WEAK (F)-AMENABILITY OF R(X)

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1. INTRODUCTION

In this paper we shall discuss the amenability and weak amenability of certain commutative Banach algebras. We begin by recalling the basic definitions.

1.1 DEFINITION Let A be an algebra, and let X be an A-bimodule. Then X is *commutative* if

$$a.x = x.a \qquad (a \in A,\ x \in X).$$

If A is commutative, then an A-*module* is a commutative A-bimodule.

Note that an algebra A is always itself an A-bimodule, with module operations given by multiplication in A.

1.2 DEFINITION Let A be a Banach algebra. A *Banach A-bimodule* is an A-bimodule X, equipped with a complete norm $\|\cdot\|$, satisfying

$$\|a.x\| \le \|a\|\|x\|,\ \|x.a\| \le \|a\|\|x\| \qquad (a \in A,\ x \in X).$$

If A is commutative, then a *Banach A-module* is a commutative Banach A-bimodule.

1.3 DEFINITION Let A be an algebra. A *derivation* from A into an

A-bimodule X is a linear map $D: A \longrightarrow X$ satisfying

$$D(ab) = a.Db + Da.b \qquad (a, b \in A).$$

The derivation D is *inner* if there exists $x \in X$ with

$$D(a) = a.x - x.a \qquad (a \in A).$$

Let A be a Banach algebra, and E be a Banach A-bimodule. Set

$$(a.f)(x) = f(x.a), \ (f.a)(x) = f(a.x) \qquad (f \in E', \ a \in A, \ x \in X).$$

Then with respect to these operations, E' becomes a Banach A-bimodule. The bimodule E' is called the *dual module* of E. It is clear that if A is commutative, then the dual of any Banach A-module is a Banach A-module.

1.4 DEFINITION Let A be a Banach algebra. Then A is *amenable* if, for every Banach A-bimodule X, every continuous derivation from A into X' is inner; A is *weakly amenable* if every continuous derivation from A into the dual module A' is inner.

The notion of amenability for Banach algebras was introduced by Johnson in [6]. It has proved to be an important concept in many different areas of Banach algebra theory. (The original motivation for this definition is the following. Let G be a locally compact group. Then the Banach algebra $L^1(G)$ is amenable if and only if the group G is amenable. See [6,Theorem 2.5]).

The concept of weak amenability was introduced for commutative Banach algebras by Bade, Curtis and Dales, in [1]. The above definition is the obvious generalization of their definition, in view of the following result from the same paper.

1.5 PROPOSITION *Let A be a commutative Banach algebra, and suppose that there is a non-zero, continuous derivation from A into some Banach A-module. Then there is a non-zero, continuous derivation from A into the dual module A'.*

1.6 DEFINITION Let A be an algebra, and let ϕ be a character on A. A *point derivation* at ϕ is a linear functional d on A, satisfying

$$d(ab) = \phi(a)d(b) + \phi(b)d(a) \quad (a,b \in A).$$

Point derivations at ϕ can be identified with derivations from A into the one-dimensional commutative A-bimodule $\mathbb{C}$, where the module operations in $\mathbb{C}$ are given by

$$a.\lambda = \lambda.a = \phi(a)\lambda \quad (a \in A,\ \lambda \in \mathbb{C}).$$

Let A be a Banach algebra. It is clear that if A is amenable, then A is also weakly amenable. It is also clear that if A is commutative and weakly amenable, then there are no non-zero, continuous point derivations on A. If A is a commutative, amenable, unital Banach algebra, then every maximal ideal in A has a bounded approximate identity ([4,Proposition 3.1]). By Cohen's factorization theorem, $M^2 = M$ for each maximal ideal M, and so there are no non-zero point derivations at all on A.

In this paper we shall discuss the amenability and weak amenability of an important class of uniform algebras, the algebra of all continuous functions on a compact plane set X which can be uniformly approximated by rational functions with poles off X.

NOTATION Let S be a set, and let f be a bounded, complex-valued function on S. For each set E contained in S, the *uniform norm of* f *on* E, denoted by $|f|_E$, is defined by

$$|f|_E = \sup\{|f(x)| : x \in E\}.$$

If A is a uniform algebra on a compact space X, then A is *trivial* if $A = C(X)$, the uniform algebra of all continuous functions on X.

1.7 DEFINITION Let X be a compact subset of $\mathbb{C}$. Then $R_0(X)$ is the set of restrictions to X of rational functions with poles off X, and $R(X)$ is the closure of $R_0(X)$ in $C(X)$. The coordinate functional is denoted by Z.

It is well-known that for any compact space X, $C(X)$ is always amenable. The following result can be found, for example, on page 178 of [3].

1.8 PROPOSITION *Let* X *be a compact plane set. Then* $R(X) = C(X)$ *if and only if there are no non-zero point derivations on* $R(X)$.

It was conjectured by Browder that this result would be false if "point derivation" were replaced by "continuous point derivation". This conjecture was proved by Wermer in [12], where he gave an example of a Swiss cheese X for which $R(X)$ is non-trivial and has no non-zero, continuous point derivations. This contrasts with the following result of Sheinberg (see [7]).

1.9 THEOREM *Every amenable uniform algebra is trivial.*

We are interested in the question: Is every *weakly* amenable uniform algebra trivial?. In particular, we are interested in the case where $A = R(X)$ for some compact plane set X. We are unable to resolve this question, but we shall obtain, in Theorem 2.13, a criterion for $R(X)$ to be trivial in terms of derivations into modules more general than Banach $R(X)$-modules.

We would be interested in an example of a compact plane set X such that $R(X)$ has no non-zero, continuous point derivations, but $R(X)$ is not weakly amenable.

The following result is immediate.

1.10 PROPOSITION *Let* X *be an infinite compact plane set, let* E *be an* $R(X)$*-module, and let* D *be a derivation from* $R(X)$ *into* E. *Then*

$$D(f) = f'.D(Z) \qquad (f \in R_0(X)).$$

1.11 PROPOSITION *Let* X *be a compact subset of* $\mathbb{C}$. *Then* $R(X)$ *is not weakly amenable if and only if there exists a complex Borel measure* μ *on* X *which does not annihilate* $R(X)$, *and* $C > 0$ *with*

$$\left| \int_X f'(z)g(z)d\mu(z) \right| \le C \, |f|_X |g|_X \qquad (f,g \in R_0(X)).$$

Proof If $R(X)$ is not weakly amenable, then there is a non-zero, continuous derivation D from $R(X)$ into $R(X)'$. By the Hahn-Banach and Riesz representation theorems, there exists a non-zero, complex Borel measure μ on X with

$$D(Z)(g) = \int_X g(z)d\mu(z) \qquad (g \in R(X)).$$

The existence of the constant C for this μ follows from Proposition 1.10 and the continuity of D.

Conversely, given μ and $C > 0$ satisfying the conditions of the proposition, there is a non-zero, continuous derivation $D: R(X) \longrightarrow R(X)'$ satisfying

$$D(f)(g) = \int_X f'(z)g(z)d\mu(z) \qquad (f,g \in R_0(X)).$$

NOTATION Throughout this paper we shall denote Lebesgue measure on the plane by m.

The following result is an immediate corollary of Proposition 1.11.

1.12 COROLLARY *Let* X *be a compact plane set, and suppose that there exists* $C > 0$ *and a Borel set* E *contained in* X *with* $m(E) > 0$ *such that*

$$\left| \int_E f'(z)g(z)dm(z) \right| \le C\, |f|_X |g|_X \qquad (f,g \in R_0(X)).$$

Then $R(X)$ *is not weakly amenable.*

Let X be a compact plane set with $R(X) \neq C(X)$. By the Hartogs-Rosenthal theorem ([3,p.161]), $m(X) > 0$. We would be interested to know whether there always exists a Borel set E satisfying the conditions of the above corollary for some $C > 0$. We would also like to know whether there always exists a Borel set E of positive Lebesgue measure contained in X such that the map

$$f \mapsto f'|E,\ R_0(X) \longrightarrow L^1(E,dm),$$

is continuous.

2. TOPOLOGICAL MODULES

2.1 DEFINITION Let (E,τ) be a topological linear space. Then E is *locally bounded* if there exists a bounded neighbourhood of 0, and E is an *(F)-space* if there is a complete metric on E inducing the topology τ.

2.2 DEFINITION Let E be a linear space, and let d be a metric on E. Then d is *invariant* if $d(x,y) = d(x-y,0)$ $(x,y \in E)$.

2.3 PROPOSITION [8,p.163] *If (E,τ) is a topological linear space with a countable base of neighbourhoods at the origin, then there is an invariant metric d on E which induces the topology τ.*

2.4 PROPOSITION [8,p.165] *If (E,τ) is an (F)-space, then (E,τ) is complete, and every invariant metric inducing the topology τ is also complete.*

2.5 DEFINITION [8,p.159] Let E be a linear space. A function $\|.\|$ on E is a *quasi-norm* if it satisfies the following conditions:

(i) $\|x\| \geq 0$ $(x \in E)$;

(ii) $\|x\| = 0$ if and only if $x = 0$;

(iii) $\|\alpha x\| = |\alpha|\,\|x\|$ $(x \in E,\ \alpha \in \mathbb{C})$;

(iv) there exists $K \geq 1$, with $\|x+y\| \leq K(\|x\|+\|y\|)$ $(x,y \in E)$.

2.6 PROPOSITION [8,pp.159-160] *Let E be a linear space, and let $\|.\|$ be a quasi-norm on E. Then the sets*

$$N_k = \{f \in E: \|f\| < 2^{-k}\} \qquad (k \in \mathbb{N})$$

form a countable base of neighbourhoods at 0 *for a locally bounded topological linear space topology* τ *on* E.

2.7 DEFINITION With E, $\|.\|$, τ, as in 2.6, we call (E,τ) a *quasi-normed space* and we denote it by $(E,\|.\|)$.

Remark There are many discontinuous quasi-norms on the topological linear space $\mathbb{C}^2$. Thus it is not always true that a quasi-norm is continuous with respect to the topology it generates.

Note that by Proposition 2.3 any quasi-normed space $(E,\|.\|)$ has a topology that can be induced by an invariant metric d. The following result is elementary.

2.8 PROPOSITION *Let* $(E,\|.\|)$ *be a quasi-normed space, and let* d *be an invariant metric on* E *inducing the same topology as* $\|.\|$. *Let* $g \in E$, *and let* (f_n) *be a sequence of elements of* E. *Then* (f_n) *converges to* g *in* (E,d) *if and only if* $\|f_n - g\| \to 0$ *as* $n \to \infty$. *Also* (f_n) *is Cauchy in* (E,d) *if and only if* $\limsup_{m,n\to\infty} \|f_n - f_m\| = 0$.

2.9 DEFINITION Let A be a commutative topological algebra, and let E be a topological linear space which is a (commutative) A-module. Then E is a *topological* A-*module* if the module operation

$$(a,x) \mapsto a.x, \quad A\times E \mapsto E$$

is jointly continuous; E is an *(F)-A-module* if it is a topological

A-module which is also an (F)-space.

2.10 DEFINITION Let A be a commutative Banach algebra. Then A is *weakly (F)-amenable* if there are no non-zero, continuous derivations from A into any (F)-A-module.

The following is a slight generalization of a result in [1].

2.11 LEMMA *Let* A *be a commutative, unital Banach algebra, let* E *be a topological* A*-module, let* D *be a continuous derivation from* A *into* E, *and let* $a \in A$. *Suppose that either of the following conditions holds:*

(i) $a \in \mathrm{Inv}(A)$ *and*

$$\lim_{n\to\infty} \|a^n\|\|a^{-n}\|/n = 0;$$

(ii) $\lim_{n\to\infty} \|\exp(na)\|\|\exp(-na)\|/n = 0.$

Then $D(a) = 0$.

Proof We set $Q = \{a.Db : a,b \in A, \|a\|,\|b\| \leq 1\}$. Then, by the joint continuity of the module operation, Q is a bounded subset of E.

Firstly, consider the case where a satisfies condition (i). Then we have

$$D(a^n) = na^{n-1}.Da \qquad (n \in \mathbb{N}).$$

Thus

$$Da = \frac{1}{n}\, a^{-n+1}.D(a^n),$$

and so

$$Da \in (\|a^{-n+1}\|\|a^n\|/n)\; Q \qquad (n \in \mathbb{N}).$$

It follows from (i) and the boundedness of Q that $Da = 0$.

Secondly suppose that a satisfies (ii). Then $\exp(a)$ satisfies (i), and so $D(\exp(a)) = 0$. But a simple calculation shows that $D(\exp(a)) = \exp(a).Da$. Thus

$$D(a) = 1.D(a) = \exp(-a).\exp(a).D(a) = 0,$$

where the first equality follows from the derivation identity, and the fact that $D(1) = 0$.

2.12 COROLLARY *Let* X *be a compact space. Then there are no non-zero, continuous derivations from* $C(X)$ *into any topological* $C(X)$*-module.*

Proof Let E be a topological $C(X)$-module, and let D be a continuous derivation from $C(X)$ into E. We shall show that $D(h) = 0$ $(h \in C(X))$. Clearly it is sufficient to consider the case where h is a real-valued, continuous function on X. But then the function ih satisfies condition (ii) of Proposition 2.11. Thus $D(h) = 0$.

In particular, $C(X)$ is always weakly (F)-amenable.

The main new result in this paper is the following theorem. We shall not give a proof of this result until §5. We would be interested in any similar result for more general uniform algebras.

2.13 THEOREM *Let* X *be a compact plane set. Then* $R(X) = C(X)$ *if and only if there are no non-zero, continuous derivations from* $R(X)$ *into any locally bounded (F)–*$R(X)$*-module.*

Thus, for any compact plane set X, $R(X) = C(X)$ if and only if $R(X)$ is weakly (F)-amenable.

We shall develop the tools needed to prove this theorem in the next two sections.

3. WEAK-L^1 AND THE L^p SPACES

NOTATION Let X be a locally compact space. Then we denote by $\mathcal{M}(X)$ the Banach space of all complex, regular Borel measures on X.

3.1 DEFINITION Let X be a locally compact space, and let μ be a positive measure on X. Then *weak-L^1 with respect to* μ, denoted by $L^1_*(X,d\mu)$, is the set of all equivalence classes of Borel measurable functions f on X satisfying

$$\sup_{t>0} t\mu(\{x \in X: |f(x)| > t\}) < \infty,$$

where the equivalence relation is almost everywhere equality with respect to μ.

For the rest of this section X will be a fixed compact space, and μ a positive measure on X. We define a function $\|.\|$ on $L^1_*(X,d\mu)$ by

$$\|f\| = \sup_{t>0} t\mu(\{x \in X: |f(x)| > t\}) \quad (f \in L^1_*(X,d\mu)).$$

3.2 LEMMA

(i) *If* $f,g \in L^1_*(X,d\mu)$, *and* $\lambda \in (0,1)$, *then* $f+g \in L^1_*(X,d\mu)$ *and*

$$\|f+g\| \le (\lambda^{-1}\|f\| + (1-\lambda)^{-1}\|g\|).$$

In particular, $\|f+g\| \le 2(\|f\|+\|g\|)$.

(ii) *If* $f \in L^1_*(X,d\mu)$ *and* $\alpha \in \mathbb{C}$, *then* $\alpha f \in L^1_*(X,d\mu)$ *and* $\|\alpha f\| = |\alpha|\|f\|$.

(iii) $(L^1_*(X,d\mu),\|.\|)$ *is a quasi-normed space.*

(iv) *The quasi-norm* $\|.\|$ *is continuous on* $(L^1_*(X,d\mu),\|.\|)$.

Proof (i) This is clear from the inclusion

$$\{x \in X: |f(x)+g(x)| > t\} \subseteq \{x \in X: |f(x)| > \lambda t\} \cup \{x \in X: |g(x)| > (1-\lambda)t\}.$$

(ii) This is trivial, and (iii) follows easily, since $\|f\| = 0$ if and only if f is zero a.e. (μ) $(f \in L^1_*(X,d\mu))$.

(iv) Let $g \in L^1_*(X,d\mu)$, and let (f_n) be a sequence of elements in $L^1_*(X,d\mu)$ converging to g. Then by (i), for each $\lambda \in (0,1)$ and $n \in \mathbb{N}$, we have

$$\|g\| \le (\lambda^{-1}\|f_n-g\| + (1-\lambda)^{-1}\|f_n\|),$$

and

$$\|f_n\| \le (\lambda^{-1}\|f_n-g\| + (1-\lambda)^{-1}\|g\|).$$

Thus

$$\limsup_{n\to\infty} \|f_n\| \le (1-\lambda)^{-1}\|g\| \le (1-\lambda)^{-2} \liminf_{n\to\infty} \|f_n\| \qquad (\lambda \in (0,1)).$$

The result now follows, on letting λ tend to 0.

The following results are well known, but we are unable to give a reference, and so we include proofs for the sake of completeness.

3.3 **LEMMA** *Let* g *be a measurable function on* X, *and suppose that* (f_n) *is a sequence of elements of* $L^1_*(X,d\mu)$ *with* $f_n \to g$ *a.e.* (μ), *and with* $\sup\{\|f_n\|: n \in \mathbb{N}\} < \infty$. *Then* $g \in L^1_*(X,d\mu)$ *and*

$$\limsup_{n\to\infty} \|f_n-g\| \le \limsup_{m,n\to\infty} \|f_n-f_m\|. \tag{1}$$

Proof Take $n \in \mathbb{N}$, and $t > 0$. We have

$$\mu(\{x \in X: |f_n(x)-g(x)| > t\})$$

$$\leq \mu\left[\bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \{x \in X: |f_n(x)-f_m(x)| > t\} \right]$$

$$\leq \limsup_{m\to\infty} \mu(\{x \in X: |f_n(x)-f_m(x)| > t\})$$

$$\leq \frac{1}{t} \limsup_{m\to\infty} \|f_n - f_m\|.$$

It follows that $f_n - g \in L^1_*(X, d\mu)$, and that

$$\|f_n - g\| \leq \limsup_{m\to\infty} \|f_n - f_m\|.$$

Thus $g \in L^1_*(X, d\mu)$, and (1) holds.

3.4 LEMMA *If* μ *is finite and* $p \in (0,1)$, *then* $L^1_*(X, d\mu)$ *is contained in* $L^p(X, d\mu)$, *and the inclusion map is continuous.*

Proof Take $f, g \in L^1_*(X, d\mu)$. Then

$$\int_X |f(x) - g(x)|^p \, d\mu(x)$$

$$= \int_0^\infty py^{p-1}\mu(\{x \in X: |f(x) - g(x)| > y\}) \, dy$$

$$\leq \inf_{\gamma>0} \left[\mu(X) \int_0^\gamma py^{p-1}dy + \|f-g\| \int_\gamma^\infty py^{p-2}dy \right]$$

$$\leq (\mu(X)+p(1-p)^{-1}) \|f-g\|^p.$$

3.5 LEMMA *If* μ *is* σ*-finite, then* $(L^1_*(X, d\mu), \|.\|)$ *is complete.*

Proof Let (f_n) be a Cauchy sequence in $L^1_*(X,d\mu)$. We shall show that (f_n) converges in $L^1_*(X,d\mu)$. Clearly

$$\sup_{n\in\mathbb{N}} \|f_n\| < \infty,$$

and, by Proposition 2.8,

$$\limsup_{m,n\to\infty} \|f_n - f_m\| = 0.$$

It follows from Lemma 3.3 that we need only show that some subsequence of (f_n) converges a.e. (μ) on X. For this, we may assume that μ is finite, as the general case follows by a diagonalization argument.

By Lemma 3.4, (f_n) is a Cauchy sequence in $L^{1/2}(X,d\mu)$, and so for some subsequence $(f_{n_k})_{k=1}^{\infty}$ we have

$$\int_X \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right|^{1/2} d\mu(x) < 2^{-k} \qquad (k \in \mathbb{N}).$$

Thus

$$\sum_{k=1}^{\infty} \int_X \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right|^{1/2} d\mu(x) < \infty.$$

It follows that

$$\sum_{k=1}^{\infty} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right|^{1/2} < \infty \quad \text{a.e. } (\mu),$$

and hence that

$$\sum_{k=1}^{\infty} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right| < \infty \quad \text{a.e. } (\mu).$$

This shows that (f_{n_k}) converges a.e. (μ), as required.

Remark The last part of the above proof is essentially the proof that $L^{1/2}$ is complete.

3.6 LEMMA

(i) *For each* $p \in (0,1)$, $L^p(X,d\mu)$ *is a locally bounded* (F)-$L^\infty(X,d\mu)$*-module with respect to pointwise multiplication.*

(ii) *If* μ *is* σ*-finite, then* $L^1_*(X,d\mu)$ *is a locally bounded* (F)-$L^\infty(X,d\mu)$*-module with respect to pointwise multiplication.*

Proof Part (i) is clear. For part (ii), note that

$$\|f.g\| \le \|f\|_\infty \|g\| \qquad (f \in L^\infty(X,d\mu),\ g \in L^1_*(X,d\mu)).$$

The following result is an easy consequence of Lemma 3.6.

3.7 PROPOSITION *Let* X *be a compact plane set, and let* μ *be a finite, positive Borel measure supported on* X*. Then* $L^1_*(X,d\mu)$ *is a locally bounded* (F)-$C(X)$*-module with respect to pointwise multiplication.*

4. THE BEURLING AND CAUCHY TRANSFORMS

Let $\mu \in \mathcal{M}(\mathbb{C})$. We set

$$\tilde{\mu}(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{d|\mu|(w)}{|w-z|} \qquad (z \in \mathbb{C}),$$

where the integrand is defined as $+\infty$ when $w = z$. For those $z \in \mathbb{C}$ with $\tilde{\mu}(z) < \infty$, we set

$$\hat{\mu}(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{d\mu(w)}{w-z} .$$

Note that for all such z we have $\mu(\{z\}) = 0$.

It follows from Fubini's theorem that $\tilde{\mu} \in L^1_{loc}(\mathbb{C}, dm)$, and so $\hat{\mu} \in L^1_{loc}(\mathbb{C}, dm)$ also. The function $\hat{\mu}$ is called the *Cauchy transform* of μ.

For each $\varepsilon > 0$, we set

$$(B_\varepsilon \mu)(z) = \frac{1}{\pi} \int_{|w-z| \geq \varepsilon} \frac{d\mu(w)}{(w-z)^2} \qquad (z \in \mathbb{C}).$$

We set

$$(B_* \mu)(z) = \sup_{\varepsilon > 0} |(B_\varepsilon \mu)(z)| \qquad (z \in \mathbb{C}).$$

We also set

$$(B\mu)(z) = \lim_{\varepsilon \to 0+} (B_\varepsilon \mu)(z) \tag{2}$$

for those $z \in \mathbb{C}$ for which this limit exists.

The function $B\mu$ is called the *Beurling transform* of μ.

By the Radon-Nikodym theorem, the map

$$f \mapsto f dm, \ L^1(\mathbb{C}, dm) \longrightarrow \mathcal{M}(\mathbb{C}),$$

is an isometric linear embedding whose range is the set of elements of $\mathcal{M}(\mathbb{C})$ which are absolutely continuous with respect to m. From now on we shall identify $L^1(\mathbb{C}, dm)$ with its embedded image.

The following result is a special case of an important result of

Calderón-Zygmund theory ([11,p.42]).

4.1 PROPOSITION

(i) *Let* $f \in L^1(\mathbb{C}, dm)$, *and set* $d\mu = f dm$. *Then the limit in* (2) *exists a.e.* (m).

(ii) B_* *maps* $L^1(\mathbb{C}, dm)$ *into* $L^1_*(\mathbb{C}, dm)$, *and there exists a constant* $C_1 > 0$ *with*

$$\|B_* f\| \le C_1 \|f\|_1 \qquad (f \in L^1(\mathbb{C}, dm)).$$

Thus B and B_* both map $L^1(\mathbb{C}, dm)$ continuously into $L^1_*(\mathbb{C}, dm)$. For our purposes we need to extend this result by showing that B and B_* also map $\mathcal{M}(\mathbb{C})$ continuously into $L^1_*(\mathbb{C}, dm)$. These results are known "folk" theorems, but we include proofs because there seem to be no explicit statements of these results in the literature. First we shall modify the usual definition of dyadic square, (see, for example, [5,p.136]).

4.2 DEFINITION A *dyadic square* is a subset of $\mathbb{R}^2$ of the form

$$\left[2^n k \ , \ 2^n(k+1) \right] \times \left[2^n l \ , 2^n(l+1) \right]$$

for some $k, l, n \in \mathbb{Z}$.

Note that if two dyadic squares have non-empty intersection, then one must contain the other.

NOTATION Let Q be a dyadic square. We shall denote by Q^* the closed square which has the same centre as Q, but $2^{3/2}$ times the side length of Q. Let $a \in \mathbb{C}$, and $r > 0$. Then $\Delta(a,r)$ denotes the set $\{z \in \mathbb{C} : |z-a| < r\}$.

4.3 DEFINITION Let $\mu \in \mathcal{M}(\mathbb{C})$, and let $z \in \mathbb{C}$. Then we set

$$(M\mu)(z) = \sup\left\{\frac{1}{\pi r^2}\,|\mu|(\Delta(z,r)) : r > 0\right\}.$$

4.4 PROPOSITION [10, p. 137] *Let* $\mu \in \mathcal{M}(\mathbb{C})$. *Then* $M\mu \in L^1_*(\mathbb{C}, dm)$, *and* $\|M\mu\| \le 9\|\mu\|$.

4.5 PROPOSITION [10, p. 142] *Let* $\mu \in \mathcal{M}(\mathbb{C})$ *with* $\mu \perp m$. *Then*

$$\lim_{r\to 0+} \frac{1}{\pi r^2}\,|\mu|(\Delta(z,r)) = 0 \qquad \text{a.e. } (m).$$

NOTATION Let F be a Borel subset of $\mathbb{C}$ with $0 < m(F) < \infty$. We set

$$\mu_F = m(F)^{-1}\mu(F) \qquad (\mu \in \mathcal{M}(\mathbb{C})),$$

and we call μ_F the *average* of μ over F. For any real-valued, measurable function f on $\mathbb{C}$, and $t > 0$, we shall denote the set $\{z \in \mathbb{C} : f(z) > t\}$ by $\{f > t\}$.

Let $\mu \in \mathcal{M}(\mathbb{C})$, and let $t > 0$. We shall denote by $\mathcal{C}_t(\mu)$ the set of all dyadic squares Q satisfying:

(i) $|\mu|(Q) > tm(Q)$;

(ii) if Q' is a dyadic square strictly containing Q, then $|\mu|(Q') \le tm(Q')$.

It is clear that $\mathcal{C}_t(\mu)$ is countable, and that

$$\sum\{m(Q) : Q \in \mathcal{C}_t(\mu)\} \le \|\mu\|/t \tag{3}$$

since the dyadic squares in $\mathcal{C}_t(\mu)$ are pairwise disjoint. It is also clear that

$$|\mu|(Q) \le 4tm(Q) \qquad\qquad (Q \in \mathscr{C}_t(\mu)).$$

We are now in a position to prove the results that we shall need in §5. We shall first deal with a special case. The proof of this result is very close to the proof of Proposition 4.1 to be found in [11], with only slight modifications needed to replace functions by measures.

4.6 LEMMA *Let* μ *be a finite, positive Borel measure on* $\mathbb{C}$, *with* $\mu \perp m$. *Then* $B_*\mu \in L^1_*(\mathbb{C}, dm)$. *Furthermore, there is a constant* $C_2 > 0$ *which does not depend on* μ, *with*

$$\|B_*\mu\| \le C_2\|\mu\|.$$

Proof Take $t > 0$. We shall estimate $m(\{B_*\mu > t\})$. We set

$$\Omega = \bigcup\{Q : Q \in \mathscr{C}_t(\mu)\}, \quad \Omega^* = \bigcup\{Q^* : Q \in \mathscr{C}_t(\mu)\}.$$

By (3) we have $m(\Omega^*) \le 8\|\mu\|/t$.

We next show that $\mu(\mathbb{C} \setminus \Omega) = 0$. To see this, let F be a Borel set with $m(F) = 0$ on which μ is supported. It follows from the definition of Ω that for any dyadic square Q,

$$\mu(Q \setminus \Omega) \le tm(Q). \tag{4}$$

Given $\varepsilon > 0$, we can cover F with countably many dyadic squares (Q_n) satisfying

$$\sum_{n=1}^{\infty} m(Q_n) < \varepsilon/t. \tag{5}$$

Thus, by (4) and (5),

$$\mu(\mathbb{C} \setminus \Omega) \le \sum_{n=1}^{\infty} \mu(Q_n \setminus \Omega) \le t \sum_{n=1}^{\infty} m(Q_n) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\mu(\mathbb{C} \setminus \Omega) = 0$, as claimed.

We now set

$$g = \sum \{\mu_Q \chi_Q : Q \in \mathscr{C}_t(\mu)\}, \quad \beta = \mu - gdm.$$

Thus β is supported on Ω, $\|g\|_1 \le \|\mu\|$, $\|\beta\| \le 2\|\mu\|$, and

$$\beta(Q) = 0 \qquad (Q \in \mathscr{C}_t(\mu)). \tag{6}$$

Also

$$B_*\mu \le B_*g + B_*\beta,$$

and so

$$m(\{B_*\mu > t\}) \le m(\{B_*g > t/2\}) + m(\{B_*\beta > t/2\}).$$

Thus, by Proposition 4.1,

$$m(\{B_*\mu > t\}) \le 2C_1\|\mu\|/t + m(\{B_*\beta > t/2\}). \tag{7}$$

Since $m(\Omega^*) \le 8\|\mu\|/t$, it follows that

$$m(\{B_*\mu > t\}) \le (2C_1+8)\|\mu\|/t + m(\{z \in \mathbb{C} \setminus \Omega^* : (B_*\beta)(z) > t/2). \tag{8}$$

We now estimate $B_*\beta$ on $\mathbb{C} \setminus \Omega$. We first enumerate $\mathscr{C}_t(\mu)$ as $\{Q_1, Q_2, \dots\}$, and for each j, we denote the centre of the square Q_j by w_j. The fact that $\mathscr{C}_t(\mu)$ may be finite will not cause any difficulty in the following.

Take $\varepsilon > 0$, and $z \in \mathbb{C} \setminus \Omega$. Then

$$(B_\varepsilon\beta)(z) = \frac{1}{\pi} \int_{|w-z|\ge\varepsilon} \frac{d\beta(w)}{(w-z)^2} = \sum_j \frac{1}{\pi} \int_{Q_j\setminus\Delta(z,\varepsilon)} \frac{d\beta(w)}{(w-z)^2}. \tag{9}$$

To estimate $|(B_\varepsilon\beta)(z)|$, we proceed with a slight modification of the argument on pages 43, 44 of [11]. For each Q_j we consider separately the two cases:

(a) $Q_j \cap \Delta(z,\varepsilon) \neq \phi$;

(b) $Q_j \subseteq \mathbb{C} \setminus \Delta(z,\varepsilon)$.

In case (a), since $z \in \mathbb{C} \setminus Q^*$, we have $\operatorname{diam}(Q_j) \leq 2 \operatorname{dist}(z,Q_j) < 2\varepsilon$. Thus

$$Q_j \setminus \Delta(z,\varepsilon) \subseteq \Delta(z,3\varepsilon) \setminus \Delta(z,\varepsilon).$$

In case (b), we have $Q_j = Q_j \setminus \Delta(z,\varepsilon)$.

It now follows from (9) that

$$|(B_\varepsilon\beta)(z)|$$

$$\leq \frac{1}{\pi} \sum_j \left| \int_{Q_j} \frac{d\beta(w)}{(w-z)^2} \right| + \frac{1}{\pi} \int_{\Delta(z,3\varepsilon)\setminus\Delta(z,\varepsilon)} \frac{d|\beta|(w)}{|w-z|^2}$$

$$\leq \frac{1}{\pi} \sum_j \left| \int_{Q_j} \frac{d\beta(w)}{(w-z)^2} \right| + \frac{1}{\pi\varepsilon^2} |\beta|(\Delta(z,3\varepsilon)).$$

Thus

$$|(B_\varepsilon\beta)(z)| \leq \frac{1}{\pi} \sum_j \left| \int_{Q_j} \frac{d\beta(w)}{(w-z)^2} \right| + 9(M\beta)(z).$$

We now take the supremum over ε to obtain

$$|(B_*\beta)(z)| \leq \frac{1}{\pi} \sum_j \left| \int_{Q_j} \frac{d\beta(w)}{(w-z)^2} \right| + 9(M\beta)(z). \tag{10}$$

We set

$$f(z) = \frac{1}{\pi}\sum_j \left| \int_{Q_j} \frac{d\beta(w)}{(w-z)^2} \right| \qquad (z \in \mathbb{C} \setminus \Omega^*).$$

Since $\beta(Q_j) = 0$ for each j, we obtain

$$f(z) = \frac{1}{\pi}\sum_j \left| \int_{Q_j} \left[\frac{1}{(w-z)^2} - \frac{1}{(w_j-z)^2} \right] d\beta(w) \right| \qquad (z \in \mathbb{C} \setminus \Omega^*).$$

We now estimate the size of this integrand for each j. Clearly

$$|w-z| \geq |w_j-z|/2 \qquad (w \in Q_j,\ z \in \mathbb{C} \setminus Q_j^*).$$

Thus

$$\left| \frac{1}{(w-z)^2} - \frac{1}{(w_j-z)^2} \right| \leq |w-w_j| \sup_{\xi\in Q_j} \left\{ \frac{2}{|\xi-z|^3} \right\}$$

$$\leq \frac{16\ |w-w_j|}{|w_j-z|^3} \qquad (w \in Q_j).$$

This gives us the estimate

$$f(z) \leq \frac{16}{\pi} \sum_j \int_{Q_j} \frac{|w-w_j|}{|w_j-z|^3}\, d|\beta|(w) \qquad (z \in \mathbb{C} \setminus \Omega^*).$$

From (10) we have

$$m(\{z \in \mathbb{C} \setminus \Omega^* : (B_*\beta)(z) > t/2\})$$

$$\leq m(\{z \in \mathbb{C} \setminus \Omega^* : f(z) > t/4\}) + m(\{9(M\beta) > t/4\})$$

$$\leq \frac{4}{t} \left[\int_{\mathbb{C}\setminus\Omega^*} f(z)\ dm(z) + 162\ \|\mu\| \right],$$

by Proposition 4.4, since $\|\beta\| \le 2\|\mu\|$. But

$$\int_{\mathbb{C}\setminus\Omega^*} f(z)\, dm(z)$$

$$\le \frac{16}{\pi}\sum_j \int_{Q_j} \int_{\mathbb{C}\setminus Q_j^*} \frac{|w-w_j|}{|w_j-z|^3}\, dm(z)\, d|\beta|(w)$$

$$\le \frac{16}{\pi}\sum_j \int_{Q_j} \int_{|z-w_j|\ge \operatorname{diam}(Q_j)} \frac{|w-w_j|}{|w_j-z|^3}\, dm(z)\, d|\beta|(w)$$

$$= 32\sum_j \int_{Q_j} \frac{|w-w_j|}{\operatorname{diam}(Q_j)}\, d|\beta|(w) \le 16\|\beta\| \le 32\|\mu\| .$$

Thus

$$m(\{z \in \mathbb{C} \setminus \Omega^* : (B_*\beta)(z) > t/2\}) \le 776\|\mu\|/t \ . \tag{11}$$

To conclude the proof, we combine (8) and (11) to obtain

$$m(\{B_*\beta > t\}) \le (2C_1+784)\|\mu\|/t.$$

Take $C_2 = 2C_1+784$.

The general result is now an easy consequence of the above preliminary results.

4.7 THEOREM *There is a constant* $C_3 > 0$ *such that*

$$B_*\mu \in L^1_*(\mathbb{C}, dm) \quad \textit{and} \quad \|B_*\mu\| \le C_3\|\mu\| \qquad (\mu \in \mathcal{M}(\mathbb{C})).$$

Proof This follows from the previous result, and Proposition 4.1, on applying the Lebesgue and Jordan decomposition theorems.

We conclude this section with a proof that the Beurling transform maps $\mathcal{M}(\mathbb{C})$ into $L^1_*(\mathbb{C},dm)$.

4.8 THEOREM *Let* $\mu \in \mathcal{M}(\mathbb{C})$. *Then the limit in* (2) *exists a.e.* (m), *and the Beurling transform maps* $\mathcal{M}(\mathbb{C})$ *continuously into* $L^1_*(\mathbb{C},dm)$.

Proof By the Lebesgue decomposition theorem, $\mu = \sigma + f dm$ for some $\sigma \perp m$ and $f \in L^1(\mathbb{C},dm)$.

For each $\nu \in \mathcal{M}(\mathbb{C})$, and for each $w \in \mathbb{C}$ with $(B_*\nu)(w) < \infty$, we set

$$E\nu(w) = \limsup_{\varepsilon\to 0+} \mathrm{Re}(B_\varepsilon\nu)(w) - \liminf_{\varepsilon\to 0+} \mathrm{Re}(B_\varepsilon\nu)(w) + \limsup_{\varepsilon\to 0+} \mathrm{Im}(B_\varepsilon\nu)(w) - \liminf_{\varepsilon\to 0+} \mathrm{Im}(B_\varepsilon\nu)(w).$$

Clearly $0 \le (E\nu)(w) \le 4(B_*\nu)(w)$ a.e. (m).

We shall show that $(E\mu)(w) = 0$ a.e. (m). By Proposition 4.1, $E\mu = E\sigma$ a.e. (m), so we shall work with σ.

Set

$$\mathcal{S} = \left\{ z \in \mathbb{C} : \lim_{r\to 0+} \frac{1}{\pi r^2} |\sigma|(\Delta(z,r)) = 0 \right\},$$

$$\mathcal{T}_\varepsilon = \{w \in \mathbb{C}: (E\sigma)(w) \le \varepsilon\} \qquad (\varepsilon > 0),$$

By Proposition 4.5, $m(\mathbb{C} \setminus \mathcal{S}) = 0$. Take $z \in \mathcal{S}$, and $\varepsilon > 0$. We shall show that $\mathcal{T}_\varepsilon$ has full area density at z. Given $\eta > 0$, there exists $\delta > 0$ such that

$$\frac{1}{\pi r^2}|\sigma|(\Delta(z,r)) < \varepsilon\eta \qquad (0 < r < \delta).$$

Take $r \in (0,\delta)$, and set $\alpha = \chi_{\Delta(z,r)}\, d\sigma$. Then

$$(E\sigma)(w) = (E\alpha)(w) \le 4(B_*\alpha)(w)$$

a.e. (m) on $\Delta(z,r)$. Thus

$$m(\{w \in \Delta(z,r):(E\sigma)(w) > \varepsilon\}) \le m(\{B_*\alpha > \varepsilon/4\}) \le 4C_3\|\alpha\|/\varepsilon < 4C_3\pi r^2\eta.$$

and so

$$\frac{1}{\pi r^2}\, m(\mathcal{T}_\varepsilon \cap \Delta(z,r)) \ge 1 - 4C_3\eta \qquad (r < \delta).$$

This shows that $\mathcal{T}_\varepsilon$ has full area density at z.

Thus for each $\varepsilon > 0$, $\mathcal{T}_\varepsilon$ has full area density at a.e. (m) point of $\mathbb{C}$, and hence $m(\mathbb{C} \setminus \mathcal{T}_\varepsilon) = 0$. It follows that $(E\sigma)(w) = 0$ a.e. (m), as claimed.

The continuity of the Beurling transform is now a trivial consequence of Theorem 4.7.

5. DERIVATIONS FROM R(X)

With the theory developed in sections 3 and 4, we can now work towards a proof of Theorem 2.13.

In the following, let X be a compact plane set for which $R(X) \neq C(X)$, and let μ be a non-zero element of $\mathcal{M}(\mathbb{C})$ supported on X, and annihilating $R(X)$. It is standard that $\hat{\mu} = 0$ off X, and that it is not true that $\hat{\mu} = 0$ a.e. (m) on $\mathbb{C}$. Thus $\hat{\mu}|X$ is a non-zero element of $L^1_*(X,dm)$. The statement of equality in the next lemma is due to O'Farrell: it can be found on page 379 of [9], where the Beurling transform of a distribution with compact support is defined as a distribution. We shall supply a proof of the equality of the corresponding elements of $L^1_*(\mathbb{C},dm)$.

5.1 LEMMA *Let* $f \in R_0(X)$. *Then*

$$f'(z)\,\hat{\mu}(z) = B(f\mu)(z) - f(z)\,(B\mu)(z),$$

and

$$|f'(z)\,\hat{\mu}(z)| \le B_*(f\mu)(z) + |f(z)|(B_*\mu)(z)$$

a.e. (m) *on* X.

Proof We set

$$\mathcal{S} = \{z \in X : \tilde{\mu}(z) < \infty\}.$$

By the comment immediately following the definition of $\tilde{\mu}$, $|\mu|(\{z\}) = 0 \quad (z \in \mathcal{S})$.

Let $z_0 \in \mathcal{S}$. Then there exists $g \in R_0(X)$ with

$$g(w) = \frac{f(w) - f(z_0) - (w-z_0)f'(z_0)}{(w-z_0)^2} \qquad (w \in X \setminus \{z_0\}).$$

We know that both g and $f'(z_0)/(w-z_0)$ belong to $L^1(X,d|\mu|)$, that

$$\frac{1}{\pi}\int_X g(w)\,d\mu(w) = 0,$$

and that

$$\frac{1}{\pi}\int_X \frac{f'(z_0)}{w-z_0}\,d\mu(w) = f'(z_0)\,\hat{\mu}(z_0).$$

Since $|\mu|(\{z_0\}) = 0$, it follows that

$$(f(w)-f(z_0))/(w-z_0)^2 \in L^1(X,d|\mu|)$$

and that

$$f'(z_0)\,\hat{\mu}(z_0) = \frac{1}{\pi}\int_X \frac{f(w)-f(z_0)}{(w-z_0)^2}\,d\mu(w)$$

$$= \lim_{\varepsilon\to 0+} \frac{1}{\pi} \int_{X\setminus\Delta(z_0,\varepsilon)} \frac{f(w)-f(z_0)}{(w-z_0)^2}\, d\mu(w)$$

$$= \lim_{\varepsilon\to 0+} \left(B_\varepsilon(f\mu)(z_0) - f(z_0)\,(B_\varepsilon\mu)(z_0)\right),$$

because $|\mu|(\{z_0\}) = 0$ and μ is supported on X.

It is now clear that $f'(z)\,\hat{\mu}(z) = B(f\mu)(z) - f(z)(B\mu)(z)$ a.e. (m) on $\mathscr{S}$, and that

$$|f'(z)\,\hat{\mu}(z)| \le B_*(f\mu)(z) + |f(z)|(B_*\mu)(z) \quad (z \in \mathscr{S}).$$

Since $m(X \setminus \mathscr{S}) = 0$, the result now follows.

5.2 THEOREM *Let* X *be a compact plane set such that* $R(X) \neq C(X)$, *let* μ *be a non-zero element of* $\mathcal{M}(\mathbb{C})$ *supported on* X *such that* $\mu \perp R(X)$, *and set* $h = \hat{\mu}|X$. *Then the map*

$$D: f \mapsto f'.h,\quad R_0(X) \longrightarrow L^1_*(X, dm)$$

is a non-zero, continuous linear operator, and the extension $\bar{D}$ *of* D *to* $R(X)$ *is a non-zero, continuous derivation, given by*

$$\bar{D}(f) = B(f\mu)|X - f.(B\mu)|X \qquad (f \in R(X)).$$

Proof The continuity of the linear operator $\bar{D}$, and the fact that $\bar{D}$ extends D are immediate consequences of Lemma 5.1, and Theorems 4.7, 4.8. The derivation identity is obvious. Since $D(Z) = h$, a non-zero element of $L^1_*(X, dm)$, the result is proved.

Note that the continuity of D follows from Theorem 4.7, and does not need Theorem 4.8.

Theorem 2.13 now follows, because if X is a compact plane set with $R(X) \neq C(X)$, then Theorem 5.2 provides a non-zero continuous derivation from $R(X)$ into a locally bounded (F)-$R(X)$-module.

We shall conclude with another condition that $R(X)$ satisfies whenever it is non-trivial.

5.3 THEOREM *Let* X *be a compact plane set such that* $R(X) \neq C(X)$. *Then there exists* $C > 0$ *with*

$$\inf\{|f'(z)|: z \in X\} \leq C\ |f|_X \qquad (f \in R_0(X)). \tag{12}$$

Proof Let μ and D be as in 5.2. Then $D(f) = f'.D(Z)$ $(f \in R_0(X))$.

Set $Q = \{D(f): f \in R_0(X),\ |f|_X \leq 1\}$. Clearly Q is a bounded subset of $L^1_*(X, dm)$. Suppose, for a contradiction, that (12) is not satisfied for any $C > 0$. Let (f_n) be a sequence in $R_0(X)$ satisfying

$$|f_n|_X \leq 1, \quad \inf\{|f_n'(z)|: z \in X\} \geq n \qquad (n \in \mathbb{N}),$$

and set $g_n = 1/f_n'$ $(n \in \mathbb{N})$. Then $g_n \to 0$ uniformly on X, and so

$$D(Z) = g_n.f_n'.D(Z) = g_n.D(f_n) \to 0 \text{ as } n \to \infty.$$

Thus $D(Z) = 0$, a contradiction of 5.2.

Note in particular that if X is Wermer's Swiss cheese [12] for which $R(X)$ has no non-zero continuous point derivations, then condition (12) still holds.

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