BOUNDARY VALUE PROBLEMS OF LINEAR ELASTOSTATICS AND HYDROSTATICS ON LIPSCHITZ DOMAINS

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Section 1 Introduction

In this note I will report on some recent progress in the study of boundary value problems for systems of equations on Lipschitz domains D in $\mathbb{R}^n$, with boundary data in $L^2(\partial D, d\sigma)$. The specific problems I will discuss here arise from elastostatics and hydrostatics.

The Dirichlet problem for a single equation (the Laplacian) on a Lipschitz domain D with $L^2(\partial D, d\sigma)$ data and optimal estimates was first treated by B. E. J. Dahlberg (see [3], [4], and [5]). His approach relied on positivity, Harnack's inequality and the maximum principle, and thus, it could not be used to study for example the Neumann problem, or systems of equations. Shortly afterwards, E. Fabes, M. Jodeit, Jr., and N. Riviere [6] were able to utilize A. P. Calderon's ([1]) theorem on the boundedness of the Cauchy integral on C^1 curves, to extend the classical method of layer potentials to the case of C^1 domains. In this work they were able to resolve the Dirichlet and Neumann problem with $L^2(\partial D, d\sigma)$ data, and to obtain optimal estimates, for C^1 domains. They relied on the Fredholm theory, exploiting the compactness of the layer potentials in the C^1 case. In 1979, D. Jerison and C. Kenig [9] were able to give a simplified proof of Dalhberg's results, using an integral identity that goes back to Rellich ([15]). However, the method still relied on positivity. Shortly afterwards, they were also able to treat the Neumann problem on

Lipschitz domains, with $L^2(\partial D, d\sigma)$ data and optimal estimates [10]. To do so they combined the Rellich type formulas with Dahlberg's results. This still restricted the applicability of the method to a single equation.

In 1981, R. Coifman, A. McIntosh, and Y. Meyer [2] established the boundedness of the Cauchy integral on any Lipschitz curve, opening the door to the applicability of the layer potential method to Lipschitz domains. This method is very flexible, and does not in principle differentiate between a single equation or a system of equations. The difficulty becomes the solvability of the integral equations. Unlike the C^1 case, on a Lipschitz domain operators like the double layer potential are not compact and so Fredholm theory is precluded.

For the case of a single equation (the Laplacian) this difficulty was overcome by G. Verchota ([16]) in his doctoral dissertation. He made the key observation that the Rellich identities mentioned before are the appropriate substitute to compactness, in the case of Lipschitz domains. Thus, he was able to recover the results of Dahlberg [4], and of Jerison and Kenig [10], for Laplace's equation on a Lipschitz domain, but using the method of layer potentials.

This note sketches the extension of the ideas of G. Verchota to the case of systems of equations. The results thus obtained had not been previously available for general Lipschitz domains, although a lot of work had been devoted to the case of piecewise linear domains. For the case of the systems of elastostatics, the result that we are about to state had been previously obtained for C^1 domains by A. Gutierriez [7], using the Fredholm theory as in [6]. Once again, compactness was a crucial element in his analysis. This is, of course, not available for Lipschitz domains.

The organization of the paper is as follows. Section 2 treats the systems of elastotastics. This is the work of B. Dahlberg, C. Kenig, and G. Verchota. Section 3 considers the Stokes problem of hydrostatics. This is joint work with C. Kenig and G. Verchota. Full proofs of the results stated here will appear in future publications.

It is a pleasure to express my gratitude to B. Dahlberg, C. Kenig, and G. Verchota for allowing me to announce here their unpublished results.

Section 2 Linear elastostatics on a Lipschitz domain.

For simplicity, in the rest of this note we will treat domains D above the graph of a Lipschitz function φ, i.e., $D = \{(x,y): y > \varphi(x)\}$, where $\varphi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ is a Lipschitz function and $n = 3$. Points $(x,\varphi(x))$ or $(y,\varphi(y))$ on ∂D will usually be denoted by P or Q. Points (x,y) in D or ${}^c\bar{D}$ will be denoted by X. The surface measure on ∂D will be denoted by $d\sigma$, and the inward unit normal will be n. By $\Gamma^+(Q)$, $Q \in \partial D$ we will denote a vertical circular cone completely contained in D. Note that the opening of $\Gamma^+(Q)$

can (and will) be taken to depend only on the Lipschitz constant of φ. By $\Gamma^-(Q)$ we will denote the reflection of $\Gamma^+(Q)$ contained in ${}^c\bar{D} = D-$. For a function $u(X)$ defined on D, $(u)^*(P) = \sup_{X \in \Gamma^+(P)} |u(X)|$, $P \in \partial D$. We will say that $u(X)$ converges non-tangentially at P to a limit ℓ if $\lim_{X \in \Gamma^+(P) \to P} u(X) = \ell$. If u is defined in $\mathbb{R}^n \setminus \partial D$ and converges non-tangentiably at $P \in \partial D$ from D and $D-$, we will denote the respective limits by $u^+(P)$ and $u^-(P)$.

Let $\lambda \geq 0$, $\mu > 0$ be constants (Lame moduli). We will seek to solve the following boundary value problems, where $\vec{u} = (u_1, u_2, u_3)$

$$(1) \qquad \begin{cases} \mu\Delta\vec{u} + (\lambda+\mu)\,\nabla\,\mathrm{div}\,\vec{u} = 0 & \text{in } D \\ \vec{u}\big|_{\partial D} = \vec{f} \in L^2(\partial D, d\sigma) \end{cases}$$

$$(2) \qquad \begin{cases} \mu\Delta\vec{u} + (\lambda+\mu)\nabla\mathrm{div}\,\vec{u} = 0 \quad \text{in } D \\ (\lambda+\mu)\, n_i(X)\, \dfrac{\partial u_j}{\partial x_j}(X) + k\left(n_j(X)\, \dfrac{\partial u_j}{\partial x_i}(X) - n_i(X)\, \dfrac{\partial u_j}{\partial x_i}(X)\right) + \\ \mu\, \delta_{ij}\, \dfrac{\partial u_j}{\partial n}(X)\Big|_{\partial D} = f_i \in L^2(\partial D, d\sigma) \end{cases}$$

Here and in the sequel we will use the summation convention. Problem (1) is the Dirichlet problem, while Problem (2) is a Neumann-type problem in which k is an arbitrary, but fixed, positive number. To ease the notation we introduce the operator

$T^k\vec{u} =$

$$(\lambda+\mu)\, n_i(X)\, \frac{\partial u_j}{\partial X_j}(X) + k\left(n_j(X)\, \frac{\partial u_j}{\partial X_i}(X) - n_i(X)\, \frac{\partial u_j}{\partial X_i}(X)\right) + \mu\delta_{ij}\, \frac{\partial u_j}{\partial n}(X) .$$

The operator T^k is called the generalized stress.

In the particular case $k = \mu$ of problem (2), the operator $T \equiv T^{\mu}$ is called the stress.

<u>Theorem 2.1</u>: a) There exists a unique solution of problem (1) in D, with $(\vec{u})^* \in L^2(\partial D, d\sigma)$ and $\vec{u}$ having non-tangential limit $\vec{f}(P)$ for almost every $P \in \partial D$. The solution $\vec{u}$ belongs to the Sobolev space $H^{1/2}(D)$.

b) For every $k > 0$ there exists a unique solution of problem (2) in D, which is 0 at infinity, with $(\nabla\vec{u})^* \in L^2(\partial D, d\sigma)$, and with $T^k\vec{u}$ having non-tangential limit $\vec{f}(P)$ for almost every $P \in \partial D$. The solution $\vec{u}$ belongs to the Sobolev space $H^{3/2}(D)$.

In what follows we will outline the proofs of part a and part b in the case $k \neq \mu$. The case of stress boundary conditions is considerably more involved and an outline of the proof would take us afield of the main ideas. The primary difficulty in the stress case will be pointed out in the course of the argument.

To begin the proof of Theorem 2.1, we first introduce the Kelvin matrix of fundamental solutions (see [11] for example), $\Gamma(X) = (\Gamma_{ij}(X))$, where $\Gamma_{ij}(X) = \frac{A}{4\pi} \frac{\delta_{ij}}{|X|} + \frac{C}{4\pi} \frac{X_i X_j}{|X|^3}$, and

$A = \frac{1}{2}\left[\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right]$, $C = \frac{1}{2}\left[\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right]$. Our solution of (1) will be given in the form of a double layer potential

$$\vec{u}(X) = \mathfrak{D}\vec{g}(X) = \int_{\partial D} \{T^k(Q)\,\Gamma(X-Q)\}^t\ \vec{g}(Q)\ d\sigma(Q) ,$$

where the operator T^k is applied to each column of the matrix Γ and the superscript t denotes the transpose matrix.

Our solution of (2) will be given in the form of a single layer potential

$$\vec{u}(X) = S\vec{g}(X) = \int_{\partial D} \Gamma(X-Q)\,\vec{g}(Q)\ d\sigma(Q) .$$

<u>Lemma 2.3</u>: Let $\mathfrak{D}\vec{g}(X)$, $S(\vec{g})(X)$ be defined as above, with $\vec{g} \in L^2(\partial D, d\sigma)$. Then, they both solve the system $\mu\Delta\vec{u} + (\lambda+\mu)\,\nabla\,\mathrm{div}\,\vec{u} = 0$ in D and $D-$. Moreover,

(a) $$\|(\mathfrak{D}\vec{g})^*_+\|_{L^2(\partial D, d\sigma)} + \|(\mathfrak{D}\vec{g})^*_-\|_{L^2(\partial D, d\sigma)} + \|\mathfrak{D}\vec{g}\|_{H^{1/2}(D)} \le C\|\vec{g}\|_{L^2(\partial D, d\sigma)}$$

(b) $$(\mathfrak{D}\vec{g})^{\pm}(P) = \pm\frac{1}{2}\,\vec{g}(P) + \text{p.v.}\int_{\partial D} \{T^k(Q)\,\Gamma(P-Q)\}^t\vec{g}(Q)\,d\sigma(Q)$$

(c) $$\|(\nabla S\vec{g})^*_+\|_{L^2(\partial D, d\sigma)} + \|(\nabla S\vec{g})^*_-\|_{L^2(\partial D, d\sigma)} + \|S\vec{g}\|_{H^{3/2}(D)} \le C\|\vec{g}\|_{L^2(\partial D, d\sigma)}$$

(d) $$\left(\frac{\partial}{\partial X_i}(S\vec{g})_j\right)^{\pm}(P) = \mp\left\{\frac{(A+C)}{2}\ n_i(P)\ g_j(P) - (n_i(P)\ n_j(P)\ \langle n(P), \vec{g}(P)\rangle)\right\} + \left(\text{p.v.}\int_{\partial D} \frac{\partial}{\partial P_i}\Gamma(P-Q)\ \vec{g}(Q)\ d\sigma(Q)\right)_j .$$

Therefore,

$$(T^k S\vec{g})^{\pm}(P) = \pm\frac{1}{2}\,\vec{g}(P) + \text{p.v.}\int_{\partial D} T^k(P)\,\Gamma(P-Q)\,\vec{g}(Q)\,d\sigma(Q) \quad .$$

The proof of Lemma 2.3 follows from the theorem of R. Coifman, A. McIntosh, and Y. Meyer ([2]). See [16] for the details in a similar situation.

Thus, the proof of Theorem 2.1 reduces to the invertibility on $L^2(\partial D, d\sigma)$ of the operators

$$\pm\frac{1}{2}\,I + K^k$$

$$\pm\frac{1}{2}\,I + (K^k)^* \quad , \text{ where}$$

$$K^k\vec{g}(P) = \text{p.v.}\int_{\partial D} \{T^k(Q)\,\Gamma(P-Q)\}^t\ \vec{g}(Q)\,d\sigma(Q) \quad .$$

This is accomplished by means of the following lemma:

<u>Lemma</u> 2.4: There exists a constant C, which depends only on the Lipschitz constant of ∂D, and on the number k, such that, if $k \neq \mu$ we have, for all $\vec{g} \in L^2(\partial D, d\sigma)$

$$\left\|\left(\frac{1}{2}\,I - (K^k)^*\right)\vec{g}\right\|_{L^2(\partial D, d\sigma)} \leq C\left\|\left(\frac{1}{2}\,I + (K^k)^*\right)\vec{g}\right\|_{L^2(\partial D, d\sigma)}$$

and,

$$\left\|\left(\frac{1}{2}\,I + (K^k)^*\right)\vec{g}\right\|_{L^2(\partial D, d\sigma)} \leq C\left\|\left(\frac{1}{2}\,I - (K^k)^*\right)\vec{g}\right\|_{L^2(\partial D, d\sigma)}$$

To show that Lemma 2.4 implies the invertibility of the operators in question, we follow Verchota's ([16]) ideas. First of all the inequalities clearly show that $\frac{1}{2}\,I + (K^k)^*$ and $\frac{1}{2}\,I - (K^k)^*$ are one to one. A simple argument using the continuity of $(K^k)^*$ shows that these operators have closed range. We can, therefore, attach an index to these operators which might possibly be infinite. Now, for each t, $0 \leq t \leq 1$, we consider the Lipschitz domain D_t given by the graph of $t\varphi$. By the theorem of Coifman - McIntosh - Meyer ([2]), the operators $(K_t^k)^*$, corresponding to the domains D_t, are continuous in norm. At $t = 0$ we are in the case of the upper half plane, and, therefore, the index is 0. Therefore, the index is also 0 at $t = 1$, and the desired invertibility follows. We are indebted to A. McIntosh for pointing out to us this simple argument using the index.

We, therefore, pass to the proof of Lemma 2.4. In order to do so, we will first explain the boundary conditions in problem (2) from the point of view of second order elliptic systems.

Let A_{ij}^{rs}, $1 \leq r$, $s \leq M$, $1 \leq i$, $j \leq n$ be constants satisfying the ellipticity condition

$$A_{ij}^{rs} \xi_i \xi_j \eta^r \eta^s \geq C|\xi|^2 |\eta|^2 ,$$

and the symmetry condition $A_{ij}^{rs} = A_{ji}^{sr}$. We consider vector valued functions $\vec{u} = (u^1, \ldots, u^M)$ on $\mathbb{R}^n$, satisfying the divergence form system

$$\frac{\partial}{\partial X_i} A_{ij}^{rs} \frac{\partial}{\partial X_j} u^s = 0 \quad \text{in} \quad D$$

From variational consideration, the most natural boundary conditions are Dirichlet conditions $(\vec{u}|_{\partial D} = \vec{f})$ or the

Neumann-type condition $\frac{\partial \vec{u}}{\partial \nu} = n_i \, A_{ij}^{rs} \frac{\partial}{\partial X_j} u^s \big|_{\partial D} = f_r$. The interpretation of problem (2) in this framework is the following: given $k > 0$, there exist constants $A_{ij}^{rs}(k) = A_{ij}^{rs}$, $1 \leq i , j \leq 3$, $1 \leq r , s \leq 3$ satisfying the ellipticity and symmetry conditions, and such that $\mu \Delta \vec{u} + (\lambda+\mu) \nabla \operatorname{div} \vec{u} = 0$ in D if and only if $\frac{\partial}{\partial x_i} A_{ij}^{rs} \frac{\partial u^s}{\partial X_j} = 0$ in D , and with $T^k \vec{u} = \frac{\partial}{\partial \nu} \vec{u} = n_i \, A_{ij}^{rs} \frac{\partial}{\partial X_j} u^s$.

<u>Lemma 2.5</u>: (The Rellich, Payne-Weinberger, Necas identities (see [15], [14] and [13]). Let $\vec{h}$ be a constant vector in $\mathbb{R}^n$, and suppose that $\frac{\partial}{\partial X_i} A_{ij}^{rs} \frac{\partial}{\partial X_j} u^s = 0$ in D , $A_{ij}^{rs} = A_{ji}^{sr}$, and $\vec{u}$ and its derivatives are suitably small at ∞ . Then

$$\int_{\partial D} h_\ell \, n_\ell \, A_{ij}^{rs} \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j} \, d\sigma = 2 \int_{\partial D} h_i \frac{\partial u^r}{\partial X_i} \, n_\ell \, A_{\ell j}^{rs} \frac{\partial u^s}{\partial X_j} \, d\sigma$$

<u>Proof</u>: Apply the divergence theorem to

$$\frac{\partial}{\partial X_\ell} [(h_\ell \, A_{ij}^{rs} - h_i \, A_{\ell j}^{rs} - h_j \, A_{i\ell}^{rs}) \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j}] = 0 .$$

<u>Corollary</u> 2.6: If $A_{ij}^{rs} \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j} \geq C \sum_r |\nabla u^r|^2$, then ,

$$\frac{\partial \vec{u}}{\partial \nu} = n_i \, A_{ij}^{rs} \frac{\partial u^s}{\partial X_j}$$

satisfies

$$\int_{\partial D} \left| \frac{\partial}{\partial \nu} \vec{u} \right|^2 \simeq \sum_r \int_{\partial D} |\nabla_t \, u^r|^2 ,$$

where $\nabla_t \, u^r$ denotes the tangential components of the gradient of u^r, and the comparability constants depend only on the Lipschitz constant of ∂D.

<u>Proof</u>: Take $\vec{h} = e_n$. Because of the Lipschitz character of ∂D, $h_\ell n_\ell \geq C$. Then,

$$\sum_r \int_{\partial D} |\nabla u^r|^2 \, d\sigma < C \int_{\partial D} h_\ell \, n_\ell \, A_{ij}^{rs} \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j} \, d\sigma =$$

$$= C \int_{\partial D} h_i \frac{\partial u^r}{\partial X_i} \; n_\ell \, A_{ij}^{rs} \frac{\partial u^s}{\partial X_j} \, d\sigma \leq$$

$$\leq C \left(\sum_r \int_{\partial D} |\nabla u^r|^2 d\sigma \right)^{1/2} . \left(\int_{\partial D} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 \, d\sigma \right)^{1/2} .$$

Thus, $\displaystyle\sum_r \int_{\partial D} |\nabla_t \, u^r|^2 \, d\sigma \leq C \int_{\partial D} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 \, d\sigma$.

For the opposite inequality, observe that, for each r,s,j fixed, the vector $h_i \, n_\ell \, A_{ij}^{rs} - h_\ell \, n_\ell \, A_{ij}^{rs}$ is perpendicular to n. Because of lemma 2.5,

$$\int_{\partial D} h_\ell \, n_\ell \, A_{ij}^{rs} \frac{\partial u^r}{\partial X_i} \frac{\partial u^s}{\partial X_j} \, d\sigma =$$

$$= 2 \int_{\partial D} (h_\ell \, n_\ell \, A_{ij}^{rs} - h_i \, n_\ell \, A_{\ell j}^{rs}) \frac{\partial u^\ell}{\partial X_i} \frac{\partial u^s}{\partial X_j} \, d\sigma .$$

Hence, $\displaystyle\int_{\partial D} |\nabla \vec{u}|^2 \, d\sigma \leq C \left(\sum_r \int_{\partial D} |\nabla_t u^r|^2 d\sigma\right)^{1/2} \left(\sum_r \int_{\partial D} |\nabla u^r|^2 d\sigma\right)^{1/2}$,

and so,

$$\int_{\partial D} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 \, d\sigma \leq C \int_{\partial D} |\nabla \vec{u}|^2 \, d\sigma \leq C \sum_r \int_{\partial D} |\nabla_t u^r|^2 \, d\sigma .$$

<u>Remark 2.7</u>: At this point we can explain the difference between problem (2) for $k \neq \mu$ and for $k = \mu$. In the case of problem

(2) with $k \neq \mu$, $A_{ij}^{rs}(k)$ satisfy the hypothesis of Corollary 2.6. On the other hand, when $k = \mu$, $A_{ij}^{rs} \frac{\partial u^s}{\partial X_i} \frac{\partial u^r}{\partial X_j} = \lambda(\operatorname{div} \vec{u})^2 + \frac{\mu}{2} \sum_{i,j} (\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j})^2$, which obviously does not satisfy the hypothesis of 2.6.

<u>Proof of Lemma 2.4</u>: Let $\vec{u}(X) = S\vec{g}(X)$. We will apply Corollary 2.6 to $\vec{u}$, which we can in the case $k \neq \mu$. We will do so in D and also in $D-$. First note that $T^k\vec{u} = \frac{\partial \vec{u}}{\partial \nu}$. Then note that because of Lemma 2.3 (d), $(\nabla_t\, u_j)_+ = (\nabla_t u_j)_-$. Therefore, $\int_{\partial D} |(T^k\vec{u})^+|^2\, d\sigma \simeq \int_{\partial D} |(T^k\vec{u})^-|^2\, d\sigma$.
But again using Lemma 2.3 (d), we see that Lemma 2.4 follows immediately. We have thus established Lemma 2.4 and hence Theorem 2.1.

<u>Section 3</u>: Linear hydrostatics on a Lipschitz domain

We will continue utilizing the notation introduced in Section 2. We will discuss the so-called Stokes problem of hydrostatics.

We seek a vector valued function $\vec{u} = (u_1, u_2, u_3)$ and a scalar valued function p satisfying

$$(4) \qquad \begin{cases} \Delta\vec{u} = \nabla p & \text{in } D \\ \operatorname{div} \vec{u} = 0 & \text{in } D \\ \vec{u}_{|\partial D} = \vec{f} \in L^2(\partial D, d\sigma) & \end{cases}$$

<u>Theorem</u> 3.1: There exists a unique solution of problem (4) in D, with $(\vec{u})^* \in L^2(\partial D, d\sigma)$, and $\vec{u}$ having non-tangential limit $\vec{f}(P)$, for almost every $P \in \partial D$. The solution $\vec{u}$ belongs to the Sobolev space $H^{1/2}(D)$.

In order to sketch the proof of Theorem 3.1 (which parallels that of Theorem 2.1), we introduce the matrix of fundamental solutions (see the book of Ladyzhenskaya, [12]) $\Gamma(X) = (\Gamma_{ij}(X))$, where $\Gamma_{ij}(X) = -\frac{1}{8\pi}\frac{\delta_{ij}}{|X|} - \frac{1}{8\pi}\frac{X_i X_j}{|X|^3}$, and its corresponding pressure vector $\vec{q}(X) = (q^i(X))$, where $q^i(X) = \dfrac{-X_i}{4\pi|X|^3}$. Observe that $\Delta\Gamma_{ij}(X) = D_{X_i}\, q^j(X)$.

Our solution of (4) will be given in the form of a double layer potential

$$\vec{u}(X) = \mathcal{D}\vec{g}(X) = \int_{\partial D} \{T'(Q)\Gamma(X-Q)\}\vec{g}(Q)\,d\sigma(Q) ,$$

where $T'(Q)$ is a matrix of first order boundary operators. As we have already seen in Section 2 there are several possibilities for $T'(Q)$. In the case of elastostatics any $T'(Q)$ connected with a pseudo-stress $(k \neq \mu)$ could have been used to formulate an elastostatic double layer potential. Let's make clear at this point the procedure for constructing double layer potentials.

We first look for a Neumann-type boundary condition, $\frac{\partial}{\partial\nu}$, for which the conclusion of Corollary 2.6 is valid. We then apply $\frac{\partial}{\partial\nu}$ to each column of the fundamental matrix, $\Gamma(X-Y)$, as a function of Y. This defines a matrix of kernels denoted by

$\{T'(Q)\Gamma(X-Q)\}$ with $Q \in \partial D$. The corresponding integral operator is called a double layer potential.

In the case of Stokes problem, we will show that for functions, $\vec{u}(X)$, satisfying $\Delta\vec{u}(X) = \nabla p(X)$, an appropriate Neumann-type condition is

$$\frac{\partial\vec{u}}{\partial\nu} = \frac{\partial\vec{u}}{\partial n} - pn .$$

Having now chosen this boundary operator, we write down the corresponding double layer potential

$$\mathfrak{D}\vec{g}(X) = \int_{\partial D} \{T'(Q)\Gamma(X-Q)\}\vec{g}(Q)\,d\sigma(Q) .$$

where $(T'(Q)\Gamma(X-Q))_{i\ell} = \delta_{ij}q^{\ell}(X-Q) \quad n_j(Q) + \frac{\partial\Gamma_{i\ell}}{\partial Q_j}(X-Q) \quad n_j(Q)$.

We also introduce the single layer potential,

$$\vec{u}(X) = S\vec{g}(X) = \int_{\partial D} \Gamma(X-Q)\,\vec{g}(Q)\,d\sigma(Q) ,$$

and observe that $\Delta S\vec{g} = \nabla P_{S\vec{g}}$ where

$$P_{S\vec{g}}(X) = \int_{\partial D} g^{\ell}(X-Q)\,g_{\ell}(Q)\,d\sigma(Q) .$$

<u>Lemma</u> 3.2: Let $\mathfrak{D}\vec{g}$, $S(\vec{g})$ be defined as above, with $\vec{g} \in L^2(\partial D, d\sigma)$. Then $\vec{u}(X) = \mathfrak{D}(\vec{g})(X)$ solves

$$\Delta\vec{u} = \nabla p$$

$$\operatorname{div}\vec{u} = 0 \quad \text{in } D \text{ and } D_- . \quad \text{Moreover}$$

(a) $$\|(\mathfrak{D}\vec{g})^*_{\pm}\|_{L^2(\partial D, d\sigma)} + \|\mathfrak{D}\vec{g}\|_{H^{1/2}(D)} \leq C\|\vec{g}\|_{L^2(\partial D, d\sigma)}$$

(b) $(\mathcal{D}g)^{\pm}(P) = \pm \frac{1}{2}\vec{g}(P) + \text{p.v.} -\int_{\partial D} \{T'(Q)\Gamma(P-Q)\}\vec{g}(Q)\,d\sigma(Q)$

(c) $\|(\nabla S\vec{g})^{*}_{+}\|_{L^2(\partial D, d\sigma)} + \|(\nabla S\vec{g})^{*}_{-}\|_{L^2(\partial D, d\sigma)} \le C\|\vec{g}\|_{L^2(\partial D, d\sigma)}$

(d) $$\left(\frac{\partial}{\partial X_1}(S\vec{g})_j\right)^{\pm}(P) = \pm\left\{\frac{n_i(P)\,g_j(P)}{2} - \frac{n_i(P)\;n_j(P)}{2}\langle\, n(P), \vec{g}(P)\,\rangle\right\} + \left(\text{p.v.} \int_{\partial D} \frac{\partial}{\partial P_i}\Gamma(P-Q)\,\vec{g}(Q)\;d\sigma(Q)\right), \text{ and}$$

$$\lim_{\substack{X \to P_{\pm} \\ X \in \Gamma^{\pm}(P)}} \frac{\partial}{\partial n} S\vec{g}(X) - P_{S\vec{g}}(X)\,n = \pm\frac{1}{2}\vec{g}(P) + \text{p.v.}\int_{\partial D} \{T(P)\Gamma(P-Q)\}\vec{g}(Q)\,d\sigma(Q)$$

where

$$(T(P)\Gamma(P\;Q))_{i\ell} = n_j(P)\,\frac{\partial \Gamma_{i\ell}}{\partial P_j}(P-Q) - \delta_{ij}\;q^{\ell}(P-Q)\;n_j(P)\;.$$

The proof of Lemma 3.2 follows, as the one in Lemma 2.3, from [2]. See [12] for the case of smooth domains. Thus, the proof of Theorem 3.1 reduces to the invertibility in $L^2(\partial D, d\sigma)$ of the operator

$$\frac{1}{2} I + K\;,\; \text{where}$$

$K\vec{g}(P) = \text{p.v.} -\int_{\partial D} \{T'(Q)\Gamma(P-Q)\}\;\vec{g}(Q)\;\;d\sigma(Q)$. As in section 2, this in turn follows from

<u>Lemma</u> 3.3: There exists a constant C, which depends only on the Lipschitz constant of ∂D, such that, for all $\vec{q} \in L^2(\partial D, d\sigma)$,

$$\left\|\left(\frac{1}{2} I - K\right)\;\vec{g}\right\|_{L^2(\partial D, d\sigma)} \le C\left\|\left(\frac{1}{2} I + K^{*}\right)\vec{g}\right\|_{L^2(\partial D, d\sigma)}\;,$$

and

$$\|(\tfrac{1}{2} I + K^*)\vec{g}\|_{\lambda^2(\partial D, d\sigma)} \leq C\|(\tfrac{1}{2} I - K^*)\vec{g}\|_{L^2(\partial D, d\sigma)}$$

We turn now to the proof of Lemma 3.3. The proof relies on two integral identities.

<u>Lemma</u> 3.4: Let $\vec{h}$ be a constant vector in $\mathbb{R}^n$, and suppose that $\Delta\vec{u} = \nabla p$, $\operatorname{div} \vec{u} = 0$ in D, and that $\vec{u}, p$ and their derivatives are suitably small at ∞. Then,

$$\int_{\partial D} h_\ell \, n_\ell \cdot \frac{\partial u^s}{\partial X_j} \frac{\partial u^s}{\partial X_j} \, d\sigma = 2 \int_{\partial D} \frac{\partial u^s}{\partial n} \cdot h_\ell \frac{\partial u^s}{\partial X_\ell} \, d\sigma - 2 \int_{\partial D} p \cdot n_s h_\ell \frac{\partial u^s}{\partial X_\ell} \, d\sigma$$

<u>Lemma</u> 3.5: Let $\vec{h}$, $\vec{u}$ and p be as in Lemma 3.4. Then,

$$\int_{\partial D} h_\ell n_\ell p^2 \, d\sigma = 2\int_{\partial D} h_r \frac{\partial u^r}{\partial n} \cdot p \, d\sigma - 2\int_{\partial D} h_r \frac{\partial u^r}{\partial X_i} \frac{\partial u^i}{\partial n} \, d\sigma + 2\int_{\partial D} h_r \, n_s \frac{\partial u^s}{\partial X_j} \frac{\partial u^r}{\partial X_j} \, d\sigma$$

The proofs of Lemmas 3.4 and 3.5 are simple applications of the properties of $\vec{u}$, p, and the divergence theorem.

An immediate consequence of Lemma 3.5 is

<u>Corollary</u> 3.6: Let $\vec{u}$, p be as in Lemma 3.4, D a Lipschitz domain. Then, $\int_{\partial D} p^2 \, d\sigma \leq C \int_{\partial D} |\nabla\vec{u}|^2 \, d\sigma$, where C depends only on the Lipschitz constant of ∂D.

A consequence of Corollary 3.6 and Lemma 3.4 is

Corollary 3.7: Let $\vec{u}$, p be as in Lemma 3.4, D a Lipschitz domain. Then,

$$\int_{\partial D} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 d\sigma \simeq \sum_r \int_{\partial D} |\nabla_t u^r|^2 d\sigma,$$

where, by definition $\frac{\partial \vec{u}}{\partial \nu} = \frac{\partial \vec{u}}{\partial n} - pn$.

Proof: Lemma 3.4 clearly implies that

$$\int_{\partial D} |\nabla \vec{u}|^2 d\sigma \leq C \int_{\partial D} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 d\sigma$$

Arguing as in the proof of Corollary 2.6, using Lemma 3.4, we see that

$$\int_{\partial D} |\nabla \vec{u}|^2 d\sigma < C(\sum_r \int_{\partial D} |\nabla_t u^r|^2 d\sigma) + \left|\int_{\partial D} p\ n_s\ h_\ell \frac{\partial u^s}{\partial X_\ell} d\sigma\right| .$$

Since $\vec{u}$ is divergence free (i.e., div $\vec{u} = 0$)

$$n_s\ h_\ell \frac{\partial u^s}{\partial X_\ell} = n_s\ h_\ell \frac{\partial u^s}{\partial X_\ell} - n_\ell\ h_\ell \frac{\partial u^s}{\partial X_s} .$$

It is easy to check that for s fixed the above operator is a tangential vector field on u^s . From this fact and Corollary 3.6 we have the bound,

$$\left|\int_{\partial D} p\ n_s\ h_s \frac{\partial u^r}{\partial X_\ell} d\sigma\right| \leq C\left[\int_{\partial D} |\nabla \vec{u}|^2 d\sigma\right]^{1/2} \left[\sum_r \int_{\partial D} |\nabla_t u^r|^2 d\sigma\right]^{1/2} .$$

Hence $\int_{\partial D} |\nabla \vec{u}|^2 d\sigma$ is equivalent to both $\int_{\partial D} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^2 d\sigma$ and $\sum_r \int_{\partial D} |\nabla \vec{u}^r|^2 d\sigma$; and so these last two quantities are themselves equivalent.

We can now prove Lemma 3.3. In fact, if we set $\vec{u} = S\vec{g}$, Lemma 3.3 is an immediate consequence of Corollary 3.7 and the second part of (d) in Lemma 3.2.

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