THE RAIKOV CONVOLUTION MEASURE ALGEBRA

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1. INTRODUCTION

What follows is propaganda for the study of a particular commutative Banach algebra, A – one not inappropriate to a conference which emphasises automatic continuity. In fact A is that subalgebra of the measure algebra, $M(\mathbb{T})$, of all regular bounded Borel measures on the circle under the total variation norm and convolution multiplication, which is characterised by the automatic continuity of measurable characters:

$$A = \{\mu \epsilon M(\mathbb{T}) : \text{ If } \nu << \mu \text{ and } \chi \in (\mathbb{T}_d)^\wedge \text{ is } \nu\text{-measurable, then } \chi \text{ is continuous}\}$$

Here $\mathbb{T}_d$ denotes the circle group with the discrete topology and $(\mathbb{T}_d)^\wedge$ its (compact) dual group.

The challenge of A is that its Gelfand structure exhibits the delicate interplay of harmonic analysis and Banach algebra theory which one finds in the full measure algebra $M(\mathbb{T})$, while the cruder pathology which arises from thin sets is necessarily absent. Indeed A admits an alternative characterization as the collection, B, of all **basic measures** defined by

$$B = \{\mu \epsilon M(\mathbb{T}) : E \text{ Borel, } |\mu|(E) > 0 \Rightarrow gp(E) = \mathbb{T}\}$$

Here $gp(E)$ denotes the intersection of all subgroups of $\mathbb{T}$ which contain E. By way of an exercise let us note that, for a symmetric Borel set E, some n-fold sum, $(n)E$, has positive Haar measure if and only if $gp(E) = \mathbb{T}$.

It turns out that $A(= B)$ can further be described as the intersection of all those ideals of measures which annihilate symmetric Raikov systems. For this reason we like to call A the **Raikov convolution measure algebra** or the **Raikov kernel.** This makes it interesting to re-investigate the pioneering work of Raikov and Šreider defining certain complex homomorphisms (in fact idempotent generalized characters) of $M(\mathbb{T})$. It has long been known that the Šreider scheme for generating splittings of $M(\mathbb{T})$ into a direct sum of an ideal and a subalgebra is not subsumed by the Raikov scheme. Here we make the simple observation that nevertheless the Raikov and Šreider kernels coincide. Also we sketch a proof of the apparently new result that the simplest Raikov splitting is indeed obtained by the Šreider method.

It is by no means obvious that A contains interesting non-singular measures. We offer a novel and particularly simple proof of that fact in Proposition 1. Apart from the examples just cited, most of the other results mentioned have already appeared in some form in [3], [4] or [8]. Some related questions are discussed in Chapter VIII of [7] where basic measures appear as "very strongly continuous" measures. There is no analogue there of the defining property of A or of the Raikov kernel, but the reader will find other points of interest which are ignored here. For further background see [5], [6].

Finally we remark that the discussion permits extension from $\mathbb{T}$ to general locally compact abelian G. For propaganda purposes it is better to exhibit all phenomena in the simplest case.

2. MEASURABLE CHARACTERS

Theorem 1. *With the notation of the introduction, $A = B$ and this is a closed ideal of $M(\mathbb{T})$.*

Proof Suppose $\mu \epsilon M(\mathbb{T})\backslash B$. Then there is a Borel set E and some non-zero $\nu << \mu$ which is concentrated on $H = gp(E)$, $H \subsetneq \mathbb{T}$. Without loss of generality we may suppose that H is dense in $\mathbb{T}$. Now choose some $\chi \in (\mathbb{T}_d/H)\hat{}$ which is not identically one. Then χ is discontinuous but ν-measurable. Thus $\mu \notin A$ and we have checked that $A \subseteq B$.

Suppose next, with a view to obtaining a contradiction, that $B \not\subseteq A$. There exists some $\mu \in B$, $\nu << \mu$, and ν-measurable χ such that χ fails to be continuous. Since B is stable with respect to absolute continuity, we see that $\nu \in B$. From Lusin's theorem, we know that there is some Borel set E with positive ν-measure such that the restriction of χ is continuous on $gp(E)$. The set $gp(E)$ equals $\mathbb{T}$ because $\nu \in B$, so χ is everywhere continuous. This is the required contradiction.

Observe finally that B is evidently norm closed and that the ideal property of B follows because

$$\mu * \nu(E) = \int \mu(E - x)d\nu(x) \neq 0$$

implies that

$$\mu(E - x) \neq 0 \text{ for some } x.$$

When $\mu \in B$, we deduce that $\mathbb{T} = gp(E - x) = gp(E)$.

The next result can be deduced from [4], [1]:

Theorem 2. *The Raikov kernel, A, contains the Riesz product measures. Hence A exhibits the Wiener-Pitt phenomenon, A (even $A \cap M_o(\mathbb{T})$) has a maximal ideal off the Šilov* *boundary, and only analytic functions operate on A.*

REMARK 1) For present purposes a Riesz product is given by

$$d\mu = \lim_{n\to\infty} \prod_{k=1}^{n} (1 + r_k \cos 2\pi n_k t)dt, \text{ where } \inf \frac{n_k + 1}{n_k} > 3.$$

2) Cantor middle-third measure belongs to A – in fact for Borel sets E, F (see [3] and elaborations in [2]),

$$\mu(E)^\alpha \mu(F)^\alpha \leq m(E + F),$$

where μ is Cantor measure, m is Haar measure, $\alpha = \log 3/\log 4$.

Here is a simple proof that A contains non-trivial measures and, in consequence, exhibits the Wiener-Pitt phenomenon.

Proposition 1. $\nu_1 = \mathop{\ast}\limits_{n=1}^{\infty} \frac{1}{2}(\delta_{4^{-n}} + \delta_{-4^{-n}})$ *belongs to A.*

Proof Note first that Haar measure m can be expressed as follows:

$$m = \mathop{\ast}\limits_{n=1}^{\infty} \frac{1}{2}(\delta_{2^{-n}} + \delta_{-2^{-n}}).$$

(Of course δ_x denotes the point probability located at x and the quickest check is to consider Fourier transforms).

Now define ν_2 by

$$\int f(t)d\nu_2(t) = \int f(2t)d\nu_1(t) \quad (f \in C(\mathbb{T}))$$

so that

$$\nu_2 = \mathop{\ast}\limits_{n=1}^{\infty} \frac{1}{2}(\delta_{2.2^{-2n}} + \delta_{-2.2^{-2n}}).$$

Considering alternate terms in the infinite convolution for m, we now see that

$$m = \nu_1 * \nu_2.$$

Now suppose that $\chi \in (\mathbb{T}_d)\hat{}$ is ω_1-measurable where $\omega_1 << \nu_1$. We see that χ^2, given by $\chi^2(t) = \chi(2t)$, is ω_1-measurable and therefore that χ is ω_2-measurable, where $\omega_2 << \nu_2$ and ω_2 is related to ω_1 as ν_2 is related to ν_1. Now we see that $\chi(s+t) = \chi(s)\chi(t)$ is $(\omega_1 \times \omega_2)$-measurable and hence that χ is $(\omega_1 * \omega_2)$-measurable. But $\omega_1 * \omega_2$ is absolutely continuous with respect to m, and it follows that χ is continuous.

The following corollary is now standard:

Corollary. *The spectrum in A of ν_1 is the unit disc while the range of its Fourier-Stieltjes transform lies on the real line.*

Proof sketch Consider $\gamma_n(t) = \exp(\pi i 4^n t)$, $\gamma(t) = \exp(\pi i m t)$, for some fixed m. We see that

$$\begin{aligned}\hat{\nu_1}(\gamma_n\gamma) &= \prod_{k=1}^{\infty} \cos(\pi 4^{n-k} + \pi m 4^{-k}) \\ &= \prod_{k=1}^{\infty} \left[\cos(\pi 4^{n-k})\cos(\pi m 4^{-k}) - \sin(\pi 4^{n-k})\sin(\pi m 4^{-k})\right].\end{aligned}$$

Therefore, as $n \to \infty$,

$$\hat{\nu_1}(\gamma_n\gamma) \to (\prod_{j=1}^{\infty} \cos(\pi 4^{-j})) \prod_{k=1}^{\infty} \cos(\pi m 4^{-k}) = a\hat{\nu_1}(\gamma),$$

for some $0 < a < 1$.

We have just shown that γ_n converges to the constant a in the $\sigma(L^\infty(\nu_1), L^1(\nu_1))$-topology. This topology is consistent with that of the Gelfand space ΔA of A. Indeed elements ϕ of $\triangle A$ can be represented as $\phi = (\phi_\mu) \in \prod_{\mu \in A} L^\infty(\mu)$, and the Gelfand topology is the appropriate product topology. At this stage we have exhibited an element Ψ of $\triangle A$ with the property that $\Psi_{\nu_1} \equiv a$.

It remains to recall that certain algebraic operations are permissible on $\triangle A$. In fact, given $\phi = (\phi_\mu) \in \triangle A$, it turns out that $|\phi|$, defined coordinatewise by

$$|\phi|_\mu(t) = |\phi_\mu(t)| \quad (\mu \in A),$$

belongs to $\triangle A$. Moreover, for z with $\mathrm{re}(z) > 0, |\phi|^z$ given by

$$(|\phi|^z_\mu(t) = |\phi_\mu(t)|^z,$$

is a member of $\triangle A$.

Now we note that, for $\mathrm{re}(z) > 0, |\Psi|^z(\nu_1) = a^z$ belongs to the spectrum of ν_1. Varying z we see that the spectrum of ν_1 is indeed the entire unit disc.

3. RAIKOV SYSTEMS AND ŠREIDER SPLITTINGS.

A **Raikov system**, $\mathcal{R}$, is a family of F_σ subsets of $\mathbb{T}$ closed under countable unions, algebraic sums, translates, and F_σ subsets. Given such a system we define

$$A(\mathcal{R}) = \{\mu \in M(\mathbb{T}) : \mu \text{ is concentrated on some subset of } \mathcal{R}\}$$

$$I(\mathcal{R}) = \{\mu \in M(\mathbb{T}) : |\mu|(R) = 0 \quad (R \in \mathcal{R})\}.$$

Then

$$M(\mathbb{T}) = A(\mathcal{R}) \oplus I(\mathcal{R}),$$

where the direct sum is orthogonal, $A(\mathcal{R})$ is a subalgebra and $I(\mathcal{R})$ an ideal. This is called a Raikov splitting of $M(\mathbb{T})$.

In fact we shall consider only **symmetric** Raikov systems, that is, those where $R \in \mathcal{R} \Rightarrow -R \in \mathcal{R}$. While making this restriction we should note that there is extraordinary pathology associated with asymmetry in this context. In fact the present authors showed in [3] that there exists a basic measure μ concentrated on a compact set K which is thin in the sense that $m((n)K) = 0$ for every positive integer n. Thus **any** Borel set E which is charged by μ must satisfy $gp(E) = \mathbb{T}$.

In the example just discussed μ and all its convolution powers are supported by thin sets. The study of A emphasises rather the distinction between the distribution of mass associated with a measure and the support set of that measure. For example the measure ν_1, discussed in Proposition 1, has mutually singular convolution powers all of which are singular to Haar measure. (This is because the generalised character Ψ constructed in the proof of the corollary takes the constant value a^n on ν_1^n). At the same time the algebraic sums of any set on which ν_1 (or any power of ν_1) is concentrated expand to cover $\mathbb{T}$. Inasmuch as Raikov systems are designed to encapsulate the properties of the families of sets of concentration of typical L-subalgebras of $M(\mathbb{T})$ this warns us to expect them to describe only a small part of the Banach algebra structure.

We say that the Raikov system $\mathcal{R}$ is **proper** if $\mathbb{T}$ itself does not belong to $\mathcal{R}$.

Proposition 2. $A = B = \bigcap\{I(\mathcal{R}) : \mathcal{R}$ *is a proper symmetric Raikov system*$\}$

Proof Suppose that $\mu \in \bigcap I(\mathcal{R})$ but that $\mu \notin B$. Then there exists some Borel set E such that $|\mu|(E) > 0$ but $m(gp(E)) = 0$. We may, of course, replace E by some compact subset K which is charged by μ and consider the Raikov system, $\mathcal{R}(K)$, generated by that set. $\mathcal{R}(K)$ is proper and $\mu \notin I(\mathcal{R}(K))$, so we have a contradiction.

Suppose next that μ is basic and that $\mathcal{R}$ is a proper symmetric Raikov system. If $\mu \notin I(\mathcal{R})$ then $\mu_{|E} \neq 0$ for some E in $\mathcal{R}$. It follows that $gp(E) = \mathbb{T}$ and again we have a contradiction.

NOTATION Now let $\mathcal{S}$ be a subgroup of $(\mathbb{T}_d)\hat{}$.

$$A(\mathcal{S}) = \{\mu \in M(\mathbb{T}) : \gamma \text{ is } \mu - \text{measurable } (\gamma \in \mathcal{S})\}, I(\mathcal{S}) = A(\mathcal{S})^{\perp}.$$

Then

$$M(\mathbb{T}) = A(\mathcal{S}) \oplus I(\mathcal{S}),$$

where the direct sum is orthogonal and $A(\mathcal{S})$ is an algebra, $I(\mathcal{S})$ is an ideal. This is called a Šreider splitting of $M(\mathbb{T})$ and the proof is an immediate consequence of the next two propositions.

Proposition 3. *If $\mathcal{S}$ is countable then there is a Raikov system $\mathcal{R}$ such that $A(\mathcal{R}) = A(\mathcal{S}), I(\mathcal{R}) = I(\mathcal{S})$.*

Proof. We take $\mathcal{R}$ to be the collection of all F_σ subsets R of $\mathbb{T}$ such that $R = \bigcup_{n=1}^{\infty} R_n$, where R_n is compact and $\chi_{|R_n}$ is continuous for all χ in $\mathcal{S}$ and all n.

If ν is concentrated on $\bigcup_{n=1}^{\infty} R_n$ then, of course, χ is ν-measurable for every χ in $\mathcal{S}$. Thus $A(\mathcal{R}) \subseteq A(\mathcal{S})$.

Suppose next that μ is a probability measure belonging to $A(\mathcal{S})$. Fix some (countable) collection $(a(\chi))_{\chi \in \mathcal{S}}$ of positive numbers such that $\sum a(\chi) = 1$. Now for $\chi \in \mathcal{S}$, χ is μ-measurable so, for all n, there exists a compact set, $K(\chi, n)$, such that

$$\mu(K(\chi, n)) > 1 - a(\chi)n^{-1},$$

and the restriction of χ to $K(\chi,n)$ is continuous. Define $K_n = \bigcap_{\chi\in\mathcal{S}} K(\chi,\ n)$. Then $\mu(K_n) > 1 - n^{-1}$, K_n is compact and the restriction of χ to K_n is continuous for each χ in $\mathcal{S}$. Now, of course, $\bigcup_{n=1}^{\infty} K_n \in \mathcal{R}$ (by definition) and $\mu(\bigcup_{n=1}^{\infty} K_n) = 1$. Thus μ is concentrated on $\mathcal{R}$, i.e. $\mu \in A(\mathcal{R})$. Because $A(\mathcal{S})$ is obviously an L-space there was no real loss of generality in choosing μ to be a probability measure. Thus $A(\mathcal{S}) \subseteq A(\mathcal{R})$. Since we already know that $I(\mathcal{R}) = A(\mathcal{R})^{\perp}$, we now see that $I(\mathcal{S}) = I(\mathcal{R})$.

Proposition 4. *For general $\mathcal{S}$, (i) $A(\mathcal{S}) = \bigcap A(\mathcal{C})$ (ii) $I(\mathcal{S}) = \bigcup I(\mathcal{C})$, where $\mathcal{C}$ ranges over the countable subgroups of $\mathcal{S}$.*

Proof. (i) follows at once from the definition, so we concentrate on (ii). The first observation is that $\bigcup I(\mathcal{C})$ is an L-subspace of $M(\mathbb{T})$. From the foregoing each $I(\mathcal{C})$ is an L-subspace. For any sequence $(\mathcal{C}_n)$, it is clear that

$$\bigcup_n I(\mathcal{C}_n) \subseteq I(\bigcup \mathcal{C}_n),$$

and all the conditions to be satisfied by an L-space are stated in terms of (at most) sequences of elements. Now we have

$$\bigcup I(\mathcal{C}) \subseteq I(\bigcup \mathcal{C}) = I(\mathcal{S}),$$

where we know that both sides are L-spaces. Suppose that the inclusion is strict. There must exist some $\mu \in I(\mathcal{S})$ with $\mu \perp I(\mathcal{C})$ for every $\mathcal{C}$. This μ belongs therefore to $\bigcap A(\mathcal{C}) = A(\mathcal{S})$, and we have a contradiction.

Corollary. $\bigcap I(\mathcal{S}) = B = A$.

4. RELATIONSHIP BETWEEN THE SYSTEMS.

First we recall an old example:

EXAMPLE *Let K be a perfect independent subset of $\mathbb{T}$ and let μ be some continuous probability measure on K. Define*

$$\mathcal{S} = \{\chi \in (\mathbb{T}_d)\hat{}: \chi \textit{ is } \mu\text{-measurable }\}.$$

Then $A(\mathcal{S}) \neq A(\mathcal{R})$ for any Raikov system $\mathcal{R}$.

Sketch of justification Fix $\mathcal{R}$ and suppose that $A(\mathcal{S}) \subseteq A(\mathcal{R})$. Necessarily $\mu \in A(\mathcal{R})$ and so μ must be concentrated on some subset E of K which belongs to $\mathcal{R}$. Choose some continuous probability measure ν concentrated also on E but orthogonal to μ. Then $\nu \in A(\mathcal{R})$. There is some perfect set $K_1 \subseteq E$ such that $\nu(K_1) > 0$ but $\mu(K_1) = 0$, and K_1, in turn, contains a subset F, say, which is not ν-measurable. Since K is independent it is possible to find some $\chi \in (\mathbb{T}_d)\hat{}$ which is identically one on F and identically equal to some other number of unit modulus on $K \backslash F$. This χ is μ-measurable so belongs to $\mathcal{S}$ but is not ν-measurable so $\nu \notin A(\mathcal{S})$, and $A(\mathcal{R}) \neq A(\mathcal{S})$.

Proposition 5. *The simplest Raikov splitting of $M(\mathbb{T})$, as a direct sum of the discrete and continuous measures, is a Šreider splitting in which $\mathcal{S}$ can be taken as generated by a singleton.*

Proof sketch Let H be a Hamel basis for $\mathbb{T}$ over $\mathcal{Q}$ and list all perfect subsets of $\mathbb{T}$ as $\mathcal{K} = \{K_1, K_2, \cdots, K_\alpha, \cdots\}$. Then $H, \mathcal{K}$, and each member of $\mathcal{K}$, all have c elements. Take $H_1 \subseteq H$ so that H_1 is countable and $\text{span}(H_1) \cap K_1$ is a countable dense subset of K_1. Let x belong to $K_1 \backslash$ span (H_1) and fix some $\tilde{H}_1$ such that $\tilde{H}_1 \backslash H_1$ is countable but

$x \epsilon$ span $(\tilde{H}_1)$. Let $x_n \in K_1 \cap$ span (H_1) be such that x_n converges to x. Now fix some character χ_1 of span (H_1) and extend it to $\tilde{\chi}_1$ on span $(\tilde{H}_1)$ in such a way that $\tilde{\chi}_1(x_n) \not\to \tilde{\chi}_1(x)$. (The point is, of course, that we have associated a discontinuous character with K_1 and we want to do this for every K_α).

Suppose next that for all $\alpha < \beta$, we have $\tilde{H}_\alpha, \tilde{\chi}_\alpha$ so that $\tilde{\chi}_\alpha$ is a character of span $(\tilde{H}_\alpha)$ which is discontinuous on span $(\tilde{H}_\alpha) \cap K_\alpha$. In fact we further suppose that, for $\alpha < \alpha' < \beta$, $\tilde{H}_\alpha \subseteq \tilde{H}_{\alpha'} \subseteq H$, that card $\tilde{H}_\alpha < c$, and that $\tilde{\chi}_\alpha = \tilde{\chi}_{\alpha'}$ on H_α. Now let $J_\beta = \bigcup_{\alpha<\beta} \tilde{H}_\alpha$ so that card $J_\beta < c$. Choose $H_\beta \supseteq J_\beta$ with $H_\beta \subseteq H$, card $H_\beta < c$, and span $(H_\beta) \cap K_\beta$ a dense subset of K_β (with cardinality less than c). Let χ_β be a character on span (H_β) which extends each $\tilde{\chi}_\alpha, \alpha < \beta$. Next take $\tilde{H}_\beta \supseteq H_\beta$ such that $\tilde{H}_\beta \backslash H_\beta$ is countable and there is y in span $(\tilde{H}_\beta) \cap K_\beta$ outside span H_β. Let $y_n \in$ span $(H_\beta) \cap K_\beta$ converge to y and take some $\tilde{\chi}_\beta$ which extends χ_β but is such that $\tilde{\chi}_\beta(y_n) \not\to \tilde{\chi}_\beta(y)$.

We continue the transfinite induction just sketched until $\mathcal{K}$ is exhausted and we have arrived at $\tilde{\chi}$, say. It may be the case that H has not been exhausted. If so extend $\tilde{\chi}$ to χ a character of $\mathbb{T}_d = \text{span} H$ in any way. We have now found a single χ with the property that $\chi_{|K}$ is discontinuous for every K in $\mathcal{K}$.

Finally let μ be a continuous measure on $\mathbb{T}$ such that χ is μ measurable. By Lusin's theorem there is some compact set – and hence some perfect set, K, such that $\chi_{|K}$ is continuous. This contradiction establishes the result.

REMARK In order to focus attention on the Raikov kernel we have just devoted considerable discussion to structure of $M(\mathbb{T})$ **outside** that kernel. There is a further important remark of that type. The methods used to establish structural properties of the Gelfand space of the kind stated in Theorem 2 depend on the notion of 'tame' measure (cf.[1]). On the other hand we have shown that all the natural examples of tame measures belong to the Raikov kernel. That raises the possibility that some of the structural complexity of the Gelfand space of $M(\mathbb{T})$ is **confined** to the Raikov kernel! However Jane Lake, [8], has shown how to construct tame measures which are not basic; so the phenomenon persists outside the Raikov kernel.

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