AXISYMMETRIC ASYMPTOTICALLY FLAT RADIATIVE SPACE-TIMES WITH ANOTHER SYMMETRY: THE GENERAL DEFINITION AND COMMENTS

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The title of my talk at the conference referred directly to the *boost-rotation symmetry*. The title above is, in fact, a "synonym":

THEOREM *Suppose that an axially symmetric vacuum space-time is asymptotically flat in the sense that it admits local smooth null infinity, i.e., suppose that the Bondi coordinates can be introduced and that the metric is asymptotically of standard Bondi's form [1] for $\phi \in [0, 2\pi)$ and some open interval θ. Suppose that this space-time admits an additional Killing vector which forms a 2-dimensional Lie algebra with the axial Killing vector. Assume that this additional symmetry allows gravitational radiation (i.e. Bondi's news function is non-vanishing). Then the additional symmetry has to be the boost symmetry and the additional Killing vector is the boost Killing vector.*

Roughly speaking, in axially symmetric asymptotically flat space-times, the only second allowable symmetry that does not exclude radiation is the boost symmetry. In the proof of the theorem (which we gave with B.G. Schmidt in [2] but which has a longer history — see references in [2]), it was assumed that the rotational Killing vector is hypersurface orthogonal. Recently, together with R. Muschall [3], we generalized the theorem to the case of the rotational Killing field which need not be hypersurface orthogonal. Moreover, although no rigorous proof is available so far, it appears that all other radiative space-times with two symmetries are not asymptotically flat (see the discussion of the 2-dimensional group of null rotations and some further remarks in [2]). In this sense, the boost-rotation symmetric space-times would play a unique role among asymptotically flat-radiative space-times. An infinite number of various boost-rotation symmetric space-times can be constructed *explicitly*, which, very probably, will not be feasible in the (much more realistic) case of space-times with one symmetry only.

The boost-rotation symmetric solutions represent the fields of "uniformly accelerated sources" in general relativity. At present only explicit vacuum solutions describing accelerated singularities (of the Curzon-Chazy-Scott type, and of all the other Weyl types), or the solutions representing accelerated black holes (as, e.g., the C-metric) are known. One can gain a useful intuition in understanding these space-times by first looking at uniformly accelerated sources in special relativity.

INTERMEZZO It was in Prague in 1911–1912 when Einstein started a systematic quest for a new theory of gravity based on the equivalence principle. In the second paper on gravitation from 1912 [4], he first clearly formulates what he understands about the transformation from an inertial frame to an accelerated frame in a relativistic theory. One can easily find out that, in the first approximation (for small times), his formulas given in [4] follow from the exact transformation relations between an inertial frame and an uniformly accelerated frame (which we often call the "Rindler frame" today) in special relativity. The reference points of such a frame move along hyperbolas $z^2 - t^2 = \text{const.}$ (we assume the motion along the z-axis of an inertial frame with Lorentzian coordinates $\{t, x, y, z\}$). The world-lines $z^2 - t^2 = B = \text{const.}$, $\quad x, y = \text{const.}$, are the orbits of the boost Killing vector $\eta = z(\partial/\partial t) + t(\partial/\partial z)$ in Minkowski space. Imagine that some sources (say charged particles) are axially symmetric about $z = 0$ and move with a uniform acceleration along this axis. The fields produced by such sources will have the boost-rotation symmetry.

THE DEFINITION OF THE ASYMPTOTICALLY FLAT BOOST-ROTATION SYMMETRIC SPACE-TIMES

Although these space-times have a long history and their general properties were reviewed recently [5], until now they were not defined and treated geometrically from a unified point of view. Here we shall only sketch their definition and mention some new results on their global behaviour. We refer to our forthcoming paper with Bernd Schmidt [6] for a detailed systematic treatment.

Locally the space-times we wish to define are characterized by the existence of two non-null, hypersurface orthogonal Killing vectors. (A more general case of the axial Killing vector which is not hypersurface orthogonal has not yet been treated systematically.) In a small region in curved space-times it is not possible to distinguish between boost-rotation symmetry and cylindrical symmetry. If, however, we demand that space-times become asymptotically flat somewhere and that there the Killing vector fields behave as the boost and the rotation Killing fields, we get geometrically an essentially different picture from the cylindrical case.

As we indicated above, a good insight into how the boost-rotation symmetry operates in a curved space-time can be gained by considering first Minkowski space. Denote the norms squared of the rotation Killing vector $\xi = x(\partial/\partial y) - y(\partial/\partial x)$ and of the boost Killing vector η by $(-A)$ and B:

$$A = -\xi_\alpha \xi^\alpha = x^2 + y^2, \qquad B = \eta_\alpha \eta^\alpha = z^2 - t^2. \tag{1}$$

The 2-dimensional group orbits are spacelike if $B < 0$, null if $B = 0$ and timelike if $B > 0$. We call the two null hyperplanes $B = 0 (z = \pm t)$ "the roof". Points with $B < 0$ are "above the roof", points with $B > 0$ "below the roof". Points with $A = 0$ form the axis.

Since we are primarily interested in the radiative properties and the existence of (at least local) asymptotically smooth null infinity, we shall concentrate on the region above the roof where the boost Killing vector is spacelike and where — as can be easily seen — "almost all" null geodesics come. Introducing here the coordinates $\{b, \rho, \phi, \chi\}$, adapted to the boost-rotation symmetry, by relations

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = b\sinh\chi, \quad t = \pm b\cosh\chi, \tag{2}$$

$b \geq 0, \quad \chi \in \mathbf{R}$, and null coordinates u, v by

$$u = b - \rho, \quad v = b + \rho, \tag{3}$$

we find the metric form

$$ds^2 = du\,dv - \tfrac{1}{4}(v^2 - u^2)\left[\frac{v-u}{v+u}\,d\phi^2 + \frac{v+u}{v-u}\,d\chi^2\right]. \tag{4}$$

The axis is given by $v = u$, the roof by $v = -u$, the lines $u = u_0$, $\chi = \chi_0$, v changing, are null geodesics which go to $\mathcal{I}^+$ as $v \to +\infty$.

Before generalizing (4) to the case of a curved space-time we shall state the following (see, e.g., [6] for the proof):

PROPOSITION *The metric of a curved space-time admitting two spacelike, hypersurface orthogonal, commuting Killing vectors $\partial/\partial\xi$ and $\partial/\partial\eta$, and satisfying the vacuum Einstein equations can, in a region where $W_{,\alpha}W^{,\alpha}$ does not change the sign, W being the "volume element" of the group orbits, be transformed into one of the following forms:*

$$ds^2 = e^\Lambda dU\,dV - W\left[e^{-\Psi}d\xi^2 + e^\Psi d\eta^2\right], \tag{5}$$

where Λ, Ψ are functions of U, V, and $W > 0$ is such that a) $W = \frac{1}{2}(U+V)$ *if $W_{,\alpha}$ is timelike,* b) $W = \frac{1}{2}(U-V)$ *if $W_{,\alpha}$ is spacelike. The coordinates U and V, called the* canonical *coordinates, are determined uniquely up to translations. (In the cases with $W_{,\alpha}$ null or $W =$* const. *the space-time can be shown to have more symmetries.)*

The almost unique canonical coordinates can be used to distinguish between cylindrical and boost-rotation symmetry. The canonical coordinates are related to u and v in (4) as follows: $V = \frac{1}{2}v^2$ everywhere above the roof; $U = \frac{1}{2}u^2$ for $u \geq 0$, i.e., inside the null cone of the origin, and $U = -\frac{1}{2}u^2$ for $u < 0$, i.e., outside the null cone. Now we adopt the following *definition of the boost-rotation symmetric curved space-time.*

DEFINITION 1 A space-time admitting two space-like, hypersurface orthogonal Killing vectors is called "boost-rotation symmetric" if in canonical coordinates $\frac{1}{2}v^2$, $\frac{1}{2}u^2\big(\text{resp.} - \frac{1}{2}u^2 \quad \text{if} \quad u < 0\big)$ the metric has the form

$$ds^2 = e^{\lambda} du\, dv - \tfrac{1}{4}\left(v^2 - u^2\right)\left[\frac{v-u}{v+u}\, e^{-\mu} d\phi^2 + \frac{v+u}{v-u}\, e^{\mu} d\chi^2\right], \tag{6}$$

$\phi \in [0, 2\pi)$, $\chi \in \mathbf{R}$, the functions $\lambda(u,v)$, $\mu(u,v)$ are defined for $v \in (0 < v_0, +\infty)$, $u \in (u_0, u_1)$, $u < v$, $u \neq -v$, and

$$\lim_{\substack{v \to +\infty \\ u \text{ fixed}}} \lambda(u,v) = \lambda_0(u)\,, \qquad \lim_{\substack{v \to +\infty \\ u \text{ fixed}}} \mu(u,v) = \kappa = \text{const.} \tag{(7)}$$

Should we not just demand $\lambda \to 0$, $\mu \to 0$ at $v \to +\infty$ so that (6) goes manifestly over into flat metric (4)? No, since the field equations would then imply that the metric is flat everywhere. However, the weaker asymptotic conditions (7) can be shown to be compatible with the asymptotic flatness and, indeed, one can show rigorously that Definition 1 implies that a boost-rotation symmetric space-time admits a local $\mathcal{I}^+$. (See Ref. 6 for a proof and a discussion of the concept of local $\mathcal{I}$ and its relation to asymptotic flatness in the case of the boost-rotation symmetric space-times.)

Now it is easy to see that the Einstein vacuum equations for the metric (6) imply the ordinary wave equation for function μ in Minkowski space in coordinates $\{u, v, \phi, \chi\}$ and two first-order equations for λ. The linear equation for μ is the integrability condition for the two equations for λ. Once μ is given, λ can be determined by integration. By going back to the Minkowskian-type coordinates $\{t, x, y, z\}$ by the same relations (2), (3) as in the flat space-time, we find that the function $\mu(A, B)$ — where again $A = x^2 + y^2$, $B = z^2 - t^2$, and μ is denoted in the same manner as the original function $\mu(u, v)$ — satisfies the ordinary wave equation, $\Box\mu = 0$. If the function $\lambda =$

$\lambda(A, B)$, instead of $\lambda(u, v)$, is introduced analogously, we can transform metric (6) into the following form:

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\begin{aligned}
ds^2 = &- (x^2 + y^2)^{-1} \left[e^{\lambda}(x\,dx + y\,dy)^2 + e^{-\mu}(x\,dy - y\,dx)^2 \right] \\
&- (z^2 - t^2)^{-1} \left[e^{\lambda}(t\,dt - z\,dz)^2 - e^{\mu}(t\,dz - z\,dt)^2 \right].
\end{aligned} \tag{8}
$$

Of course, we have to bear in mind that the metric was obtained by the transformation of the original metric (6) by which we geometrically defined the boost-rotation symmetric space-times above the roof only, i.e., for $t^2 > z^2$. Naturally, we want to analyze the extension to all values of $\{t, x, y, z\}$. As it is clear from (8), the axis and the roof are critical in this respect.

Since, however, the function μ is fundamental, λ being, again, determined by integration, we have first to analyze the boost-rotation symmetric solutions of the wave equation in the flat space-time. The detailed discussion shows that (i) no asymptotically regular (in the sense of Penrose's treatment of null infinity — see, e.g., [7]) *vacuum* solution exists except for $\mu = 0$, (ii) solutions given in terms of retarded and advanced potentials generated by flat-space sources which occur along boost-symmetric world-lines with $0 \leq A < A_0$, $0 < B_0 \leq B \leq B_1 < \infty$, are analytic everywhere, except for A, B where sources occur; in particular, they are analytic on the roof and everywhere on the axis outside the sources. And they are asymptotically regular everywhere on $\mathcal{I}^{\pm}$, except for two points of $\mathcal{I}^{+}$ and two points of $\mathcal{I}^{-}$, where the world-lines of the sources "start" and "end"; these points, given by $t = \pm z$, $z \to \pm\infty$, are the fixed points of the boost-rotation symmetry. By using the field equations for λ it is then not difficult to prove that given μ, such a $\lambda(A, B)$ exists which satisfies vacuum field equations and is analytic in all the regions in which μ is analytic.

Now we can already adopt the following:

DEFINITION 2 The boost-rotation symmetric vacuum solutions have the metric (8) with $(t, x, y, z) \in \mathbf{R}^4$, $x^2 + y^2 > 0$, $z^2 - t^2 \neq 0$, and

$$\mu = \mu(A = x^2 + y^2, B = z^2 - t^2), \quad \lambda = \lambda(A = x^2 + y^2, B = z^2 - t^2).$$

The function μ is, up to an additive constant, an analytic, asymptotically regular solution of the flat-space wave equation, except for the regions where sources uniformly accelerated with respect to the Minkowskian background and defining μ occur. The function λ is, up to an additive constant, the solution of the field equations which determine λ in terms of quadratures; it is analytic in all regions where μ is analytic.

Notice that the boost-rotation symmetric vacuum solutions are defined "geometrically" since coordinates $\{t, x, y, z\}$ have an invariant geometrical meaning through their relation to the canonical coordinates introduced in Definition 1. Moreover, it can easily be seen that μ and λ, described in Definition 2, in the canonical coordinates satisfy the boundary conditions required in Definition 1.

Since both λ and μ are determined uniquely up to additive constants, all boost-rotation symmetric solutions decompose into two-parameter classes determined by $\mu + c_1$ and $\lambda + c_2$. A detailed analysis shows that by choosing the constants such that $\lambda(0,0) = \mu(0,0)$ we make the *roof* ($z^2 = t^2$) *regular*. Then we still have a one-parameter freedom in adding the same constant to both λ and μ. In this manner we influence the distribution of *nodal (conical) singularities* along the z-axis. Physically they are quite understandable: they cause the sources to move with an acceleration. There exist cases in which *no* nodal singularities occur. It can be shown that this requires an asymptotically regular μ (of course, except for the points with $z = \pm t$, $t \to \infty$) such that $\mu(0,0) = 0$. An infinite number of the different solutions with such a property is available in *explicit* forms [8]. A detailed analysis of null infinity (see [6]) shows that in these space-times all $\mathcal{I}^{\pm}$, $I^{\pm}$, and i^0 are smooth, except for four points at which the boost orbits "start" and "end". Therefore, we can choose arbitrarily *strong* boost-rotation symmetric data on a hyperboloidal hypersurface above the roof, which lead to a complete smooth null and timelike infinity in its future. With these specific space-times it is thus not necessary to assume weak-field data which are required in the deep work of Friedrich and others on the existence of general asymptotically flat radiative solutions. (I thank Piotr Chrusciel for this remark.)

The main drawback of the asymptotically flat boost-rotation symmetric space-times is the vanishing of the ADM mass. This result is physically understandable — it has its counterpart in the electromagnetic case in flat space-time (see [5] for details). One thus cannot learn much about the structure of i^0 in realistic general space-times with positive ADM mass. On the other hand, the radiative properties of the boost-rotation symmetric space-times [9] (such as the peeling-off properties, the news function, the radiation patterns, etc.) make them to be the only non-trivial explicit exact examples of the gravitational radiation theory known at present. They are also available for testing various approximation methods and complicated codes in numerical relativity. Inspired by the null cone version of numerical relativity, Bičák, Reilly and Winicour [10] recently gave the explicit boost-rotation symmetric "initial null cone solution", which solves the Bondi hypersurface and evolution equations. This solution is already being used to improve the accuracy of the numerical codes [11].

There exist "generalized" boost-rotation symmetric space-times which, though not asymptotically flat, are of a physical interest. No conical singularities or negative masses are necessary to cause accelerations. A charged black hole accelerated by an electric field which is "uniform" at infinity [12] is a good example of the exact generalized boost-rotation symmetric solution which does not contain singularities or negative masses. Although in such space-times non-stationary regions occur and accelerated objects radiate in external fields, the radiation is difficult to analyze because the space-times are not asymptotically flat. Only if the external field is weak, there exists a region in which the space-times are approximately flat, and here their radiative properties might be investigated. We refer to the review [5] for more details and references. However, no systematic treatment of these space-times has been undertaken so far.

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