0^{1,p}(\Omega;\mathbb{R}^m)\right\}$$

Then Inf(*P*) = Inf(*PR*). *Moreoever, if* $\bar{u}$ *is a solution of* (*PR*), *there exists a sequence* (u_h) *minimizing* (*P*) *which converges weakly to* $\bar{u}$ *in* $W^{1,p}$. *Conversely, if* (u_h) *is a minimizing sequence of* (*P*), *there exists a subsequence which converges to a solution of* (*PR*).

Using the regularity arguments of [11] and [9], from the above theorem one can easily deduce the following.

COROLLARY 2.5 - *Under the hypothesis of* Theorem 2.4, *if* $a(x) \in L^{\sigma}$ *for some* $\sigma > 1$ *, therefore for any solution* $\bar{u}$ *of the problem* (PR) *there exists a minimizing* (u_h) *of* (P) *such that* $u_h \to \bar{u}$ *weakly in* $W^{1,q}_{loc}(\Omega;\mathbb{R}^m)$, *with* $q \in [p,p+\varepsilon)$ *and* $\varepsilon \equiv \varepsilon(a(x),\sigma,p,C)$.

3.PROOFS

In order to prove the results stated in the previous section, following an idea introduced in [10], we shall look first at the case $p = +\infty$. Let us suppose then that f verifies

$$(3.1) \qquad 0 \leqq f(x,\xi) \leqq g(x,|\xi|) ,$$

where g is a Carathéodory function, non decreasing in $|\xi|$ and $g(\cdot,|\xi|) \in L^1_{loc}(\mathbb{R}^n)$ for any ξ. If $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ we shall write: $\bar{F}(u;\Omega) = \operatorname{Inf}\{\liminf_h F(u_h;\Omega) : u_h \to u \text{ weakly* in } W^{1,\infty}(\Omega;\mathbb{R}^m)\}$. Our main goal is to prove the following

THEOREM 3.1 - *If* f *verifies* (3.1) *, then there exists a Carathéodory function* $\bar{f}(x,\xi) : \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}$ *quasi-convex in* ξ *, such that for any* Ω *and any* $u \in C^1_0(\mathbb{R}^n;\mathbb{R}^m)$

$$(3.2) \qquad \bar{F}(u;\Omega) = \int_{\Omega} \bar{f}(x,Du(x))dx .$$

In order to prove this result we shall prove some preliminary lemmas. First, if $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ with $\|Du\|_{L^{\infty}(\Omega;\mathbb{R}^{mn})} \leqq r$ let us put

$$F(r,u;\Omega) = \operatorname{Inf}\{\liminf_h F(u_h;\Omega) : u_h \to u \text{ weakly* in } W^{1,\infty}(\Omega;\mathbb{R}^m) \text{ and } \|Du_h\|_{L^{\infty}} \leqq r\}$$

REMARK 3.2 - By a standard diagonalization argument it is easy to check that the above infimum is actually a minimum and that the functional $F(r,u;\Omega)$ is weakly* s.l.s.c. on the set $\{u \in W^{1,\infty}(\Omega;\mathbb{R}^m) : \|Du\|_{L^\infty} \leqq r\}$. If now $u \in W^{1,\infty}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$, and $\|Du\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{mn})} \leqq r$, for any Ω we shall denote

$$\Phi(r,u;\Omega) = \lim_{r'\downarrow r} F(r',u;\Omega) = \sup_{r'>r} F(r',u;\Omega) .$$

Then we may prove

LEMMA 3.3 - *If* $u \in W^{1,\infty}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ *and* $\|Du\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^{mn})} \leqq r$, *there exists a function* $h_u \in L^1_{loc}(\mathbb{R}^n)$ *such that for any* Ω

$$\Phi(r,u;\Omega) = \int_\Omega h_u(x)\,dx$$

PROOF: Let us fix u and prove that

$$\Phi(r,u;\Omega) = \lim_{r'\downarrow r} F_0(r',u;\Omega) \tag{3.3}$$

where F_0 is defined by

$$F_0(r',u;\Omega) = \mathrm{Inf}\,\{\liminf_h F(u_h;\Omega) : u_h \to u \text{ weakly* in } W^{1,\infty}(\Omega;\mathbb{R}^m),$$

$$u_h = u \text{ on } \partial\Omega \text{ and } \|Du_h\|_{L^\infty} \leqq r'\}.$$

If we fix $\varepsilon > 0$ there exists $\delta > 0$ such that for any $r' \in (r,r+\delta]$

$$\lim_{r'\downarrow r} F_0(r',u;\Omega) \leqq F_0(r',u;\Omega)+\varepsilon ; \quad \lim_{r'\downarrow r} F(r'u;\Omega) \geqq F(r',u;\Omega)-\varepsilon .$$

If we fix now $r' \in (r,r+\delta)$, let (u_h) be a sequence such that $u_h \to u$ weakly*, $\|Du_h\|_{L^\infty} \leq r'$ and $F(r',u;\Omega) = \lim_h F(u_h;\Omega)$. Let us take a compact set $K \subset \Omega$ such that

$$\int_{\Omega - K} g(x,r+\delta)\,dx < \varepsilon$$

and let ϕ be a $C_0^1(\Omega)$ function such that $\phi(x) \equiv 1$ on K, $0 \leq \phi(x) \leq 1$; if we denote $v_h = u+\phi(u_h-u)$, then $v_h \to u$ weakly*, $v_h \equiv u$ on $\partial\Omega$ and there exists h_0 such that for any $h \geq h_0$ $\|Dv_h\|_{L^\infty} \leq r+\delta$. So we have :

$$\lim_{r' \downarrow r} F_0(r'u;\Omega) \leq F_0(r+\delta,u;\Omega) + \varepsilon \leq \liminf_{h \geq h_0} F(v_h;\Omega) + \varepsilon$$

$$\leq \liminf_h \, [F(v_h;\Omega) - F(u_h;\Omega)] + F(r'u;\Omega) + \varepsilon$$

$$\leq \Phi(r,u;\Omega) + \int_{\Omega-K} g(x,r+\delta)\,dx + 2\,\varepsilon \quad .$$

Then letting $\varepsilon \to 0^+$, we get $\lim_{r' \downarrow r} F_0(r'u;\Omega) \leq \phi(r,u;\Omega)$. Since the reverse inequality is obviously verified by definition, we have proved (3.3).
Now, let us denote by $\mathcal{F}$ the class of all the finite unions of cubes of the type $\{a_i \leq x_i \leq a_i+\ell \; : \; i=1,\dots,n\}$ and define $\mu(P) = \Phi(r,u;P-\partial P)$ for any $P \in \mathcal{F}$. From (3.3) it follows that $\mu(P)$ is finitely additive, since it is easy to verify that $F(r,u;\Omega)$ is sub-additive with respect to Ω, while F_0 is super additive. Let us now extend μ to the class of all Lebesgue measurable sets in $\mathbb{R}^n$. If we denote still by μ the resulting extension, then using again (3.3) it is easy to check that $\mu(\Omega) = \Phi(r,u;\Omega)$ for any open set Ω. Finally, the existence of h_u comes easily from the fact that for any Ω

$$0 \leq \mu(\Omega) \leq \int_\Omega g(x,r)\,dx \; . \qquad \blacksquare$$

<u>LEMMA 3.4</u> - *If* $u_1, u_2 \in W^{1,\infty}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$, $\|Du_i\|_{L^\infty(\mathbb{R}^n;\, \mathbb{R}^{mn})} \leq r$, $i=1,2$,

Then for any Ω :

$$|\Phi(r,u_1;\Omega) - \Phi(r,u_2;\Omega) \leq \int_\Omega \omega(x, 2r, \|Du_1 - Du_2\|_{L^\infty(\Omega;\mathbb{R}^{mn})})\, dx\ ,$$

where

$$\omega(x,2r,\delta) = \sup\{|f(x,\xi_1) - f(x,\xi_2)| \ :\ |\xi_i| \leq 2r \ \textit{and}\ |\xi_1 - \xi_2| \leq \delta\}\ .$$

<u>PROOF</u>: Let us take $r' \in (r,2r)$ and (u_h) such that $u_h \to u$ weakly*, $\|Du_h\|_{L^\infty} \leq r'$ and $F(r'u_1;\Omega) = \lim_h F(u_h;\Omega)$. If we take $v_h = u_h + (u_2 - u_1)$ we obtain

$$F(r',u_2;\Omega) - F(r',u_1;\Omega) \leq \liminf_h [F(v_h;\Omega)\ F(u_h;\Omega)]$$

$$\leq \int_\Omega \omega(x,2r, \|Du_1 - Du_2\|_{L^\infty})\, dx$$

Then the result follows by changing u_1 with u_2 and taking the limit as $r' \to r^+$ ∎

<u>PROOF OF THEOREM 3.1</u> - Let us fix r and consider the class A_r of all linear functions $u(x) = \xi \cdot x$, with $|\xi| \leq r$, $\xi \in \mathbb{Q}^{mn}$. Let us say L the set of all the Lebesgue points for the functions $h_u(x)$ with $u \in A_r$. then for any $x \in L$, $\xi \in \mathbb{Q}^{mn}$ with $|\xi| \leq r$ we may put :

$$\phi_r(x,\xi) = h_u(x)$$

where $u(x) = \xi \cdot x$. From Lemma 3.4 we can deduce that for a.e. $x \in L$, $\xi_1, \xi_2 \in \mathbb{Q}^{mn}$ with $|\xi_i| \leq r$

$$|\phi_r(x,\xi_1) - \phi_r(x,\xi_2)| \leq \omega(x,2r,|\xi_1-\xi_2|) \ .$$

This means that for a.e. $x \in L, \phi_r(x,\cdot)$ can be extended by continuity to the set $\{\xi \in \mathbb{R}^{mn} : |\xi| \leq r\}$. Moreover, using again Lemma 3.4, it is clear that for such an extension of ϕ_r we still have $\phi_r(x,\xi) = h_u(x)$, for any $u(x) = \xi \cdot x$ with $|\xi| \leq r$. If $u(x) \in C_0^1(\mathbb{R}^n; \mathbb{R}^m)$, with $|Du(x)| \leq r$, there exists a sequence (u_h) of piecewise affine functions, such that $u_h \to u$ and $Du_h \to D_u$ uniformly in $\mathbb{R}^n$, and $|Du_h(x)| \leq r$ (see [8], Ch.X, prop.2.1). Then by Lemmas 3.3 and 3.4 and by the definition of ϕ_r we get :

$$(3.4) \qquad \Phi(r,u;\Omega) = \int_\Omega \phi_r(x,Du(x)dx$$

for any $u \in C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ with $|Du(x)| \leq r$. So by the weakly* s.l.s.c. of the functional $\Phi(r,u;\Omega)$ on the set $\{u \in W^{1,\infty}(\Omega;\mathbb{R}^m): \|Du\|_{L^\infty} \leq r\}$, and by the representation formula (3.4), using the same argument as Theorem II.2 in [2], we have that ϕ_r is quasi-convex in ξ , where $|\xi| \leq r$, i.e. for any x_0 a.e., any $\xi \in \mathbb{R}^{mn}$ and any $z(y) \in C_0^1(\Omega;\mathbb{R}^m)$ such that $|\xi| + |Dz(y)| \leq r$

$$(3.5) \qquad \phi_r(x_0,\xi)(\text{meas } \Omega) \leq \int_\Omega \phi_r(x_0, \xi + Dz(y))dy \ .$$

Finally, if we define for any x a.e. and any $\xi \in \mathbb{R}^{mn}$ $\bar{f}(x,\xi) = \lim\limits_{r \geq |\xi|} \phi_r(x,\xi)$, Then by (3.5), $\bar{f}$ is clearly a Carathéodory function quasi-convex in ξ . Moreover (3.4) implies that $\bar{f}$ verifies (3.2). ∎

REMARK 3.5 - Since $\bar{f}$ is quasi-convex in ξ , then (see [2]) the functional $\int_\Omega \bar{f}(x,Du)dx$ is weakly* s.l.s.c.. So from Theorem 3.1 it is clear that it is the greatest functional defined on $C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ which is weakly* s.l.s.c. and less than or equal to $\int_\Omega f(x,Du)dx$.

<u>LEMMA 3.6</u> - *For a.e.* $x \in \mathbb{R}^n$ $\xi \to \bar{f}(x,\xi)$ *is the greatest quasi-convex function less than or equal to* $\xi \to f(x,\xi)$

<u>PROOF:</u> Let us fix Ω . Using the same argument as in the proof of Theorem 3.1, we deduce that for a.e. $x_0 \in \Omega$ there exists a continuous function $g_r^{(x_0)}(\xi)$ such that for any $u \in C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ with $|Du(x)| \leq r$

$$\int_\Omega g_r^{(x_0)}(Du(x)dx = \sup_{r'>r} \text{Inf}\,\{\liminf_h \int_\Omega f(x_0,Du(y))dy : u_h \to u \text{ weakly* and } \|Du_h\|_{L^\infty(\Omega;\mathbb{R}^{mn})} \leq r'\}.$$

Since f is a Carathéodory function, for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \in \Omega$ such that f is continuous on $K_\varepsilon \times \mathbb{R}^{mn}$ and meas$(\Omega - K_\varepsilon) < \varepsilon$. Let us put $g_r(x,\xi) = g_r^{(x)}(\xi)$ for any $x \in K_\varepsilon$ and any ξ . By the uniform continuity of f on the bounded subsets of $K_\varepsilon \times \mathbb{R}^{mn}$ it follows that $g_r(x,\xi)$ is continuous on $K_\varepsilon \times \{\xi : |\xi| \leq r\}$. So, because of the arbitrariness of ε , we may define $g_r(x,\xi)$ for a.e. $x \in \Omega$. Moreoever g_r will be a Carathéodory function. Then if we define for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^{mn}$

$$g(x,\xi) = \lim_{r \geq |\xi|} g_r(x,\xi) ,$$

from the Remark 3.5 it follows that for a.e. $x_0 \in \Omega$ the functional $u \to \int_\Omega g(x_0,Du(x)dx$ is the greatest functional on $C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ which is weakly* s.l.s.c. and less than or equal to $u \to \int_\Omega f(x_0,Du(x))dx$. This implies that $\xi \to g(x_0,\xi)$ is the greatest quasi-convex function less than or equal to $\xi \to f(x_0,\xi)$. So $g(x_0,\xi) \geq \bar{f}(x_0,\xi)$. But also $\int_\Omega g(x,Du(x))dx$ is weakly* s.l.s.c., since g is quasi-convex in ξ . So by the Remark 3.5 it follows that for any $u \in C_0^1(\mathbb{R}^n;\mathbb{R}^m)$ $\int_\Omega g(x,Du) \leq \int_\Omega \bar{f}(x,Du)dx$, which implies $\bar{f}(x,\xi) \geq g(x,\xi)$ for a.e. x and ξ . This inequality , combined with the previous one shows then that $\bar{f} = g$, thus proving the Lemma. ∎

PROOF OF THEOREM 2.1 - From the Theorem 1.1 we have that the functional $\int_\Omega \bar{f}(x,Du)dx$ is weakly s.l.s.c. on $W^{1,p}(\Omega;\mathbb{R}^n)$. So

$$(3.6) \qquad \int_\Omega \bar{f}(x,Du)dx \leq \bar{F}_p(u;\Omega) \qquad \text{for any } u \in W^{1,p}(\Omega;\mathbb{R}^m).$$

But if $u \in C_0^1(\mathbb{R}^n;\ \mathbb{R}^m)$, from Theorem 3.1 it follows that for any $\varepsilon > 0$ there exists a sequence (u_h) such that $u_h \to u$ weakly* and

$\int_\Omega \bar{f}(x,Du)dx \geq \liminf_h \int_\Omega f(x,Du_h) - \varepsilon$. From this we get :

$$\int_\Omega \bar{f}(x,Du)dx \geq \liminf_h \bar{F}_p(u_h;\Omega) - \varepsilon \geq \bar{F}_p(u;\Omega) - \varepsilon .$$

This inequality, together with (3.6), proves the theorem when u is a C_0^1 function on $\mathbb{R}^n$. The general case, when Ω is a regular open set, follows easily by approximation. ■

PROOF OF THEOREM 2.2 - Follows at once from Theorem 3.1 and the proof of Lemma 3.6 ■

PROOF OF THEOREM 2.4 - If $\bar{u}$ is a solution of (PR) , then for any h there exists a $C_0^1(\Omega;\mathbb{R}^m)$ function v_h such that $\|Dv_h\|_{W^{1,p}} \leq \frac{1}{h}$ and $\left|\int_\Omega \bar{f}(x,D\bar{u}(x)Dx - \int_\Omega \bar{f}(x,D\bar{u}(x) + Dv_h(x))dx\right| \leq \frac{1}{h}$. If we apply Theorem 3.1 and the formula (3.3) to the function $K(x,\xi) = f(x,D\bar{u}(x)+\xi)$, we may say that for any h there exists $w_h \in W^{1,\infty}(\Omega;\mathbb{R}^m)$, $w_h \equiv 0$ on $\partial\Omega$ such that

$$\left|\int_\Omega \bar{f}(x,D\bar{u}(x) + Dv_h(x))dx - \int_\Omega f(x,D\bar{u}(x) + Dw_h(x))dx\right| \leq \frac{1}{h}$$

and $\|v_h - w_h\|_{L^\infty} \leq \frac{1}{h}$. So if we put $u_h = \bar{u} + w_h$, then obviously $u_h \to \bar{u}$ in $L^p(\Omega;\mathbb{R}^m)$. Moreover from (2.3) we have also that $\|Du_h\|_{L^p} \leq$ constant, so we may suppose that (u_h) converges weakly in $W^{1,p}$ to $\bar{u}$. And by construction we have also

$$\int_\Omega \bar{f}(x,D\bar{u})dx = \lim_h \int_\Omega f(x,Du_h)dx .$$

This proves that $\mathrm{Inf}(\mathcal{P}) = \mathrm{Inf}(\mathcal{PR})$ and also that for any solution $\bar{u}$ of $(\mathcal{PR})$ there exists a minimizing sequence of $(\mathcal{P})$ which converges to the solution $\bar{u}$ weakly in $W^{1,p}$. The converse is then obvious ■

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