CENTRALLY TRIVIAL AUTOMORPHISMS OF C^*-ALGEBRAS

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We continue our study of central sequences in, and automorphisms of separable C^*-algebras begun in [7]. We would like to attempt, as far as possible, to follow the plan of attack developed by A. Connes in [2,3] for von Neumann algebras. Unfortunately, very few of the general ideas survive in the C^*-algebra setting. The main reason for this is the overabundance of nontrivial central sequences in a general separable C^*-algebra. Despite this, we are able to sufficiently analyze some large classes of C^*-algebras so that those automorphisms which behave trivially on central sequences can be computed. Complete proofs and more detailed examples will appear elsewhere.

§ 1. Preliminaries

Let A denote a separable unital C^*-algebra over the complex numbers. Let $\mathrm{Aut}A$ denote the group of all *-automorphisms of A, and $\mathrm{Inn}A$ the normal subgroup of all inner automorphisms. Let $\varepsilon : \mathrm{Aut}A \to \mathrm{Aut}A/\mathrm{Inn}A = \mathrm{Out}A$ be the quotient.

A central sequence in A is a bounded sequence $\{x_n\}$ of elements of A with the property that $[x_n,a] = x_na - ax_n \to 0$ in norm for each $a \in A$. A uniformly central sequence is a bounded sequence $\{x_n\}$ for which the operators (on A) $\mathrm{ad}x_n(\cdot) = [x_n,\cdot]$ converge to 0 in norm. A central sequence $\{x_n\}$ is called hypercentral if $\|[x_n,y_n]\| \to 0$ for every central seqeunce $\{y_n\}$ of A. A central sequence $\{x_n\}$ is called trivial if there is a sequence $\{\lambda_n\}$ of central elements in A so that $\|x_n-\lambda_n\| \to 0$. It is evident that any trivial sequence is uniformly central and any uniformly central sequence is hypercentral. Two central sequences are called equivalent if their difference converges to zero in norm.

If $\alpha \in \text{Aut } A$, then we say that α is centrally trivial if $\|\alpha(x_n)-x_n\| \to 0$ for every central sequence $\{x_n\}$ of A. We denote the normal subgroup of centrally trivial automorphism of A by $\text{Ct } A$ and note that $\text{Inn } A \subseteq \text{Ct } A$. We let $\text{Inn } A^-$ denote the closure of $\text{Inn } A$ in the topology of pointwise norm convergence. Following A. Connes we define $\chi(A) = \varepsilon(\text{Ct } A \cap \text{Inn } A^-)$. A straightforward adaptation of an argument of A. Connes gives us the following.

1.1 Proposition: If A is a separable unital C*-algebra then $\chi(A)$ is abelian. In fact, $\varepsilon(\text{Ct } A)$ commutes with $\varepsilon(\text{Inn } A^-)$.

Now, if we let $\text{Inn } A^{-\|\cdot\|}$ denote the uniform norm closure of Inn A in Aut A then a simple $\frac{\varepsilon}{2}$ - argument shows that $\text{Inn } A^{-\|\cdot\|} \subseteq \text{Ct } A$. Since $\text{Inn } A^{-\|\cdot\|} \subseteq \text{Inn } A^-$ we have $\varepsilon(\text{Inn } A^{-\|\cdot\|}) \subseteq \chi(A)$. We have proved:

1.2 Corollary: $\varepsilon(\text{Inn } A^{-\|\cdot\|})$ is abelian!

Now, by an argument of C. Akemann and G. Pedersen [1], given any central element $z \in A^{**}$ we can find a central sequence $\{x_n\}$ in A so that $x_n \to z$ strongly. We easily conclude that if $\alpha \in \text{Ct } A$ then $\alpha^{**} \in \text{Aut } A^{**}$ fixes the centre of A^{**}. We have proved:

1.3 Proposition: Under the natural embedding $\text{Aut } A \to \text{Aut } A^{**}$, Ct A gets sent into $\text{Aut}_Z A^{**}$, the subgroup of centre-fixing automorphisms.

1.4 Corollary: If A is type I, $\text{Ct } A \subseteq \pi(A)$ the subgroup of π-inner automorphisms.

1.5 <u>Corollary</u>: If A is simple, Ct A = Inn A .

Since α^{**} fixes the center of A^{**} we see that α acts trivially on $\hat{A}$, the space of irreducible representations of A . Combining this with a result of G.A. Elliott's [4] gives us 1.4: combining it with A. Kishimoto's result [6] gives us 1.5. It is easy to produce type I examples, A , for which Ct A $\subset \pi(A)$. We have the following conjecture:

1.6 <u>Conjecture</u>: Ct A $\subseteq \pi(A)$ for all separable, unital C*-algebras, A .

§2. <u>Primitive C*-algebras</u>

2.1 <u>Conjecture</u>: If A is a primitive, unital, separable C*-algebra, then Ct A = Inn A .

2.2 <u>Theorem</u>: If A is a primitive, unital, separable A.F.-algebra, then Ct A = Inn A .

The idea here is to take an $\alpha \in$ Ct A and show that given $\epsilon > 0$ there is a finite-dimensional subalgebra $B \subseteq A$ so that $||\alpha\text{-id}|| \leq \epsilon$ on the relative commutant $B^C = B' \cap A$. By representing A irreducibly on H , B^C is strongly dense in B' and so $||\alpha\text{-id}|| \leq \epsilon$ on B' : α is spatial on H by 1.3 and so α makes sense on B' . If $\alpha = \mathrm{Ad}v$ then carefully projecting v into B yields an $x \in B$ with $||v-x|| \leq \epsilon$. Thus, $v \in A$.

2.3 Theorem . If A is an extension of a separable unital C*- algebra by the compact operators, then Ct A = Inn A .

Again we represent A irreducibly on H and use 1.3 to obtain a unitary v with $\alpha = \mathrm{Adv}$. If $v \notin A$ then by Voiculescu's double commutant theorem, $\pi(v) \notin \pi(A)^{cc}$ where $\pi : B(H) \to Q(H)$ is the Calkin map and $(\ )^c$ denotes the commutant in Q(H), the Calkin algebra. Then using two carefully chosen quasicentral approximate identities in $\mathcal{K}(H)$ to successively modify v , we obtain a central sequence $\{x_n\} \subseteq \mathcal{K}(H) \subseteq A$ so that $\|\alpha(x_n)-x_n\|$ is bounded away from 0 , a contradiction.

2.4 Remark: Thus, if our primitive (separable, unital) C*-algebra A is either simple, A.F., or an extension by $\mathcal{K}(H)$ we have CtA = Inn A . This is the evidence we offer for conjecture 2.1.

§3. Hypercentral sequences

Here we discuss the existence problem for nontrivial hypercentral sequences; i.e., those central sequences which commute asymptotically with all other central sequences. First we prove by a fairly direct but tricky argument the following theorem.

3.1 Theorem: If A is a primitive (separable, unital) AF algebra then every hypercentral sequence is trivial.

If A is, in fact, a UHF algebra then this conclusion is easy to prove.

Now, a straightforward argument using partitions of unity establishes the following.

3.2 Proposition: If A is a separable, unital C*-algebra with no nontrivial hypercentral sequences and X is a compact separable space, then $C(X) \otimes A$ has no nontrivial hypercentral sequences either.

The next lemma is a simple adaptation to the C*-algebra setting of a result of A. Connes.

3.3 Lemma. If A is a separable, unital C*-algebra in which all central sequences are hypercentral, then $\mathrm{Inn} A^{-} \subseteq \mathrm{Ct}\, A$ so that $\varepsilon(\mathrm{Inn}\, A^{-})$ is abelian (= $\chi(A)$, in fact).

We then deduce:

3.4 Theorem: Let A be a separable unital C*-algebra which has an infinite-dimensional primitive quotient which is either simple, A.F. or contains the compact operators. Then not all central sequences of A are hypercentral.

Since central sequences can always be lifted from a quotient by [1] we can assume A is primitive, infinite-dimensional and either simple, A.F., or an extension by the compact operators. Thus, if all central sequences were hypercentral we would have $\mathrm{Inn}\, A^{-} \subseteq \mathrm{Ct}\, A = \mathrm{Inn}\, A$. However, this is not possible by [7].

3.5 Example: A separable, unital C*-algebra A, for which all central sequences are hypercentral but not necessarily uniformly central and hence not trivial. Define: $A = \{f:[0,1] \to M_2(C) \mid f$ is continuous and $f(1)$ is diagonal$\}$. It is easy and instructive to verify that A has the desired properties.

3.6 Example: A separable, unital A.F. algebra, A, such that

(1) $Z(A) = C$

(2) A has non-hypercentral central Sequences and also nontrivial hypercentral sequences.

(3) $\mathrm{Inn}\, A \subset \mathrm{Inn}\, A^{-\|\cdot\|} = \mathrm{Ct}\, A \subset \mathrm{Inn}\, A^{-}$

(4) $$\chi(A) \cong \frac{\{(\lambda_n) \in \prod_1^\infty S^1 \mid \lim \lambda_n \lambda_{n+1} = 1\}}{\{(\lambda_n) \mid \lim \lambda_n \text{ exists in } S^1\}}$$

This example is a modification of example 6.3 of [5]. Although the construction and proofs are not terribly difficult they are rather lengthy and so we omit them here.

3.7 Theorem: Let X be a separable compact space and let A be a separable unital C^*-algebra such that

1. $Z(A) = \mathbb{C}$
2. $\mathrm{Ct}\, A = \mathrm{Inn}\, A$
3. Every hypercentral sequence in A is uniformly central.

If $B = C(X) \otimes A$ then we have an exact sequence

$0 \to \mathrm{Inn}\, B \to \mathrm{Ct}\, B \xrightarrow{\eta} H^2(X, \mathbb{Z})$. In fact, $\mathrm{Ct}\, B = \{\alpha | x \mapsto \alpha_x : X \to \mathrm{Inn}\, A$ is uniformly continuous$\}$.

To see this last statement (which is the crux of the matter) let $\alpha \in \mathrm{Ct}\, B$. Then α preserves the centre of B and so gives rise to a map $x \mapsto \alpha_x : X \to \mathrm{Aut}\, A$ continuous in the point-norm topology. Since central sequences lift from A to B we see each $\alpha_x \in \mathrm{Ct}\, A = \mathrm{Inn}\, A$. Now, if $x \mapsto \alpha_x$ is not uniformly continuous one can (without loss of generality) obtain $\alpha_{x_n} \to \mathrm{id}$ pointwise but not uniformly. Since $\alpha_{x_n} = \mathrm{Ad}u_n$, $\{u_n\}$ is a central, but not uniformly central, sequence and therefore not hypercentral. We thus choose $\{a_n\}$ central in A so that $\|[u_n,a_n]\| \not\to 0$. But, then $\{1 \otimes a_n\}$ is a central sequence in B and $\|\alpha(1 \otimes a_n) - 1 \otimes a_n\| \not\to 0$, a contradiction.

We observe that by 2.2 and 3.1, every primitive (separable, unital) A.F. algebra satisfies the hypotheses of theorem 3.7.

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