GROUP ACTIONS ON CUNTZ ALGEBRAS

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1. INTRODUCTION

The Cuntz algebra $\mathcal{O}_n$ $(1 < n < \infty)$ is the C*-algebra generated by the range of a linear map s from C^n to the bounded linear operators on an infinite dimensional Hilbert space which satisfies

$$s(h_1)^*s(h_2) = \langle h_1, h_2 \rangle 1 , \quad h_j \in C^n , \; j = 1,2 \tag{1.1}$$

$$\Sigma_{j=1,n} s(e_j)s(e_j)^* = 1 , \tag{1.2}$$

where $\langle \, , \rangle$ is an inner product on C^n, $\{e_j\}_{j=1,n}$ an orthornormal basis with respect to this inner product and 1 the identity operator. One may think of $\mathcal{O}_n$ as a 'non-commutative version' of the unit sphere in C^n. This analogy is reinforced by the fact that the noncompact lie group U(n,1) acts automorphically on $\mathcal{O}_n$ by generalised Mobius transformations. This U(n,1) action was introduced by Voiculescu [6], however, understanding his proof of its existence requires some stamina on the part of the reader. We show here that the action may be defined using just elementary algebra and the result of Cuntz [3] that $\mathcal{O}_n$ is uniquely determined by the relations (1.1) and (1.2) satisfied by s .

2. THE U(n,1) ACTION

Define a row vector $s = (s(e_1),\ldots,s(e_n))$. Then with s^* denoting the column vector with entries $s(e_j)^*$ $(j = 1,\ldots,n)$ one has from (1.1) and (1.2) the relations

$$ss^* = 1 , \quad s^*s = \mathrm{diag}(1,\ldots,1) \tag{2.1}$$

If A, B are n x n matrices over C and sAs^* denotes the obvious matrix product then

(2.2) $$sAs^*sBs^* = sABs^* \ , \ sAs^*s(h) = s(Ah) \ , \quad h \in C^n$$

and also

(2.3) $$s(h) = s.h^t .$$

Now note that if $U(n,1)$ denotes the group of $(n+1)x(n+1)$ matrices A such that

(2.4) $$AJA^* = J \ , \quad J = \mathrm{diag}(-1,1,\ldots,1)$$

then each such A may be written

(2.5) $$A = \begin{bmatrix} a_0 & \langle h_1, \rangle \\ h_2 & A_1 \end{bmatrix}$$

with $a_0 \in C$, $h_1, h_2 \in C^n$ and A_1 an nxn matrix. Now (2.4) implies for example:

(2.6) $$|a_0|^2 - \|h_1\|^2 = 1 \ , \ a_0h_2^* = h_1A_1^* - h_2h_2 = 1_n .$$

Now define

(2.7) $$u_A = (a_0 - s(h_2))^{-1}(-s(h_1)^* + sA_1s^*)$$

LEMMA 2.1. u_A *is a well defined unitary in* O_n

Proof. Using $A^*JA = J$ and the ensuing relations, corresponding to (2.6), one checks that $(a_0-s(h_2))^{-1}$ exists in O_n. Then these relations also give, after some elementary algebraic manipulations, $u_A^*u_A = 1 = u_Au_A^*$.

LEMMA 2.2. *(Takesaki) There is a bijection between unitaries* u *in* O_n *and unital endormorphisms* α *of* O_n *given by*

$$u = \textstyle\sum_j \alpha(s(e_j))s(e_j)^* \quad \text{and} \quad \alpha(s(e_i)) = u\, s(e_i)$$

Moreover α *is an automorphism if and only if there exists a unitary* u' *in* $\mathcal{O}_n$ *with* $\alpha(u') = u^*$.

Proof. See [4]. (The proof uses only elementary algebra and uniqueness of the Cuntz algebra).

From the preceding lemmas and (2.7) we now have a map $A \to \alpha_A$ from $U(n,1)$ into the initial endormorphisms of $\mathcal{O}_n$. But now it is an easy matter to verify the relation

$$\alpha_A(u_B)u_A = u_{AB} \tag{2.8}$$

so that with $B = A^{-1}$ one has $u_A' = u_A^{-1}$ satisfying $\alpha_A(u_A') = u_A^*$ and hence $A \to \alpha_A$ is a homomorphism into $\mathrm{Aut}\,\mathcal{O}_n$. So we have proved:

THEOREM 2.3. *(Voiculescu) The map* $A \to \alpha_A$ *with*

$$\alpha_A(s(h)) = (a_0 - s(h_2))^{-1}(-< h_1, h > + s(A_1 h))$$

is a homomorphism of $U(n,1)$ *into* $\mathrm{Aut}\,\mathcal{O}_n$.

Remark 2.4. For $n = 1$ define $\mathcal{O}_n$ to be $C(T)$ then the corresponding action of $U(1,1)$ is of course well known. Let

$$A = e^{i\theta}\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad 0 \le \theta \le 2 \quad , \alpha , \beta \in C \text{ with } |\alpha|^2 - |\beta|^2 = 1$$

then we have

$$A.z = (\alpha - \beta z)^{-1}(-\beta + \alpha z) \ , \ z \in T \ .$$

Similarly there exists an action of $U(n,1)$ on the unit ball in C^n by such fractional linear transformations for which the $\mathcal{O}_n$ action may be regarded as a non-commutative analogue.

$$F_n = \bigoplus_{k=0}^{\infty} (\otimes^k C^n) \quad \text{where} \quad \otimes^0 C^n = C$$

Define for $h \in C^n$, $o(h) : F_n \to F_n$ by

$$o(h): h_1 \otimes \dots \otimes h_m \to h \otimes h_1 \otimes \dots \otimes h_m$$

Then the map $h \to o(h)$ satisfies

(3.1) $$o(h_1)^* o(h_2) = \langle h_1, h_2 \rangle .1 \qquad h_1, h_2 \in C^n ,$$

(3.2) $$\textstyle\sum_j o(e_j) o(e_j)^* = 1 - P_\Omega$$

where P_Ω is the projection onto $C \subseteq F_n$ and 1 is the identity operator. The C*-algebra T_n is generated by the range of o and is generated by the range of o and is uniquely determined by the relations (3.1) and (3.2) [5]. Moreover we have the exact sequence

$$0 \to K \to T \to O_n \to 0$$

where K denotes the ideal of compact operators on F_n . The following is an analogue of lemma 2.2.

LEMMA 3.1. *There is a bijection between partial isometries* v *in* T_n *satisfying*

$$v^*v = 1 - P_\Omega , \quad vv^* < 1$$

and unital endormorphisms β *of* T_n *given by*

$$v = \textstyle\sum_j \beta(o(e_j) o(e_j)^*) , \quad \beta(o(h)) = vo(h) .$$

Moreover β *is an automorphism if and only if there exists a partial isometry* $v' \in T_n$ *such that*

This action has some interesting properties, for example:

THEOREM 2.5. *The* $U(n,1)$ *action on* $\mathcal{O}_n$ *is ergodic.*

(By ergodic we mean that the only fixed points for the action are multiples of the identity operator). The preceding theorem follows from the stronger result:

THEOREM 2.6. *For each* $A \in U(n,1)$ *of the form*

$$A = \exp \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix} , \quad h \in C^n$$

there exists a state ψ_A *on* $\mathcal{O}_n$ *such that for all* $x,y,z \in \mathcal{O}_n$

$$\lim_{n\to\infty} \psi_A(y\alpha_A^{\,n}(x)z) = \psi_A(x)\, \psi_A(yz) \tag{2.9}$$

(this is known as a 3-point cluster property [1]).

To see that theorem 2.5 follows from theorem 2.6 one needs the fact that $\mathcal{O}_n$ is a simple C*-algebra so that the G.N.S. cyclic representation π_A, corresponding to ψ_A is faithful. If Ω_A is the G.N.S. cyclic vector then (2.8) says that if x is fixed by the α_A-action:

$$\begin{aligned} < \pi_A(y^*)\Omega_A \,, \; \pi_A(x)\,\pi_A(z)\,\Omega_A > \;&= \; < \pi_A(y^*)\Omega_A \,, \; \pi_A(\alpha_A^{\,n}(x)z)\,\Omega_A > \\ &= \; \psi_A(x) < \pi_A(y^*)\Omega_A \,, \; \pi_A(z)\Omega_A > \end{aligned}$$

from which $\pi_A(x) = \psi_A(x).1$ and hence that $x = \psi_A(x)$ using faithfulness. More details of this and other properties of the action may be found in [2].

3. A PROBLEM

There is an extension of $\mathcal{O}_n$ by the compacts defined as follows. Let F_n denote the Hilbert space direct sum of the tensor powers of C^n :

$$v'^*v' = 1 - P_\Omega\ , \quad v'v'^* < 1 \quad \text{and} \quad \beta(v') = v^*\ .$$

The proof of this lemma is much the same as that for lemma 2.2. Now F_n is often called the full Fock space so that our next result should be contrasted with the corresponding results for the CAR and CCR algebras acting on their respective Fock spaces.

LEMMA 3.2. *If* β *is an automorphism of* T_n *then there is a unitary* U *on* F_n *such that* $U\mathrm{o}(h)U^* = \beta(\mathrm{o}(h))$ *if and only if one of the following equivalent conditions holds:*

(i) *there is a unitary* u *in* T_n *such that* $u(1 - P_\Omega) = v$

(ii) *the projections* $\beta(P_\Omega)$ *and* P_Ω *are equivalent in* T_n .

Using these facts $U(n,1)$ action on T_n may be defined. Firstly note that T_n is isomorphic to the subalgebra of $\mathcal{O}_{n+1}$ generated by $s(e_1),\ldots,s(e_n)$ with $P_\Omega = s(e_{n+1})s(e_{n+1})^* = p$ [5]. Then we apply:

LEMMA 3.3. *If* β *is a unital endomorphism of* T_n *corresponding to* $v \in T_n$ *as in lemma* 3.1 *then the following are equivalent:*

(i) β *extends to an endormorphism of* $\mathcal{O}_{n+1}$,

(ii) *there is a unitary* u *in* $\mathcal{O}_{n+1}$ *such that* $u(1 - p) = v$,

(iii) *the projections* $\beta(p)$ *and* p *are equivalent in* $\mathcal{O}_{n+1}$.

Now for $A \in U(n,1)$ the existence of an automorphism β_A follows from lemma 3.3 applied to the partial isometry v_A:

$$v_A = (a_0 - \mathrm{o}(h_2))^{-1}(-\mathrm{o}(h_1)^* + \mathrm{o}A_1\mathrm{o}^*)$$

(here we use the same notation for o as we used for s) and the unitary u_A

$$\mathcal{O}_{n+1} \ni u_A = (a_0 - s(h_2))^{-1}(-s(h_1) + sA_1s^* + s(e_{n+1})s(e_{n+1})^*)$$

which are clearly related by $v_A = u_A(1 - p)$. Note that here we regard A as an element of $U(n+1,1)$:

$$A \equiv \begin{pmatrix} a_0 & \langle h_1 , \rangle & 0 \\ h_2 & A_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If follows immediately from lemma 3.2 that there is a representation $A \to U_A$ of $U(n,1)$ by

$$U_A g_1 \otimes \dots \otimes g_m \equiv U_A \sigma(g_1)\dots\sigma(g_m)\Omega$$

$$= \beta_A(\sigma(g_1)\dots\sigma(g_m))(a_0-\sigma(h_2))^{-1}\Omega$$

where $g_j (j = 1,\dots m)$ lie in C^n , the notation for A is as in section 2 and

$$\beta_A(\sigma(g_j)) = (a_0-\sigma(h_2))^{-1}(-\langle h_1,g_j \rangle + \sigma(A_1g_j)) .$$

(Here Ω is the element $1 \oplus 0 \oplus 0 \oplus \dots$ in F_n) .

PROBLEM. (Voiculescu) What are the irreducibles and their multiciplicities in this representation?

It is known that this representation does decompose into a direct sum of irreducibles each occurring with finite multiplicity [6]. Moreover this problem can be formulated in a purely algebraic way since the action of the Lie algebra of $U(n,1)$, by derivations on the tensor algebra over C^n , is easily computed from the preceeding formulae. The problem then becomes one of finding certain lowest weight vectors for the Lie algebra action.

4. THE ANALOGY WITH THE HOMOGENEOUS SPACE SU(n,1)/U(n)

The preceeding is perhaps easier to understand by analogy with the well known $U(n,1)$ action on the bounded symmetric domain

$$D = \{z \in C^n : z^*z = 1\} = SU(n,1)/U(n)$$

To see how this analogy carries through we will show that there is a Hilbert space of analytic functions on D which carries a representation of $U(n,1)$ equivalent to the cyclic representation of $U(n,1)$ on $\mathcal{F}_n$ generated from Ω . Introduce the functions

$$(4.1) \qquad e_w : z \to (1-\bar{w}.z)^{-1} \ , \quad w,z \in D \ .$$

These are holomorphic and linearly independent on D and a pre-Hilbert space structure is obtained on their linear span by writing

$$< e_{w'} \, , e_w > \ = (1-\bar{w}'.w)^{-1}$$

For $A \in U(n,1)$ define an action on D by

$$z \to (a_0-h_2.z)^{-1} (-\bar{h}_1 + A_1 z) = A.z$$

for

$$A = \begin{pmatrix} a_0 & < h_1, . > \\ h_1 & A_1 \end{pmatrix}$$

Now the functions e_w satisfy the identity

$$(a_0 - h_2 z)^{-1} e_w(Az) = (a_0 + \bar{w}.\bar{h}_1)^{-1} \ e_{A^{-1}w}(z)$$

from which it follows that one has a unitary representation $A \to W_A$ of $U(n,1)$ on the completion $\mathcal{H}_D$ of the linear span of e_w via

$$W_A : e_w \to (a_0 + \bar{w}.\bar{h}_1)-1 \quad e_{A^{-1}w} .$$

Then e_0 is clearly a cyclic vector for the representation W. (In fact one deduces easily that W is an irreducible representation). Moreover there is an isometric map $\eta : H_D \to F_\eta$ such that

$$\eta : W_A e_0 \to (a_0 - o(h_2))^{-1}\Omega$$

as

$$< (a_0 - o(h_2))^{-1}\Omega, (b_0 - o(k_2))^{-1}\Omega >$$

$$= (\bar{a}_0 b_0 - < h_2, k_2 >)^{-1}$$

$$= < e_{\bar{a}_0^{-1}\bar{h}_2} , e_{\bar{b}_0^{-1}\bar{k}_2} >$$

$$= < W_A e_0 , W_B e_0 >$$

for
$$A = \begin{pmatrix} a_0 & < h_1 , . > \\ h_2 & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_0 & < k_1 , . > \\ k_2 & B_1 \end{pmatrix}$$

So the cyclic subrepresentation of $U(n,1)$ on F_η generated from Ω is equivalent via η to a representation on holomorphic functions on D. Notice that the function $(z,w) \to e_w(z)$ is related to the Bergman kernel of the domain D.

For the case $n = 1$ from remark 2.4 it is not hard to see that the corresponding representation on F_1 is the usual one of $U(1,1)$ on the Hardy space. Ergodicity may be verified by elementary arguments in this case.

REFERENCES

[1] O. Bratteli and D.W. Robinson, *Operator algebras and quantum statistical mechanics II,* Springer, New York, 1979.

[2] A.L. Carey and D.E. Evans, *On an automorphic action of* U(n,1) *on* $\mathcal{O}_n$ *,* preprint.

[3] J. Cuntz, *Simple C*-algebras generated by isometries,* Commun. Math. Phys. 57, 1977, 173-185.

[4] J. Cuntz, *Automorphisms of certain simple C*-algebras,* in L. Streit, ed., Quantum Fields-Algebras, Processes, Springer, Wien-New York, 1980, 187-196.

[5] J. Cuntz, *K-theory for certain simple C*-algebras,* Ann. of Math. 113, 1981, 181-197.

[6] D. Voiculescu, *Symmetries of some reduced free product C*-algebras,* Preprint, INCREST, 1983.

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