BILINEAR INTEGRALS AND RADON-NIKODYM DERIVATIVES

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Dedicated to Igor Kluvánek

1. INTRODUCTION

A basic problem in the treatment of random evolutions via the non-commutative Feynman-Kac formula is to construct a random multiplicative operator valued functional M, and show that the perturbed semigroup of the random evolution is represented as the integral of M with respect to the associated operator valued measure [3]. The solutions of certain partial differential equations can thereby be represented as a bilinear integral, giving further insight into their behaviour.

A related problem is to determine when a process described by an operator valued measure may be expressed as the integral of an operator valued function with respect to another such process.

The purpose of this paper is to consider the above question in a general setting. Given measures m, n with values in the vector spaces Y, Z respectively, there is a given continuous bilinear "multiplication" defined on the product of the vector spaces X and Y with values in Z, and we ask whether n can be expressed as the integral of an X- valued function with respect to m.

The present approach differs from previous work [1], [6] in a number of ways. The locally convex setting is used, because ultimately, the case of spaces of operators on Banach spaces, with the strong operator topology is to be treated. The bilinear multiplication - the composition of operators - is only separately continuous in that case.

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The conditions imposed in [1] and [6] require that, in the operator setting, the vector measure m has finite variation in the uniform norm, a property too restrictive in practice. By working with weak vector valued integrals, the more general situation is treated. The results are formulated in terms of bornologies, because they form a natural structure for viewing the variation of a vector measure.

The types of integrals considered are introduced in Section 2; they are modelled along the lines of the Pettis integral in Banach spaces. The main result (Theorem 3.6) on the differentiation of a vector measure with respect to another vector measure is given in Section 3. The assertion is used to establish the existence of operator functionals associated with Markov processes, and it is also applied to the existence of conditional expectations with respect to operator valued measures in Section 4.

Let E be a locally convex space. The standard notation of [7] is generally followed. In particular, for a disked set $B \subseteq E$ such that $x = 0$ whenever $p_B(x) = 0$, E_B denotes the normed space consisting of $\cup_{n\geq 1} nB$ equipped with the gauge p_B. of B.

A saturated family [7] p 81, $\mathfrak{B}$ of bounded subsets of E is called a *convex bornology* in E. If for each closed disked set $B \in \mathfrak{B}$, the normed space E_B is complete, then $\mathfrak{B}$ is said to be *complete*.

Let $\mathfrak{S}$ be a σ-algebra of subsets of a set Ω. The collection of all finite partitions of a set $A \in \mathfrak{S}$ by elements of $\mathfrak{S}$ is denoted by $\Pi(A, \mathfrak{S})$. Given an additive set function $m : \mathfrak{S} \to E$ and a disked set $D \subseteq E$ the D*-variation* $V(m,D) : \mathfrak{S} \to [0,\infty]$ of m is the set function defined by

$$V(m,D)(A) = \sup\{\Sigma_{B\in\pi}\, p_D(m(B)) : \pi \in \Pi(A,\mathfrak{S})\},\ A \in \mathfrak{S}.$$

The set function m is said to have *finite D-variation on* $A \in \mathfrak{S}$ if $V(m,D)(A) < \infty$. If m is a vector measure with finite D-variation on Ω, then $V(m,D)$ is σ-additive whenever D is closed, because then p_D is lower semicontinuous on E.

Now let $\mathfrak{F}$ be a family of subsets of E. A vector measure $m : \mathfrak{S} \to E$ is said to have *σ-finite $\mathfrak{F}$-variation* if there exists a partition Ω_i, $i = 1,2,\dots$ of Ω by elements of $\mathfrak{S}$ such that for each $i = 1,2,\dots$, there exists a disked set $F_i \in \mathfrak{F}$ such that m has finite F_i-variation on Ω_i. It is often convenient to replace the family $\mathfrak{F}$ by the property which defines it; for example, in the case that $\mathfrak{F}$ is the family of all bounded subsets of E, the vector measure is said to have σ-finite *bounded* variation.

2. INTEGRATION

In the context that the following results are to apply, it is important not to place too many restrictions on the variation of the vector measure with respect to which we are differentiating, so the weak form of vector valued integrals is the most appropriate type to use.

Let E be a locally convex space. The integral $f\mu : \mathfrak{S} \to E$ of a vector valued function $f : \Omega \to E$ with respect to a scalar measure $\mu : \mathfrak{S} \to [0,1]$ is taken in the following sense. The function f is *μ-integrable* if for each $\xi \in E'$, $\langle f, \xi \rangle$ is μ-integrable, and for every $A \in \mathfrak{S}$, there exists $f\mu(A) \in E$ such that $\langle f\mu(A), \xi \rangle = \int_A \langle f, \xi \rangle \, d\mu$ for every $\xi \in E'$.

The convex bornology $\mathfrak{W}\mathfrak{C}\mathfrak{C}$ of subsets of E whose closed convex hulls are weakly compact is distinguished by the following property; if $m : \mathfrak{S} \to E$ is a vector measure with σ-finite $\mathfrak{W}\mathfrak{C}\mathfrak{C}$-variation such that $m \ll \lambda$ for some finite measure λ, then there exists a function $f : \Omega \to E$ such that $m = f\lambda$. Moreover, the density f is *λ-regular* in E_σ; that is, f is Borel measurable for the weak topology $\sigma(E, E')$ of E, and $\lambda \circ f^{-1}$ is a Radon measure for $\sigma(E, E')$. Any two such densities must be equal λ-a.e. [4].

There is another notion of measurability associated with each convex bornology $\mathfrak{B}$ on E. A function $f : \Omega \to E$ is said to be $\mathfrak{B}$*-strongly* λ*- measurable* if for every set $V \in \mathfrak{S}$ of positive λ-measure, there exists another set $W \in \mathfrak{S} \cap V$ of positive λ-measure, and a disked set $B \in \mathfrak{B}$, such that the restriction $f|W$ of f to W is strongly measurable in the normed space E_B ; that is, there exist E_B -valued $\mathfrak{S}$-simple functions s_n , $n = 1,2,\ldots$ on W converging pointwise to f on W, in the normed space E_B . It follows that f is Borel λ-measurable on E and $\lambda \circ f^{-1}$ is a Radon measure on E .

Let X be a locally convex space and suppose that $(\Omega, \mathfrak{S})$ is a measurable space. Let $m : \mathfrak{S} \to X'_\sigma$ be a vector measure with σ-finite $\mathfrak{M}\mathfrak{C}\mathfrak{S}$-variation. If X is a barrelled space, such as a Banach space, then bounded subsets of X' are $\sigma(X',X)$-compact, so it is enough to know that m has σ-finite bounded variation in X'. Then there exists a finite measure $\lambda : \mathfrak{S} \to [0,1]$ and a λ-weakly regular function function $g : \Omega \to X'_\sigma$ such that g is λ-integrable in X'_σ and $m = g\lambda$.

DEFINITION 2.1. A function $f : \Omega \to X$ is said to be *m-integrable* if for any λ-weakly regular function g such that $m = g\lambda$, $\langle f,g \rangle$ is λ-measurable and $\int_\Omega |\langle f,g \rangle| \, d\lambda < \infty$. The indefinite integral $\langle f,m \rangle$ of f with respect to m is defined by $\langle f,m \rangle(A) = \int_A \langle f,g \rangle \, d\lambda$, $A \in \mathfrak{S}$.

The definition makes sense because any two λ-weakly regular densities of m with respect to λ are equal λ-a.e. . Moreover, it is easy to see that the the choice of the measure λ is irrelevant.

It may be seen from the case of stochastic integrals that there can be significant problems associated with the definition of the bilinear integral whenever the vector measure m does *not* have σ-finite variation.

Whenever X is a Banach space, the bilinear integral defined by Bartle [1] requires that the vector measure m has finite variation in X'. The integral so defined has good convergence properties, but it is too restrictive for many applications to operator valued measures.

Now let $(x,y) \to xy$ be a separately continuous bilinear form from the product $X \times Y$ of the locally convex spaces X and Y into the locally convex space Z. Let $m : \mathfrak{S} \to Y$ be a vector measure. For each $\zeta \in Z'$, define $m_\zeta : \mathfrak{S} \to X'_\sigma$ by $m_\zeta(A)x = \langle xm(A), \zeta \rangle$, $x \in X$, $A \in \mathfrak{S}$. Suppose that for each $\zeta \in Z'$, the vector measure m_ζ has σ-finite $\mathfrak{M}\mathfrak{S}\mathfrak{S}$-variation.

DEFINITION 2.2. A function $f : \Omega \to X$ is said to be *m-integrable* if for each $\zeta \in Z'$, f is m_ζ - integrable, and for every $A \in \mathfrak{S}$, there exists $fm(A) \in Z$ such that $\langle fm(A), \zeta \rangle = \langle f, m_\zeta \rangle(A)$ for every $\zeta \in Z'$.

The set function $fm : \mathfrak{S} \to Z$ is σ-additive by the Orlicz-Pettis lemma. It is the integral of f with respect to m.

The space of all X-valued $\mathfrak{S}$-simple functions is denoted by $\mathrm{sim}(\mathfrak{S}, X)$. If $f = \sum_{i=1}^{k} c_i \chi_{A_i}$, $c_i \in X$, $A_i \in \mathfrak{S}$ is an X-valued $\mathfrak{S}$-simple function, then f is clearly m-integrable and the indefinite integral fm of f with respect to m is given by $fm(A) = \sum_{i=1}^{k} c_i m(A_i \cap A)$, $A \in \mathfrak{S}$.

3. A RADON-NIKODYM THEOREM.

Let $m : \mathfrak{S} \to Y$, $n : \mathfrak{S} \to Z$ be vector measures. A *Radon-Nikodym derivative* of n with respect to m is a function $f : \Omega \to X$ such that $n = fm$. This section gives a necessary and sufficient condition for the existence of a Radon-

Nikodym derivative in the case that m has the appropriate variational properties. The main result is preceded by some preparatory lemmas.

As before, it is assumed that m_ζ has σ-finite $\mathfrak{M}\mathfrak{C}\mathfrak{S}$-variation in X'_σ for each $\zeta \in Z'$. Let $\mathfrak{B}$ be a complete convex bornology in X. Let $\mathfrak{U}$ be a neighbourhood base of zero in Z consisting of closed disked sets.

Set $\mathfrak{S}^+ = \{A \in \mathfrak{S}: m(B) \neq 0 \text{ for some } B \subseteq A\}$ and for each disked set $D \subseteq X$ and $\zeta \in Z'$, let

$$|m|_{D,\zeta}(A) = \sup\{\, |\langle fm,\zeta\rangle|(A) : f \in \mathrm{sim}(\mathfrak{S},X),\ f(\Omega) \subseteq D \,\},\ A \in \mathfrak{S}.$$

Then $|m|_{D,\zeta} : \mathfrak{S} \to [0,\infty]$ is the D°-variation of an X'_σ-valued measure, so it is σ-additive. If $|m|_{D,\zeta}(\Omega) < \infty$, it follows that $|m|_{D,\zeta} \ll m.$.

For a closed disked subset C of Z,

$$|n|_C(A) = \sup\{|\langle n,\zeta\rangle|(A) : \zeta \in C^\circ\},\ A \in \mathfrak{S},$$

$$|m|_{D,C}(A) = \sup\{|m|_{D,\zeta}(A) : \zeta \in C^\circ\},\ A \in \mathfrak{S}.$$

LEMMA 3.1. *Suppose that Z is sequentially complete. Let $B \in \mathfrak{B}$ be a closed disked set. If $A \in \mathfrak{S}$ and $|m|_{B,U}(A) < \infty$ for all $U \in \mathfrak{U}$, then the uniform limit f in X_B of X_B-valued $\mathfrak{S}$-simple functions f_n, $n = 1,2,\ldots$ is m-integrable in Z, and for each $n = 1,2,\ldots$, $U \in \mathfrak{U}$*

$$|fm - f_n m|_U(A) \le \sup\{p_B(f - f_n(\omega)) : \omega \in A,\ n = 1,2,\ldots\}|m|_{B,U}(A).$$

Proof. For each $\zeta \in Z'$, m_ζ has σ-finite weakly compact variation in X'_σ, so there exists a measure $\lambda_\zeta : \mathfrak{S} \to [0,1]$ such that $m_\zeta \ll \lambda_\zeta$, and a λ_ζ -

weakly regular function $g_\zeta : \Omega \to X'$ such that $m_\zeta = g_\zeta \lambda_\zeta$. Furthermore, there exists a partition Ω_j, $j = 1,2,\ldots$ of Ω into sets on which g_ζ is bounded in X'.

From the Banach-Mackey theorem, the set B is strongly bounded in X, so $\langle f_n, g_\zeta \rangle$ converges uniformly on each Ω_j, $j = 1,2,\ldots$ to $\langle f, g_\zeta \rangle$. Moreover,

$$|f_j m - f_k m|_U (A) \le \sup\{p_B (f_j - f_k (\omega)) : \omega \in A, n = 1,2,\ldots\} |m|_{B,U} (A).$$

By the sequential completeness of Z, there exists a vector measure $r : \mathfrak{S} \to Z$ such that $r(S) = \lim_{j\to\infty} f_j m(S)$ uniformly for all $S \in \mathfrak{S} \cap A$. . In particular, $|\langle r, \zeta \rangle - \langle f, g_\zeta \rangle \lambda_\zeta|(\Omega_j) = 0$ for each $j = 1,2,\ldots$, so $\lambda_\zeta (|\langle f, g_\zeta \rangle|) = |\langle r, \zeta \rangle|(\Omega) < \infty$, and $\langle r, \zeta \rangle = \langle f, m_\zeta \rangle$. The function f is therefore m-integrable and the required inequality holds.

If $E \in \mathfrak{S}$, and D is a disked set in X such that $|m|_{D,\zeta} (E) < \infty$ for every $\zeta \in Z'$, then for each $\varepsilon > 0$ set

$$A_D (E, \varepsilon) = \{x \in X : |\langle n - xm, \zeta \rangle|(F) \le \varepsilon |m|_{D,\zeta} (F), F \in \mathfrak{S} \cap E, \zeta \in Z'\}.$$

DEFINITION 3.2. Let $D \subseteq X$ be a disked set. A set $E \in \mathfrak{S}^+$ is said to be *D-semilocalized in a set $K \subseteq X$ with respect to* (n,m), if $|m|_{D,\zeta} (E) < \infty$ for all $\zeta \in Z'$, and for all $\varepsilon > 0$, there exists $F \in \mathfrak{S}^+ \cap E$ with $A_D (F, \varepsilon) \cap K \ne \varnothing$.

A set $E \in \mathfrak{S}^+$ is said to be *D-localized in a set $K \subseteq X$ with respect to* (n,m) if each $F \in \mathfrak{S}^+ \cap E$ is D-semilocalized in K with respect to (n,m).

The set Ω is said to be $\mathfrak{B}$*-compactly localized in X with respect to* (n,m) if for each $E \in \mathfrak{S}^+$, there exist a disked set $B \in \mathfrak{B}$, and sets $F \in \mathfrak{S}^+ \cap E$, $K \subseteq X$ such that K is X_B-compact, and F is B-localized in K. with respect to (n,m).

In what follows, the vector measures n and m will be fixed, so the phrase "with respect to (n,m)" will be omitted.

The next lemma follows from an exhaustion argument (Lemma 1.1 [6]).

LEMMA 3.3. *If there exists a measure* $\mu:\mathfrak{S} \to [0,1]$ *such that* $m \ll \mu$, and Ω *is* $\mathfrak{B}$*-compactly localized in* X, *then there exists pairwise disjoint sets* $\Omega_i \in \mathfrak{S}$, $i = 1,2,\ldots$ *such that* $\Omega = \cup\Omega_i$, *and for some disked set* $B_i \in \mathfrak{B}$, *there exists an* X_{B_i}*-compact set* K_i, *such that* Ω_i *is* B_i*-localized in* K_i, $i = 1,2,\ldots$

LEMMA 3.4. *Suppose that* $\Gamma \subseteq \Omega$ *is* B*-localized in the* X_B*-compact set* K, $B \in \mathfrak{B}$. *If* $\mu:\mathfrak{S} \to [0,1]$ *is a measure such that* $n \ll \mu$ *and* $m \ll \mu$ *on* $\Gamma\cap\mathfrak{S}$, *then there exists a sequence* f_K, $k = 1,2,\ldots$ *of* X_B*-valued* $\mathfrak{S}$*-simple functions, such that for every* $\varepsilon > 0$, *there is a set* $A \in \Gamma\cap\mathfrak{S}$ *with* $\mu(A^c) \leq \varepsilon$, *and* f_K, $k= 1,2,\ldots$ *converges uniformly in* X_B *on the set* A.

Furthermore, $\lim_{K\to\infty} |\langle(n- f_K m),\zeta\rangle|(A) = 0$ *for all* $\zeta \in Z'$..

Proof. By induction, as in [6] Lemma 2.6, for each $j = 1,2,\ldots$ there exists a finite partition π_j of Γ such that for every $W \in \pi_j$, there exists an X_B-compact set K_W^j of diameter less than $1/2^j$, and a countable partition $\pi_j'(W)$ of W with $A_B(V,1/2^j) \cap K_W^j \neq \emptyset$, for all $V \in \pi_j'(W)$. Moreover, π_{j+1} is finer than π_j, $j = 1,2,\ldots$, and whenever $V \in \pi_{j+1}$, $U \in \pi_j$, $V \subseteq U$, then $K_V^{j+1} \subseteq K_U^j$. Let P_j be the cardinality of the set π_j. .

For every $j = 1,2,\ldots$ and $W \in \pi_j$, there exists a finite collection $\pi_j''(W) \subseteq \pi_j'(W)$ such that $\mu(W \setminus \cup_{V\in\pi_j''(W)} V) \leq 1/(P_j 2^j)$. For each $V \in \pi_j''(W)$, $W \in \pi_j$, choose $x_{V,W,j} \in A_B(V,1/2^j) \cap K_W^j$ and then define the X-valued $\mathfrak{S}$-simple function f_j by $f_j = \Sigma_{W\in\pi_j} \Sigma_{V\in\pi_j''(W)} x_{V,W,j}\chi_V$.

Let $\mathcal{Y}_j = \cup_{W \in \pi_j} \cup_{V \in \pi_j''(W)}$ for each $j = 1,2,\dots$. Then $\mu(\Gamma \setminus \mathcal{Y}_j) \le 1/2^j$ for all $j = 1,2,\dots$ and $\mu(\Gamma \setminus \cup_{k>0} \cap_{j>k} \mathcal{Y}_j) = 0$.

If $\omega \in \cap_{j>K} \mathcal{Y}_j$ and $h \ge i > K$, then $f_h(\omega), f_i(\omega)$ are both in the same X_B -compact set of diameter less than $1/2^i$, so f_i, $i = 1,2,\dots$ converges uniformly in the space X_B, on each set $\cap_{j>K} \mathcal{Y}_j$, $k = 1,2,\dots$. Given $\varepsilon > 0$, choose K so large that $\mu(\Gamma \setminus \cap_{j>K} \mathcal{Y}_j) \le \varepsilon$ and set $A = \cap_{j>K} \mathcal{Y}_j$. It remains to show that for each $\zeta \in Z'$, $\lim_{K \to \infty} |\langle (n - f_K m), \zeta \rangle|(A) = 0$.

Let f be the limit of f_h, $h = 1,2,\dots$ on A and set $S = \sup\{p_B(f(\omega)) : \omega \in A\}$. Let $\zeta \in Z'$. Now $p_B(f_h(\omega) - f_i(\omega)) \le 1/2^i$ whenever $\omega \in A$ and $h \ge i > K$, so $|\langle f_h m, \zeta \rangle|(A) \le |m|_{B,\zeta}(A)(S + 1/2^h)$. Therefore, for each $j \ge K$,

$$|\langle (n - f_j m), \zeta \rangle|(A) \le |n, \zeta\rangle|(A \setminus \mathcal{Y}_j) + |\langle f_j m, \zeta \rangle|(A \setminus \mathcal{Y}_j) + |\langle n - f_j m, \zeta \rangle|(\mathcal{Y}_j)$$

$$\le |\langle n, \zeta \rangle|(A \setminus \mathcal{Y}_j) + (S + 1)|m|_{B,\zeta}(A \setminus \mathcal{Y}_j) + \Sigma_{W \in \pi_j} \Sigma_{V \in \pi_j''(W)} 2^{-j} |m|_{B,\zeta}(V \cap A)$$

$$\le |\langle n, \zeta \rangle|(A \setminus \mathcal{Y}_j) + (S + 1)|m|_{B,\zeta}(A \setminus \mathcal{Y}_j) + 2^{-j} |m|_{B,\zeta}(A).$$

Since $n \ll \mu$ and $m \ll \mu$, the right-hand side goes to zero as $j \to \infty$.

LEMMA 3.5. *Suppose that $\Gamma \in \mathfrak{S}^+ \cap \Omega$ and $f : \Gamma \to X$ is an m-integrable function such that $n = fm$ on $\mathfrak{S} \cap \Gamma$.*

If for some closed disked set $B \in \mathfrak{B}$ there exist X_B-valued $\mathfrak{S}$-simple functions f_j, $j = 1,2,\dots$ on Γ such that $f_j \to f$ uniformly in X_B, and $|m|_{B,\zeta}(\Gamma) < \infty$ for all $\zeta \in Z'$, then Γ is B-localized in some X_B-compact set K.

Proof. Let K be the closure of $\bigcup_{j\geq 1} f_j(\Gamma)$ in X_B. Then K is X_B-compact. Suppose that $f_j = \sum_{A\in\pi_j} c_{A,j}\chi_A$, for finite partitions π_j of Γ by sets in $\mathfrak{S}^+$.

Let $\mathcal{Y} \in \mathfrak{S}^+\cap\Gamma$ and $\varepsilon > 0$. Choose $j = 1,2,\dots$ such that $p_B(f(\omega) - f_j(\omega)) \leq \varepsilon$ for all $\omega \in \Gamma$. Then for all $\zeta \in Z'$, $E \in \mathfrak{S}\cap\mathcal{Y}$,

$$|\langle n - f_j m, \zeta\rangle|(E) \leq \varepsilon |m|_{B,\zeta}(E).$$

There exists a set $A \in \pi_j$ such that $E = A\cap\mathcal{Y} \in \mathfrak{S}^+$. Since f_j is equal to $c_{A,j}$ on E, for all $\zeta \in Z'$, $F \in E\cap\mathfrak{S}$

$$|\langle n - c_{A,j} m, \zeta\rangle|(F) \leq \varepsilon |m|_{B,\zeta}(F).$$

Because $c_{A,j} \in K$, $A_B(E,\varepsilon)\cap K \neq \varnothing$. Every set $\mathcal{Y} \in \Gamma\cap\mathfrak{S}^+$ is therefore B-semilocalized in K, so Γ is B-localized in K.

THEOREM 3.6. *Let $(\Omega,\mathfrak{S})$ be a measurable space and $(x,y) \to xy$ a separately continuous bilinear form from the locally convex spaces X, $\mathcal{Y}$ into the locally convex space Z. The space X is endowed with a convex bornology $\mathfrak{B}$.*

Let $m : \mathfrak{S} \to \mathcal{Y}$, $n : \mathfrak{S} \to Z$ be vector measures such that $n \ll m$. Suppose that there exists a measure $\mu: \mathfrak{S} \to [0,1]$ such that $m \ll \mu$, and for each $\zeta \in Z'$ the vector measure $m_\zeta : \mathfrak{S} \to X'_\sigma$ has σ-finite $\mathfrak{M}\mathfrak{S}\mathfrak{C}$-variation. Furthermore, suppose that for each $E \in \mathfrak{S}^+$ there exists a disked set $B \in \mathfrak{B}$, and a set $F \in \mathfrak{S}^+\cap E$ such that $m|_{B,\zeta}(F) < \infty$ for all $\zeta \in Z'$.

Then there exists a $\mathfrak{B}$-strongly measurable function $f : \Omega \to X$ such that $n = fm$, if and only if Ω is $\mathfrak{B}$-compactly localized in X.

Proof. Suppose first that Ω is $\mathfrak{B}$-compactly localized in X. By Lemma 3.3, there exist pairwise disjoint sets $\Omega_j \in \mathfrak{S}^+$, $j = 1,2,\dots$ such that $\Omega = \bigcup_{j\geq 1} \Omega_j$

and for each $j = 1,2,\ldots$, there exists a closed disked set $B_j \in \mathfrak{B}$ and an X_{B_j}-compact set K_j, such that Ω_j is B_j-localized in K_j.

According to Lemma 3.4, for each $j = 1,2,\ldots$, there exists X_{B_j}-valued $\mathfrak{S}$-simple functions s_i, $i = 1,2,\ldots$ converging uniformly in X_{B_j} on sets of arbitrarily large μ-measure. Let $f_j : \Omega_j \to X$ be the limit on a set of full μ-measure in Ω_j, and zero off this set. By taking the weak-completion of Z if necessary, Lemma 3.1 shows that f_j is m-integrable on sets of arbitrarily large μ-measure in Ω_j, and its indefinite integral is equal to n on these sets. Consequently, f_j is m-integrable on Ω_j, and $f_j m = n$ on $\mathfrak{S} \cap \Omega_j$, $j = 1,2,\ldots$.

On setting $f = \Sigma_{j \geq 1} f_j \chi_{\Omega j}$, it follows immediately from the definition of integrability that f is m-integrable and $n = fm$.

Suppose now that $f : \Omega \to X$ is a $\mathfrak{B}$-strongly measurable function such that $n = fm$. Then there exist pairwise disjoint sets $\Omega_j \in \mathfrak{S}^+$, $j = 1,2,\ldots$ such that $\Omega = \bigcup_{j \geq 1} \Omega_j$, and for each $j = 1,2,\ldots$, there exists a closed disked set $B_j \in \mathfrak{B}$ such that $f_j = f\,|\Omega_j$ is strongly measurable in X_{B_j}. The sets B_j may also be chosen such that $|m|_{B_j,\zeta}(\Omega_j) < \infty$ for all $\zeta \in Z'$.

By the Banach space version of Egorov's theorem, there exist X_{B_j}-valued $\mathfrak{S}$-simple functions s_i, $i = 1,2,\ldots$ converging to f_j in X_{B_j}, uniformly on sets of arbitrarily large μ-measure in Ω_j.

It follows from Lemma 3.5 that Ω is $\mathfrak{B}$-compactly localized in X.

Remark. If the elements of the family $\mathfrak{B}$ of subsets of X are not bounded in X, then under slightly different conditions, a function $f : \Omega \to X$ such that $n = fm$ can still be constructed ; in Definition 3.2, instead of the condition that a set is "$\mathfrak{B}$-compactly localized in X", it is sufficient to assume that the set K is compact in X, and continuously included in the normed space X_B, instead of requiring that it is X_B-compact. For example, K may be compact, disked and metrizable in X.

It is not hard to show that the function f so constructed has good measurability properties; for example, it is Borel measurable for the original topology of X, and $\mu \circ f^{-1}$ is a Radon measure on X. Any two such densities must agree μ-a.e. [4] Corollary 4.

However, the necessary and sufficient conditions of Theorem 3.6 are more natural as stated for convex bornologies. The following definition is applied in Section 4. A disked subset D of a vector space is said to be *injective* if $x = 0$ whenever $p_D(x) = 0$.

DEFINITION 3.7. The set Ω is said to be *compactly localized in X with respect to (n,m)* if for each $E \in \mathfrak{S}^+$, there exist an injective disked set $B \subseteq X$, and sets $F \in \mathfrak{S}^+ \cap E$, $K \subseteq B$ such that K is compact in X and continuously included in the normed space X_B, and F is B-localized in K. with respect to (n,m).

4. APPLICATIONS

The conditions of Theorem 3.6 are of sufficient generality to treat the existence of operator-valued functionals associated with Markov processes. Such functionals arise in the solution of partial differential equations via the non-commutative Feynman-Kac formula. To illustrate how the result may be applied, the presentation of J.W. Hagood [3] is followed.

Let $(\Omega, \mathfrak{S}, (P^x)_{x \in \Sigma}; (X_t)_{t>0})$ be a right continuous Markov process with locally compact, second countable state space Σ. Put $\mathfrak{E} = \mathfrak{B}(\Sigma)$, the Borel σ-algebra of Σ, and for each $t > 0$, denote the σ-algebra generated by X_s, $0 \le s \le t$ by $\mathfrak{S}_t$. Let μ be a fixed excessive measure with respect to this process; that is, there exists $C > 0$ such that $\int_\Sigma P^x(X_t \in B)\, d\mu(x) \le C\,\mu(B)$ for every $B \in \mathfrak{E}$ and $t > 0$.

Let F be a Banach space and $1 \le p \le \infty$. The space $L^p(\Sigma,\mu;F)$ of (equivalence classes of) functions strongly p-integrable with respect to μ is denoted by E. If $f \in E$, then for μ-almost all $x \in \Sigma$, the function $f \circ X_t : \Omega \to F$ is strongly p-integrable with respect to P^x for every $t > 0$. Furthermore, if $P^\mu : \mathfrak{S} \to [0,\infty[$ is the measure defined by $P^\mu(S) = \int_\Sigma P^x(S)d\mu(x)$, $S \in \mathfrak{S}$, then $f \circ X_t$ is strongly p-integrable in F for every $t > 0$.

The operator valued measure $m_t : \mathfrak{S}_t \to \mathfrak{L}(E,F)$ is defined by

$$m_t(S)f = \int_\Omega f \circ X_t(\omega)\, dP^\mu(\omega) \quad , \quad f \in E\, , \ S \in \mathfrak{S}_t\, ,$$

for every $t > 0$. The space $\mathfrak{L}(E,F)$ of bounded linear operators of E into F is endowed with the strong operator topology.

The bilinear form $\mathfrak{L}(F)\times\mathfrak{L}(E,F) \to \mathfrak{L}(E,F)$ is taken to be the composition of operators: $(A,B) \to AB$, $A \in \mathfrak{L}(F)$, $B \in \mathfrak{L}(E,F)$. The product is clearly separately continuous.

The continuous dual $\mathfrak{L}(E,F)'$ of $\mathfrak{L}(E,F)$ may be identified with the tensor product $E \otimes F'$ of E with F' by the action

$(A, f \otimes \xi) \to \langle Af, \xi\rangle$, $A \in \mathfrak{L}(E,F)$, $f \in E$, $\xi \in F'$ [7] p139.

First, we will see that for each $t > 0$, the vector measure m_t is of the type considered in Section 2; namely, for each $\zeta \in \mathfrak{L}(E,F)'$, the vector measure $[m_t]_\zeta : \mathfrak{S}_t \to \mathfrak{L}(F)'$ has σ-finite $\mathfrak{M}\mathfrak{C}\mathfrak{C}$-variation in $\mathfrak{L}(F)'$. It is sufficient to consider linear functionals of the form $\zeta = f \otimes \xi$, $f \in E$, $\xi \in F'$, in which case $\langle A, [m_t]_\zeta(S)\rangle = \langle A\, m_t(S)f, \xi\rangle$, $A \in \mathfrak{L}(F)$ and $[m_t]_\zeta(S) = \dot{m}_t(S)f)\otimes\xi$ for all $S \in \mathfrak{S}_t$. .

The vector measure $m_t f : \mathfrak{S}_t \to F$ has a density $f \circ X_t : \Omega \to F$ with respect to the measure $P^\mu : \mathfrak{S}_t \to [0,\infty[$, so from [4] Corollary 4, $m_t f$ has σ-finite compact variation in F. Since a set of the form $K \otimes \xi$ for K compact in F , $\xi \in F'$ is

$\sigma(\mathfrak{L}(F)',\mathfrak{L}(F))$-compact in $\mathfrak{L}(F)'$, it follows that $[m_t]_\zeta$ has σ-finite weakly compact variation in $\mathfrak{L}(F)'$.

THEOREM 4.1. *Let* $t > 0$ *and suppose that* $n_t : \mathfrak{S}_t \to \mathfrak{L}(E, F)$ *is a vector measure . If* Ω *is compactly localized in* $\mathfrak{L}(F)$ *with respect to* (n_t, m_t), *then there exists* P^μ *-regular function* $M_t : \Omega \to \mathfrak{L}(F)$ *such that* $n_t = M_t m_t$.

Suppose that F *is separable. If there exists an* m_t*-integrable function* $M_t : \Omega \to \mathfrak{L}(F)$ *such that* $n_t = M_t m_t$, *then for each* $f \in E$, *the set* Ω *is compactly localized in* $\mathfrak{L}(F)$ *with respect to* $(n_t f, m_t f)$.

Proof. As indicated in the remark after Theorem 3.6, the function M_t can be constructed as before when Ω is compactly localized in $\mathfrak{L}(F)$ with respect to (n_t, m_t).

For the converse, note that $\mathfrak{L}(F)$ is a Suslin space whenever F is separable [8] p67. Let $M_t : \Omega \to \mathfrak{L}(F)$ be an m_t- integrable function such that $n_t = M_t m_t$. Since M_t is scalarly measurable with respect to μ, it follows from the properties of Suslin spaces [8] p67 that $P^\mu \circ M_t^{-1}$ is a Radon measure on $\mathfrak{L}(F)$.

For every $\varepsilon > 0$, there exists a set $\Gamma \in \mathfrak{S}_t \cap \Omega$ such that $P^\mu(\Gamma^c) < \varepsilon/2$ and $K = M_t(\Gamma)$ is relatively compact in $\mathfrak{L}(F)$, and $C = f \circ X_t(\Gamma)$ is relatively compact in F .

Let $x_k \in F$, $k = 1,2,\ldots$ be vectors whose linear span is dense in F , and set $a_k = \sup\{\sup_{x \in C} \|Ax\| + \|Ax_k\| : A \in K\}$, $k = 1,2,\ldots$,

$$D = \{A \in \mathfrak{L}(F) : \Sigma_{k \geq 1} (\sup_{x \in C} \|Ax\| + \|Ax_k\|)/(a_k 2^k) \leq 1\} .$$

Then D is a closed, injective, disked subset of $\mathfrak{L}(F)$. Because the topology of compact convergence coincides with the strong operator topology on equicontinuous subsets of $\mathfrak{L}(F)$, it follows that $K \subseteq D$ is continuously included in $\mathfrak{L}(F)_D$.

For each $\xi \in F'$, $|m_t f|_{D,\xi}(\Gamma) < \infty$. Furthermore, by the vector valued version of Egorov's theorem, there exists a set $\Lambda \subseteq \Gamma$ and K-valued $\mathfrak{S}_t$-simple functions S_j, $j = 1,2,\ldots$ converging uniformly in $\mathfrak{L}(F)_D$ to M_t on Λ, such that $P^{\mu}(\Lambda^c) < \varepsilon$. The conclusion now follows as in the proof of Lemma 3.5.

Remark. If the conditions of Theorem 3.6 applied, then the operator valued function M_t would be strongly measurable in the uniform operator topology - a property which is too strong to be practical. The function M_t is a candidate for a *multiplicative operator functional* [3] associated with the given Markov process.

An example of Edgar [2] p672 may be adapted to show that the separability of the Banach space F cannot be omitted in general.

The question was raised in [5], as to when there exists a conditional expectation of a scalar function with respect to a vector valued measure m and a sub σ-algebra $\mathfrak{S}$. The technique developed in [5] answered that question in the case that the restriction of m to the σ-algebra $\mathfrak{S}$ has σ-finite $\mathfrak{M}\mathfrak{C}\mathfrak{C}$-variation.

An application of Theorem 3.6 gives an alternative condition which does not require that the vector measure m admits a density. The proof follows that of [6] Theorem 3.2. In the present setting, the variation of m is replaced with the variation of $\langle m, \xi\rangle$ for the appropriate continuous linear functional ξ.

THEOREM 4.2. *Let X be a locally convex space, and $(\Omega,\mathfrak{S})$ a measurable space. Suppose that $m:\mathfrak{S} \to X$, $n:\mathfrak{S} \to X$ are vector measures such that $n \ll m$, and for some scalar measure $\mu:\mathfrak{S} \to [0,1]$, $m \ll \mu$.*

Then there exists a function $f:\Omega \to \mathbb{R}$ such that $n = fm$ if and only if for all $E \in \mathfrak{S}^+$, $\varepsilon > 0$, there exists $F \in \mathfrak{S}^+ \cap E$ such that $A_{[-1,1]}(F,\varepsilon) \neq \varnothing$.

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