DERIVATIONS OF CONVOLUTION ALGEBRAS

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1. INTRODUCTION

Let w be a radical, algebra weight on $\mathbb{R}^+ = [0, \infty)$. That is, w satisfies $w(x+y) \leq w(x)w(y)$, and $w(x)^{1/x} \to 0$ as $x \to +\infty$. We will also assume that w is continuous on $\mathbb{R}^+$, and that $w(0) = 1$. We are interested in derivations on the algebra $L^1(w)$ consisting of Lebesgue measurable functions on $\mathbb{R}^+$ which are integrable with respect to the weight w, and on related algebras. The algebra $M(w)$ of Radon measures on $\mathbb{R}^+$ which have finite total variation with respect to w will also play an important part. By identifying functions with absolutely continuous measures, $L^1(w)$ is a closed ideal in $M(w)$. It is also true that $M(w)$ is isometrically isomorphic with the multiplier algebra of $L^1(w)$, a measure μ corresponding to a multiplier T by $Tf = \mu * f$. Our interest in derivations is related to some questions about automorphisms. We will mention one such question, but this article will concentrate on derivations. Unless otherwise indicated, all integrals occurring here will be over the domain $\mathbb{R}^+$.

We begin by fixing some notation. We write X for the operation of multiplication by the coordinate function: thus, if f is a function, Xf is the function defined by $Xf(x) = x\,f(x)$, and if μ is a measure, $X\mu$ is the measure defined by $d(X\mu)(x) = x\,d\mu(x)$. Similarly, if z is a complex number, e^{zX} denotes the operation of multiplication by the function $x \mapsto e^{zx}$. If λ is a real number, it is easy to check that $e^{\lambda X}w$ is a radical, algebra weight. Let A_λ denote $L^1(e^{-\lambda X}w)$, and let M_λ denote $M(e^{-\lambda X}w)$. Note that the algebras A_λ (respectively, M_λ) are all isomorphic. In fact, $e^{(\lambda-\rho)X}$ defines an isomorphism from A_ρ onto A_λ (respectively, M_ρ onto M_λ). Also note that $A_\lambda \supseteq A_\rho$ if $\lambda \geq \rho$, and the inclusion map is a continuous embedding of A_ρ onto a dense subalgebra

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of A_λ. Now define $A_+ = \cap\{A_\lambda : \lambda > 0\}$. Then A_+, as an intersection of Banach algebras, can be given the structure of a Fréchet algebra. Note that A_+ contains A_0, and the inclusion is a continuous embedding. In general, a linear map from A_0 into A_+ is continuous if and only if, for each $\lambda > 0$, the map defines a continuous linear map from A_0 into A_λ. Also, a linear map on A_+ is continuous if and only if, for each $\lambda > 0$, there is $\rho > 0$ such that the map extends to a continuous linear map from A_ρ into A_λ; note that we can assume $\rho \leq \lambda$. Similarly, if we define $M_+ = \cap\{M_\lambda : \lambda > 0\}$, then M_+ is a Fréchet algebra, and there are similar characterizations of continuity of linear maps from M_0 into M_+ and from M_+ into itself. In particular, the map X defines a continuous derivation on each of the algebras A_+ and M_+: the derivation property of the map X is well known and easy to verify, and while X is not a continuous map on A_λ into itself for any λ, it is a continuous map from A_ρ into A_λ whenever $\rho < \lambda$; similarly for measure algebras. To see this, note that for $\rho < \lambda$ and $\mu \in M_\rho$,

$$\begin{aligned} \|X\mu\|_\lambda &= \int xe^{-\lambda x} w(x)\, d|\mu|(x) \\ &= \int xe^{-(\lambda-\rho)x} e^{-\rho x} w(x)\, d|\mu|(x) \\ &\leq C(\rho,\lambda)\|\mu\|_\rho\,, \end{aligned}$$

where $C(\rho,\lambda) = \sup\{xe^{-(\lambda-\rho)x} : x \in \mathbb{R}^+\} = [e(\lambda-\rho)]^{-1}$. It was this observation that originally led us to consider the algebras A_+ and M_+.

2. RESTRICTIONS AND EXTENSIONS OF DERIVATIONS

Let D be a derivation on $A_0 = L^1(w)$. It is known, [5], that D is continuous, and, [2], that there is a Radon measure μ on $\mathbb{R}^+$ such that

$$Df = Xf * \mu \tag{1}$$

for each f in A_0, and

$$\|D\| = \sup\left\{\frac{x}{w(x)}\int w(x+y)\, d|\mu|(y) : x > 0\right\} < \infty\,. \tag{2}$$

It is easy to see that $e^{-\lambda X}De^{\lambda X}$ defines a derivation on $A_{-\lambda}$. For $\lambda > 0$, $A_{-\lambda}$ is a dense subalgebra of A_0. It is a straightforward calculation to show that

$$(3) \qquad e^{-\lambda X}De^{\lambda X}(f) = e^{-\lambda X}(Xe^{\lambda X}f * \mu) = Xf * e^{-\lambda X}\mu$$

for f in $A_{-\lambda}$. It follows from these observations that $e^{-\lambda X}De^{\lambda X}$ is continuous on $A_{-\lambda}$ with respect to the norm of A_0, and extends by continuity to a derivation on A_0. On the other hand, if $\lambda < 0$, $A_{-\lambda}$ contains A_0, and in this case, one may ask the following question.

QUESTION 1. If $\lambda < 0$ and D is a derivation on A_0, does the derivation $e^{-\lambda X}De^{\lambda X}$ on $A_{-\lambda}$ restrict to a derivation on A_0? In other words, is $e^{-\lambda X}De^{\lambda X}(A_0)$ contained in A_0?

If so, let Δ be the restriction of $e^{-\lambda X}De^{\lambda X}$ to A_0. Then Δ is continuous on A_0; using (2) and (3), we see that this happens if and only if

$$\sup\left\{\frac{x}{w(x)}\int w(x+y)e^{-\lambda y}\, d|\mu|(y) : x > 0\right\} < \infty ,$$

that is, if and only if $e^{-\lambda X}\mu$ defines a derivation on A_0 via (1). It is then easy to see that $e^{\lambda X}\Delta e^{-\lambda X} = D$; since $\lambda < 0$, our earlier remarks show that D defines, by restriction, a derivation on A_λ. In fact, Question 1 is equivalent to the next question.

QUESTION 2. If $\lambda < 0$ and D is a derivation on A_0, does D restrict to a derivation on A_λ? That is, is $D(A_\lambda)$ contained in A_λ?

Here is a reason for being interested in the restriction question. It was shown in [3] that, at least for weights w belonging to a certain class (there denoted W^+), every automorphism of A_0 has a representation of the form $e^{i\alpha X}e^{\lambda X}e^{D}e^{-\lambda X}$, where α is real, $\lambda > 0$, and D is a derivation on A_0. Suppose D restricts to a derivation on $A_{-\lambda}$. Then $\Delta = e^{\lambda X}De^{-\lambda X}$ is a derivation on A_λ which restricts to a derivation on A_0, and $D = e^{-\lambda X}\Delta e^{\lambda X}$. It follows that $e^D = e^{-\lambda X}e^{\Delta}e^{\lambda X}$, and the above representation of an automorphism simplifies to $e^{i\alpha X}e^{\Delta}$.

It is not true that Question 2 has a positive answer for every derivation D on A_0, and every $\lambda > 0$. To see this, suppose D is determined by μ as in (1) and (2). Then, given $\lambda > 0$, D restricts to $A_{-\lambda}$ if and only if

$$\begin{aligned}&\sup\left\{\frac{x}{e^{\lambda x}w(x)}\int e^{\lambda(x+y)}w(x+y)\,d|\mu|(y) : x > 0\right\}\\ &= \sup\left\{\frac{x}{w(x)}\int e^{\lambda y}w(x+y)\,d|\mu|(y) : x > 0\right\}\\ &< \infty .\end{aligned}$$

It should be clear from this that some derivations on A_0 will restrict to $A_{-\lambda}$ for all $\lambda > 0$, some will restrict for a finite interval of values of λ, and some will restrict for no λ. Since our interest in restriction is associated with the representation of automorphisms, we refine Question 2 as follows.

QUESTION 2A. Suppose $\lambda > 0$, D is a derivation on A_0, and it is known that $e^{\lambda X}e^{D}e^{-\lambda X}$ is an automorphism of A_0. Does it follow that D restricts to a derivation on $A_{-\lambda}$?

3. DERIVATIONS ON A_+

Suppose a Radon measure μ on $\mathbb{R}^+$ defines a derivation D on A_0 as in equation (1). Then (2) holds, so for any $\lambda > 0$, the inequality obtained by replacing w by $e^{-\lambda X}w$ in (2) is also valid. Thus D extends by continuity to a derivation on A_λ, for each $\lambda > 0$, and hence to a continuous derivation on A_+. However, there are other continuous derivations on A_+: for example, we saw at the end of the Introduction that X defines a continuous derivation on A_+, but not on A_0. Our aim in this section is to characterize the continuous derivations on A_+. Here is a preliminary result.

PROPOSITION 1. *Every measure μ in M_+ determines a continuous derivation on A_+ using (1).*

Proof. Let μ belong to M_+. For any ρ and λ such that $0 \le \rho < \lambda$, and any f in A_ρ, we have Xf in A_λ, as in the Introduction, and μ in M_λ. Therefore,

$$\|Xf * \mu\|_\lambda \le \|Xf\|_\lambda\,\|\mu\|_\lambda \le C(\rho,\lambda)\|f\|_\rho\,\|\mu\|_\lambda .$$

That implies that $Df = Xf * \mu$ $(f \in A_\rho)$ defines a continuous linear map from A_ρ to A_λ, whenever $0 \le \rho < \lambda$. It is straightforward to check that D is a derivation, so the proposition is proved.

The rest of this note will be devoted to an attempt to prove the converse to the above proposition. We will see that for some weights, every derivation on A_+ is determined by a measure in M_+, while for other weights, the situation is less clear. We need some technical facts about certain linear maps between the algebras A_λ for different values of λ. Most of this is a straightforward adaptation of results in [2]. Recall that M_λ is the multiplier algebra of A_λ, and thus has both the operator norm topology and the strong operator topology (SO) as a subalgebra of $B(A_\lambda)$, the algebra of bounded linear operators on A_λ. Since it is also the dual of the space $C_0(1/w)$ of continuous functions f on $\mathbb{R}^+$ such that f/w vanishes at infinity, M_λ also has a weak-* topology (w^*). Now, for $0 \le \rho \le \lambda$, write $M_{\rho,\lambda}$ for the space of multipliers from A_ρ into A_λ; these are the linear maps $T : A_\rho \to A_\lambda$ such that $T(f * g) = Tf * g$ for any f and g in A_ρ (recall that when $\rho \le \lambda$, $A_\lambda \supseteq A_\rho$). Since $e_n = n\chi_{[0,1/n]}$ is a bounded approximate identity in A_λ, for every λ, all such multipliers are continuous. To see this, suppose $f_n \to 0$ in A_ρ. By the Varopoulos extension of Cohen's factorization theorem [[1], p.62], there are (g_n) and h in A_ρ such that $g_n \to 0$ and $f_n = h * g_n$. Then $Tf_n = Th * g_n \to 0$ in A_λ, so continuity of T follows by the closed graph theorem.

PROPOSITION 2. *For $0 \le \rho \le \lambda$, $M_{\rho,\lambda} = M_\lambda$, and the norms $\| \ \|_{\rho,\lambda}$ and $\| \ \|_\lambda$ inherited as subspaces of $B(A_\rho, A_\lambda)$ and $B(A_\lambda)$ are equivalent.*

Proof. Since A_ρ is densely and continuously embedded in A_λ by inclusion, M_λ embeds continuously into $M_{\rho,\lambda}$ by restriction. On the other hand, suppose $T \in M_{\rho,\lambda}$. Since $e_n = n\chi_{[0,1/n]}$ is a bounded approximate identity in both A_ρ and A_λ, we have $e_n * f \to f$ in A_ρ as $n \to \infty$, and therefore $Te_n * f = T(e_n * f) \to Tf$. We also have (Te_n) bounded in M_λ, so (Te_n) has a (w^*)-convergent subnet — say it is (Tf_j), where $f_j = e_{n(j)}$. Let μ be the (w^*)-limit of (Tf_j) in M_λ. Since multiplication is separately (w^*)-continuous, $(f * Tf_j)$ converges (w^*) to $f * \mu$. Since (Tf_j) is a subnet of (Te_n), we conclude that

$Tf = f * \mu$. Thus, restriction maps M_λ onto $M_{\rho,\lambda}$. It is easy to check that $M_{\rho,\lambda}$ is closed in $B(A_\rho, A_\lambda)$; in fact an (SO)-limit of multipliers is a multiplier. The equivalence of the norms now follows from the open mapping theorem, and the proof is complete.

PROPOSITION 3. *Let $0 \leq \rho \leq \lambda$ and let $D : A_\rho \to A_\lambda$ be a derivation. Then D is continuous, and extends to a continuous derivation $M_\rho \to M_\lambda$.*

Proof. The continuity of D follows from a slight modification of the result in [5]. For $\mu \in M_\rho$ and $f \in A_\rho$, define $Tf = D(\mu * f) - \mu * Df$, an element of A_λ. One checks easily that $T \in M_{\rho,\lambda}$. By Proposition 2, there is $v \in M_\lambda$ such that $Tf = f * v$; put $\Delta\mu = v$. Observe that Δ is an extension of D: the measure v is unique, and if $\mu = g \in A_\rho$, then $Dg \in A_\lambda$ satisfies the requirements for v. It is routine to check that Δ is linear; also, Δ is continuous:

$$\begin{aligned}\|\Delta\mu * f\|_\lambda &\leq \|D(\mu * f)\|_\lambda + \|\mu * Df\|_\lambda \\ &\leq 2\|D\|_{\rho,\lambda}\,\|\mu\|_\rho\,\|f\|_\rho\,,\end{aligned}$$

whence $\|v\|_{\rho,\lambda} \leq 2\|D\|_{\rho,\lambda}\,\|\mu\|_\rho$. Since $\|\ \|_{\rho,\lambda}$ and $\|\ \|_\lambda$ are equivalent norms on M_λ, the continuity of Δ follows. Finally, we show that Δ is a derivation. For $f \in A_\rho$ and $\mu, v \in M_\rho$,

$$\begin{aligned}f * \Delta(\mu * v) &= D(f * \mu * v) - \mu * v * Df \\ &= f * \mu * \Delta v + v * D(f * \mu) - \mu * v * Df \\ &= f * \mu * \Delta v + v * \big(f * \Delta\mu + \mu * Df\big) - \mu * v * Df \\ &= f * \big(\mu * \Delta v + \Delta\mu * v\big)\,,\end{aligned}$$

which implies $\Delta(\mu * v) = \mu * \Delta v + \Delta\mu * v$, as required.

In fact, we shall write D for either a derivation from A_ρ to A_λ, or its extension to a derivation from M_ρ to M_λ.

PROPOSITION 4. *Let $0 \leq \rho \leq \lambda$ and let D be a derivation from M_ρ into M_λ. Then*

(a) D maps A_ρ into A_λ;

(b) D is continuous; and

(c) D is (SO) to (SO)-continuous on bounded subsets of M_ρ.

Proof. For (a), note that, by Cohen's factorization theorem, each f in A_ρ factors: $f = g * h$. Then $Df = Dg * h + g * Dh$ belongs to A_λ, since A_λ is an ideal in M_λ.

By Proposition 3, $D \mid A_\rho$ is continuous. Also, $D\mu * f = D(\mu * f) - \mu * Df$, so D is the extension to M_λ of $D \mid A_\rho$, as constructed in Proposition 3. That proves (b).

Finally, we prove (c). Let (μ_j) be a bounded net in M_ρ, (SO)-convergent to μ. Because of the boundedness and because A_ρ is dense in A_λ whenever $\rho \leq \lambda$, we can conclude that (μ_j) is (SO)-convergent to μ in A_λ, for any $\lambda \geq \rho$, so for f in A_ρ, $(\mu_j * Df)$ is norm-convergent to $\mu * Df$. Also, $(\mu_j * f)$ is norm-convergent to $\mu * f$, so $(D(\mu_j * f))$ is norm-convergent to $D(\mu * f)$. Since $D(\mu_j * f) = D\mu_j * f + \mu_j * Df$ and $D(\mu * f) = D\mu * f + \mu * Df$, it now follows that $(D\mu_j * f)$ is norm-convergent to $D\mu * f$. Since $(D\mu_j)$ is bounded and A_ρ is dense in A_λ, it follows that $(D\mu_j * f)$ is norm-convergent to $D\mu * f$ for any f in A_λ. Thus, $(D\mu_j)$ is (SO)-convergent to $D\mu$, and (c) is proved.

By Proposition 3 and part (a) of Proposition 4, the derivations from A_ρ into A_λ, and those from M_ρ into M_λ, are the same. The next result characterizes these derivations.

THEOREM 1. *Let $0 \leq \rho \leq \lambda$. Then D is a derivation from A_ρ into A_λ if and only if there is a Radon measure μ on $\mathbb{R}^+$ such that*

(a) $Df = Xf * \mu$ for all f in A_ρ, and

(b) $S = \sup\{\frac{x}{e^{-\rho x} w(x)} \int e^{-\lambda(x+y)} w(x+y)\, d|\mu|(y) : x > 0\} < \infty$.

In such cases, $\|D\|_{\rho,\lambda} = S$.

Sketch of Proof. This is much as in [2], and many details will be omitted. One first shows $\alpha(D\delta_a) \geq a$ for any a in $\mathbb{R}^+$ (recall that $\alpha(\mu) = \inf(\text{support } \mu)$; also, δ_a denotes the point mass at a). Then one shows the existence of a measure μ such that $D\delta_x = x\delta_x * \mu$ for all x in $\mathbb{R}^+$. In passing, we note that the supremum S in (b), above, is

$$S = \sup\left\{ \left\| \frac{x\delta_x}{e^{-\rho x} w(x)} * \mu \right\|_\lambda : x > 0 \right\}.$$

Thus, finiteness of S follows from boundedness of D; in fact, $S \leq \|D\|_{\rho,\lambda}$.

Next, let μ be a measure satisfying (b). One shows that $D_\mu(f) = Xf * \mu$ defines a bounded derivation $D_\mu : A_\rho \to A_\lambda$, with $\|D_\mu\|_{\rho,\lambda} \le S$.

Finally, starting with D, produce μ as in the first paragraph, then D_μ as in the second. Observe that the two derivations are (SO) to (SO)-continuous on bounded sets, and agree on the subalgebra generated by the point masses. Since every measure in M_ρ is the (SO)-limit of a bounded net of measures with finite support, $D = D_\mu$, and the result is proved.

The existence of a measure determining the derivation D in the last result can also be deduced from Theorem 3.4 of [4]. We now give a characterization of the continuous derivations on A_+.

THEOREM 2. *Let w be a radical, algebra weight on $\mathbb{R}^+$. The following statements are equivalent.*

(a) D is a derivation on A_+.

(b) D is a derivation from A_0 to A_λ for every $\lambda > 0$.

(c) There is a Radon measure μ on $\mathbb{R}^+$ such that $Df = Xf * \mu$ for every f in A_0, and

$$S_\lambda = \sup\left\{ \frac{x}{w(x)} \int e^{-\lambda(x+y)} w(x+y)\, d|\mu|(y) : x > 0 \right\} < \infty$$

for every $\lambda > 0$.

Proof. The equivalence of (b) and (c) was proved in Theorem 1. Since A_0 is continuously embedded in A_+, it is immediate that (a) implies (b). We show that (c) implies (a). Thus, suppose that the supremum S_λ of (c) is finite for each $\lambda > 0$, and fix $\lambda > 0$ and ρ such that $0 \le \rho < \lambda$. Then

$$\begin{aligned}
&\sup\left\{ \frac{x}{e^{-\rho x} w(x)} \int e^{-\lambda(x+y)} w(x+y)\, d|\mu|(y) : x > 0 \right\} \\
&= \sup\left\{ \frac{x}{e^{-\rho x} w(x)} \int e^{-\rho(x+y)} e^{-(\lambda-\rho)(x+y)} w(x+y)\, d|\mu|(y) : x > 0 \right\} \\
&\le S_{\lambda-\rho} < \infty\,.
\end{aligned}$$

By Theorem 1, $Df = Xf * \mu$ defines a continuous derivation from A_ρ to A_λ whenever $0 \le \rho < \lambda$. It follows that $Df = Xf * \mu$ defines a continuous derivation on A_+, and the theorem is proved.

COROLLARY (to the proof). *If D is a derivation on A_+, then for any ρ and λ such that $0 < \rho < \lambda$, D extends to a continuous derivation from A_ρ to A_λ. In particular, every derivation on A_+ is continuous.*

We now return to the question of a converse to Proposition 1, and we formulate the following question.

QUESTION 3. Suppose $\mu \notin M_+$; that is,

$$\int e^{-\Lambda y} w(y)\, d|\mu|(y) = \infty$$

for some $\Lambda > 0$. For each $\lambda > 0$, define S_λ as in Theorem 2(c) by

$$S_\lambda = \sup\left\{ \frac{x}{w(x)} \int e^{-\lambda(x+y)} w(x+y)\, d|\mu|(y) \,:\, x > 0 \right\}.$$

Is there $\lambda > 0$ such that $S_\lambda = \infty$?

If such a λ exists whenever $\mu \notin M_+$, then, by Theorem 2, the continuous derivations on A_+ are exactly those determined by the measures in M_+. One way to achieve $S_\lambda = \infty$ for given λ would be to find $x > 0$ such that

$$\int e^{-\lambda(x+y)} w(x+y)\, d|\mu|(y) = \infty .$$

In some cases, we can do this.

THEOREM 3. *Let $w(y) = e^{-y^2}$. If $\mu \notin M_+$, there are $\lambda > 0$ and $x > 0$ such that*

$$\int e^{-\lambda(x+y)} w(x+y)\, d|\mu|(y) = \infty .$$

Hence, for this weight, the continuous derivations on A_+ are exactly those determined by the measures in M_+.

Proof. Suppose $\Lambda > 0$ and

$$\int e^{-\Lambda y} e^{-y^2}\, d|\mu|(y) = \infty .$$

For any λ such that $0 < \lambda < \Lambda$, we can take $x > 0$ such that $\lambda + 2x < \Lambda$. Then we have

$$\begin{aligned}
&\int e^{-\lambda(x+y)} e^{-(x+y)^2}\, d|\mu|(y) \\
&= e^{-\lambda x - x^2} \int e^{-(\lambda+2x)y} e^{-y^2}\, d|\mu|(y) \\
&> e^{-\lambda x - x^2} \int e^{-\Lambda y} e^{-y^2}\, d|\mu|(y) \\
&= \infty\,.
\end{aligned}$$

Having seen this calculation, one can immediately enlarge the set of weights for which such a result holds. For example, any of the weights $e^{-x \log x}$, or e^{-x^p} $(1 < p \leq 2)$ can be used. One can also invent weights for which the argument in Theorem 3 fails, and we end with a question.

QUESTION 4. Suppose $w(y) = e^{-y^3}$. Is there a Radon measure μ on $\mathbb{R}^+$ such that

$$\int e^{-\Lambda y} e^{-y^3}\, d|\mu|(y) = \infty$$

for some $\Lambda > 0$, while

$$\sup\left\{ \frac{x}{e^{-x^3}} \int e^{-\lambda(x+y)} e^{-(x+y)^3}\, d|\mu|(y) : x > 0 \right\} < \infty$$

for every $\lambda > 0$?

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