SOME APPLICATIONS OF HARMONIC ANALYSIS TO NUMBER THEORY

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Two sets A,B of real numbers, all exceeding unity, are termed *multiplicatively independent* if no relation of the form $r^m = s^n$ holds for $m,n \in Z^+$, $r \in A$, $s \in B$. We define a real number x to be *normal to base* r if the sequence $(r^n x)_{n=0}^{\infty}$ is uniformly distributed modulo unity. (*cf.* Mendès France [12]). In the case of an integer base this definition has been shown by Wall [21] (see also Niven [13]) to be equivalent to the more usual definition involving asymptotic frequencies of all possible digit blocks.

It has long been known that almost all numbers (in the sense of Lebesgue measure λ) are normal with respect to any given base (Weyl [22]). Schmidt [18] has established that if integers r,s are multiplicatively dependent, then normality to base r entails normality to base s. The proof of Kuipers and Neiderreiter ([11], Theorem 8.2) covers the non-integer case. The question of what class of numbers may be simultaneously normal to one base and non-normal to another is more subtle. Stimulus for research in this area has largely stemmed from work of Cassels [8] and Schmidt [18]. Schmidt was the first to prove in full generality that if r and s are multiplicatively independent integers, then the set of numbers normal to base r but not to base s is uncountable. His proof, which is fairly intricate, was based on use of the support set of the general Cantor measure on [0,1] with constant integer ratio of dissection. Shortly afterwards Schmidt [19] showed that if A,B are multiplicatively independent sets of integers, then uncountably many of the points of [0,1] are simultaneously normal to every base in A and non-normal to every base in B.

Michael Keane and one of the present authors [14] were able to provide a substantially simpler proof of the main result in [18] by exploiting the measure-theoretic motif implicit in Schmidt's construction. The relevant calculation is performed in terms of the known Fourier-Stieltjes coefficients of Cantor measure. The same tools have subsequently been used by Volkmann [20] to derive the following generalization.

THEOREM 1 *Let* ζ_j $(0 \leqslant j < s)$ *be non-negative real numbers summing to unity. Denote by* $N_j(x,n)$ *the proportion of times the digit* j *occurs in the first* n *places in the expansion of* x *to base* s.

Then there exist uncountably many numbers $x \in [0,1]$ *that are normal to base* r *whilst such that*

$$\frac{1}{n} N_j(x,n) \to \zeta_j \quad \text{as } n \to \infty, \quad 0 \leqslant j < s.$$

This approach has been carried further by the authors [4-5], who have made use of the particularly convenient properties of the Fourier-Stieltjes coefficients of Riesz products to derive appreciably stronger conclusions. The starting point is the following result, which may be deduced from a theorem of Davenport, Erdös and LeVeque [9].

PROPOSITION 1 *Let* μ *be a probability measure on* $[0,1]$ *and* r *an integer exceeding unity. For* $\ell \in Z^+$, *define quantities* $H_n = H_n(\mu,r,\ell)$ *in terms of the Fourier-Stieltjes coefficients* $\mu\hat{}(n)$ *of* μ *by*

$$H_n \equiv n^{-2} \sum_{\nu=1}^{n} \sum_{0 \leqslant j < \nu} |\mu\hat{}(\ell(r^{\nu} - r^{j}))|, \quad n \geqslant 1.$$

If, for each $\ell \in Z^+$, *there exist constants* $c = c(\mu, r, \ell) > 0$ *and* $a > 0$ *such that for all* $n \geq 3$ *we have*

(1) $$H_n \leq c\,(\ln n)^{-1}(\ln \ln n)^{-1-a},$$

then normality to base r *holds on a subset of* $[0,1]$ *of full* μ*-measure.*

By Weyl's criterion (see Cassels [7], Chapter 4), the following converse is also available.

PROPOSITION 2 *With notation as above, suppose there exists an* $\ell \in Z^+$ *for which*

(2) $$n^{-1} \sum_{j=1}^{n} \mu\hat{\ }(r^j \ell) \to b \neq 0 .$$

Then normality to base r *holds only on a* μ*-null subset of* $[0,1]$.

Consider a Riesz product measure of the form

(3) $$d\mu = \{ \prod_{\nu=1}^{\infty} [1 + \cos(2\pi w \phi_\nu x)]\}\, d\lambda$$

defined on $[0,1]$, where $w \in Z^+$ and the positive integers ϕ_ν satisfy a lacunarity condition.

The following properties for the Fourier-Stieltjes coefficients are well-known:

(i) $\mu\hat{\ }(0) = 1$,

(ii) if $0 < n \in Z^+$, then $\mu\hat{\ }(n) = 0$ unless n is of the form

(4) $$n = \sum_{\nu=0}^{m} w \varepsilon_\nu \phi_\nu \text{ with each } \varepsilon_\nu \in \{0, \pm 1\}, \text{ for some } m \geq 0,$$

in which case

$$(5) \qquad \mu^\wedge(n) = \prod_{\nu=0}^{m} (\tfrac{1}{2})^{|\varepsilon_\nu|} .$$

In [4], the following situation is considered. Suppose A,B are multiplicatively independent sets of integers, with $B = \{s\}$ a singleton set. A Riesz product is set up of form (3) with $\phi_\nu = s^\nu$ and it is shown that if $r \in A$, then there exists a set $M_{s,r} \subset [0,1]$ of full μ-measure such that each point of $M_{s,r}$ is normal to base r but non-normal to base s. The countable intersection $M_s \equiv \bigcap_{r\in A} M_{s,r}$ is then a subset of [0,1] of full μ-measure each point of which is simultaneously normal to each base $r \in A$ and non-normal to base s. There is thus an uncountable set of points on [0,1] with these normality properties.

The proof is base on Propositions One and Two, the estimations required to establish (1) and (2) being effected via (ii) above. The estimations turn on the fact that if α is irrational, then the sequence $(k\alpha)_{k=0}^{\infty}$ is uniformly distributed modulo unity. Here $\alpha = \log_s r$. In fact sharp discrepancy estimates are required on the rate at which this sequence approaches uniformity. By a theorem of Baker ([1], p. 22) the multiplicative indepence of r,s entails that $\log_s r$ is of finite transcendence type in the sense of Kuipers and Neiderreiter ([11], p. 121). Theorem 3.2 of [11] provides an appropriate discrepancy result from which we may place bounds on how often $(\ell(r^\nu - r^j))$ is of the form (4).

The argument can be carried further without computation. By virtue of Schmidt's result of the equivalence of normality to multiplicatively dependent bases, we may suppose $s \geqslant 8$. Consider two Riesz product measures μ_1, μ_2 differing only in the choices $w_1=1$, $w_2=5$. The uniqueness theorem for Fourier-Stieltjes coefficients is readily seen to imply that the convolution $\mu_1 * \mu_2$ is Lebesgue measure. It follows that *almost any* (λ) *number is the sum of two components, each of which is simultaneously non-normal to base* s *and normal to every base* $r \in A$.

But if F is a subset of [0,1] of full Lebesgue measure,

$$F + F \equiv [0,1] \pmod 1 \quad .$$

Hence we may remove the measure-theoretic scaffolding and state the result that

every real number is the sum of four components, each of which is non-normal to base s *but is normal to every base* $r \in A$.

Whether this result can be strengthened to give decompositions involving fewer than four components is not known. Some light is shed on this question from a rather different viewpoint. Brown and Moran [2] have shown that classical Cantor measure is contained in the Raikov kernel of the convolution algebra of finite regular Borel measures on [0,1]. This result has been extended [3] by a purely combinatorial argument to Cantor measures with general fixed integer ratio of dissection. In [3] the authors deduce the following result by the argument advanced above.

THEOREM 2 *For any integer* $s \geq 3$, *each real can be decomposed in the form*

$$x = \sum_{i=1}^{s-1} x_{s,i} \quad ,$$

where each $x_{s,i}$ *is normal to every integer base multiplicatively independent of* s *but is not (even simply) normal to base* s.

The argument of [4] is extended in [5] by incorporating more structure into the Riesz product employed. This enables the integer restriction on the bases involved to be removed and for the singleton restriction on B also to be lifted. Several results emerge. In particular:

THEOREM 3 *Let* A,B *be multiplicatively independent sets of algebraic numbers. Then every real number may be expressed as a sum of four numbers each of which is normal to every base in* A *and non-normal to every base in* B.

For more general sets A,B the argument encounters a difficulty in that if the analogue of M_s is not a countable intersection it may fail to be a set of full μ-measure. However, the following can be derived:

THEOREM 4 *Let* $B \subset (1,\infty)$ *be countable. Then there is a set* $W \subset (1,\infty)$ *with* $\lambda((1,\infty) \setminus W) = 0$ *such that*

(i) W *is multiplicatively independent of* B,

(ii) *for any countable set* $A \subset W$, *every real number is a sum of four components each normal to every base in* A *and non-normal to every base in* B.

A corollary of the argument is that in the situations of Theorems Three and Four, almost all (Lebesgue) reals can be expressed as a sum of two components with the stated normality properties.

When $B \subset Z^+$, the countability requirement for A in Theorems Three and Four may be removed.

The question of the Hausdorff dimension of the set of numbers with prescribed normality properties has been taken up by Pollington [17], who derives the following result.

THEOREM 5 *Let* A,B *be multiplicatively independent sets of integers. Then the subset of* [0,1] *which is simultaneously normal to every base of* A *and normal to no base of* B *has Hausdorff dimension one.*

Although the construction uses a Cantor set and two lemmas of Schmidt, the proof of this result is essentially an argument from first principles based on a result of Eggleston [10].

It is natural to take a fresh look at this and related questions from the standpoint of Riesz products. Pertinent to such an approach is the work of Peyrière [15,16], which considers the Hausdorff dimension of Borel sets of positive Riesz product measure. Existing proofs are fairly involved and current work of the present authors includes more direct probabilistic arguments [6].

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