32 Appendix The Eigenvalue Problem

In order to discuss the stabilities of a minimal surface, we need some general knowledge of the (Dirichlet) eigenvalues of a self-adjoint second order elliptic operator.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and L be a self-adjoint second order elliptic operator

$$Lu = D_i(a^{ij}D_ju + b^iu) - b^iD_iu + cu,$$

where (a^{ij}) is symmetric. We suppose that L satisfies

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbf{R}^n, \tag{32.186}$$

$$\sum |a^{ij}(x)|^2 \leq \Lambda^2, \quad 2\lambda^{-2}\sum_{i=1}^{n}\left(|b^i(x)|^2 + \lambda^{-1}|c(x)|\right) \leq \nu^2, \quad \forall x \in \Omega, \tag{32.187}$$

for some constants λ, Λ, $\nu > 0$.

Define $(u, v) = \int_\Omega uv\,dx$, and a quadratic form on $H = H(\Omega) = W_0^{1,2}(\Omega)$ by

$$\mathcal{L}(u, v) = \int_\Omega (a^{ij}D_iuD_jv + b^iuD_iv + b^ivD_iu - cuv)dx = -(Lu, v).$$

The ratio

$$J(u) = \frac{\mathcal{L}(u, u)}{(u, u)}, \quad u \not\equiv 0, \quad u \in H,$$

is called the Rayleigh quotient of L.

By (32.186) and (32.187) we see that J is bounded from below. In fact, writing $b = (b^1, \cdots, b^n)$ and $|b|^2 = \sum_i |b^i|^2$, we have

$$\begin{aligned}
\mathcal{L}(u, u) &= \int_\Omega (a^{ij}D_iuD_ju + 2b^iuD_iu - cu^2)dx \\
&\geq \int_\Omega \left[\lambda|Du|^2 - \left(\frac{1}{2}\lambda|Du|^2 + 2\lambda^{-1}|b|^2u^2 + cu^2\right)\right] dx \\
&\quad \text{(by (32.186) and Schwarz's inequality)} \\
&\geq \int_\Omega \left(\frac{1}{2}\lambda|Du|^2 - \lambda\nu^2|u|^2\right) dx \text{ (by (32.187))} \qquad (32.188) \\
&\geq \left(\frac{\lambda}{2}C^{-1} - \lambda\nu^2\right)\int_\Omega |u|^2\,dx \text{ (by Poincaré's inequality).}
\end{aligned}$$

Hence we may define

$$\lambda_1 = \inf_H J. \tag{32.189}$$

We claim now that λ_1 is the minimum eigenvalue of L on H; that is, there exists a non-trivial $u \in H$ such that $Lu + \lambda_1 u = 0$ and λ_1 is the smallest number for which this is possible. To show this we choose a minimizing sequence $\{u_m\} \subset H$ such that $\|u_m\|_{L^2} = 1$ and $J(u_m) \to \lambda_1$.

By (32.188) and $\|u_m\|_{L^2} = 1$, we have

$$\frac{\lambda}{2}\int_\Omega |Du_m|^2 dx \le 2\lambda\nu^2 + \mathcal{L}(u_m, u_m) \le 2(\lambda\nu^2 + |\lambda_1|),$$

hence $\{u_m\}$ is bounded in H. Thus by the compactness of the embedding $H \to L^2(\Omega)$, a subsequence, which we still note as $\{u_m\}$ itself, converges in $L^2(\Omega)$ to a function u with $\|u\|_{L^2} = 1$. Since $Q(u) = \mathcal{L}(u, u)$ is quadratic, we also have for any l, m,

$$Q\left(\frac{u_l - u_m}{2}\right) + Q\left(\frac{u_l + u_m}{2}\right) = \frac{1}{2}(Q(u_l) + Q(u_m)).$$

Since

$$\begin{aligned} Q\left(\frac{u_l + u_m}{2}\right) &= \mathcal{L}\left(\frac{u_l + u_m}{2}, \frac{u_l + u_m}{2}\right) \\ &\ge (\inf_H J)\left(\frac{u_l + u_m}{2}, \frac{u_l + u_m}{2}\right) = \lambda_1\left(\frac{u_l + u_m}{2}, \frac{u_l + u_m}{2}\right), \end{aligned}$$

we have

$$Q\left(\frac{u_l - u_m}{2}\right) \le \frac{1}{2}\left(Q(u_l) + Q(u_m)\right) - \lambda_1\left\|\frac{u_l + u_m}{2}\right\|_{L^2}^2 \to 0.$$

Again by (32.188),

$$\begin{aligned} \frac{\lambda}{2}\int_\Omega |D(u_l - u_m)|^2 dx &\le \mathcal{L}(u_l - u_m, u_l - u_m) + 2\lambda\nu^2\int_\Omega |u_l - u_m|^2 dx \\ &\le 4Q\left(\frac{u_l - u_m}{2}\right) + 2\lambda\nu^2\|u_l - u_m\|_{L^2}^2 \to 0, \end{aligned}$$

and so we see that $\{u_m\}$ is a Cauchy sequence in H. Hence $u_m \to u$ in H, and moreover $Q(u) = \lambda_1$.

Let $v \in H$ and consider

$$J(u + tv) = \frac{\mathcal{L}(u + tv, u + tv)}{(u + tv, u + tv)} = \frac{Q(u) + 2t\mathcal{L}(u, v) + t^2 Q(v)}{(u, u) + 2t(u, v) + t^2(v, v)}.$$

By (32.189), we have

$$0 = \left.\frac{dJ(u + tv)}{dt}\right|_{t=0} = \frac{2\mathcal{L}(u, v)(u, u) - 2(u, v)Q(u)}{(u, u)^2} = 2[\mathcal{L}(u, v) - \lambda_1(u, v)],$$

i.e.,

$$\int_\Omega (a^{ij}D_iuD_jv + b^iuD_iv + b^ivD_iu - cuv - \lambda_1 uv)dx = 0.$$

Integrating by parts we obtain

$$\int_\Omega \left[D_j\left(a^{ij}D_iu + b^ju\right) - b^iD_iu + cu + \lambda_1 u\right] vdx = \int_\Omega (Lu + \lambda_1 u)vdx = 0.$$

By the arbitrariness of $v \in H$, we must have $Lu + \lambda_1 u = 0$.

On the other hand, suppose $v \in H$ satisfies $Lv + \sigma v = 0$ (such a σ is called an *eigenvalue* and v is called an *eigenfunction* corresponding to σ). Then

$$0 = \int_\Omega (Lv + \sigma v) v dx = -\mathcal{L}(v, v) + \sigma(v, v).$$

We have

$$\sigma = J(v) \geq \inf_H J(u) = \lambda_1,$$

and thus λ_1 is the minimum eigenvalue.

Let λ be an eigenvalue, the *eigenspace* V_λ corresponding to λ is defined by

$$\{u \in H \mid Lu + \lambda u = 0\}.$$

If we arrange (as we will always do) the eigenvalues of L in increasing order λ_1, λ_2, $\cdots$, and designate their corresponding eigenspaces by V_1, V_2, $\cdots$, we may characterize the eigenvalues of L through the formula

$$\lambda_m = \inf\{J(u) \Big| u \not\equiv 0, \quad (u, v) = 0, \quad \forall v \in \{V_1, \cdots, V_{m-1}\}\}. \tag{32.190}$$

We summarize the above in the following result. Readers can refer to the books [21] (Theorem 8.37, p 214) and [10] (Chapter V, especially page 424).

Theorem 32.1 *Let L be a self-adjoint operator satisfying* (32.186) *and* (32.187). *Then L has a countably infinite discrete set of eigenvalues, $\Sigma = \{\lambda_m\}$, given by* (32.190). *Whose eigenfunctions span $W_0^{1,2}(\Omega)$. Furthermore,* $\dim V_m < \infty$ *and* $\lim_{m\to\infty} \lambda_m = \infty$.

We also need the Harnack inequality,

Theorem 32.2 (See [21] Corollary 8.21, page 199) *Assume L satisfies* (32.186) *and* (32.187), *$u \in W^{1,2}(\Omega)$ satisfies $u \geq 0$ in Ω, and $Lu = 0$ in Ω. Then for any $\Omega' \subset\subset \Omega$ we have*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u,$$

where $C = C(n, \Lambda/\lambda, \nu, \Omega', \Omega)$.

Theorem 32.3 *Given v_1, $\cdots$, $v_{k-1} \in H$, let*

$$\mu = \inf\{J(u) \mid u \in H, \ u \not\equiv 0, \ (u, v_i) = 0, \ 1 \leq i \leq k-1\}.$$

Then we have $\lambda_k \leq \mu$.

Proof. Take ϕ_i as the i-th eigenfunction corresponding to the i-th eigenvalue λ_i, $1 \leq i \leq k$. We can assume that ϕ_i's are orthonormal in $L^2(\Omega)$. We can select k constants $d_1, \cdots, d_k$, not all zero, such that

$$\sum_{i=1}^{k} d_i \int_\Omega \phi_i v_j dx = 0, \quad 1 \leq j \leq k-1.$$

Let $c_i = d_i(\sum_{j=1}^k d_j^2)^{-1/2}$ and define $f = \sum_{i=1}^m c_i\phi_i$. Then $(f, f) = \sum_{i=1}^k c_i^2 = 1$, and $(f, v_i) = 0$ for $1 \leq i \leq k-1$. By the definition of μ we have

$$\mu \leq J(f) = \mathcal{L}(f, f) = \sum_{i=1}^{k} c_i^2\lambda_i \leq \lambda_k \sum_{i=1}^{k} c_i^2 = \lambda_k.$$

□

Theorem 32.4 *Let $\Omega_1, \cdots, \Omega_m$ be pairwise disjoint domains in Ω. Considering the eigenvalue problem for each Ω_i and arrange all the eigenvalues of $\Omega_1, \cdots, \Omega_m$ in an increasing sequence*

$$v_1 \leq v_2 \leq \cdots$$

then we have

$$\lambda_k \leq v_k \quad \text{for} \quad k \geq 1.$$

Proof. Choose ψ_i to be the eigenfunction corresponding to v_i in the related domain and extend ψ_i by 0 such that $\psi_i \in H = W_0^{1,2}(\Omega)$ for $1 \leq i \leq k$. We can assume that the ψ_i's are orthonormal. For any $h_1, \cdots, h_{k-1} \in H$, as in the proof of Theorem 32.3 we can select c_i not all zero, and $f = \sum_{i=1}^k c_i\psi_i$ such that $(f, f) = 1$ and $(f, h_j) = 0$ for $1 \leq j \leq k-1$. If we select h_i as the i-th eigenfunction corresponding to λ_i, then by Theorem 32.3 and (32.190),

$$\lambda_k \leq J(f) = \mathcal{L}(f, f) = \sum_{i=1}^{k} c_i^2 v_i \leq v_k.$$

□

Combining the above with the Harnack inequality, we have an immediate corollary:

Corollary 32.5 *If $\Omega' \subset \Omega$, and the eigenvalues of L on $H(\Omega')$ are $\lambda_1', \lambda_2', \cdots$, then*

$$\lambda_m' \geq \lambda_m, \quad m = 1,\ 2,\ 3 \cdots.$$

If $\Omega' \subset \Omega$ is a proper subdomain, i.e., $\Omega - \overline{\Omega'}$ contains an non-empty open set, then

$$\lambda_m' > \lambda_m, \quad m = 1,\ 2,\ 3 \cdots.$$

Remark 32.6 We have neglected the boundary regularity of subdomains in the theorems, but it is true that if u on Ω' satisfies $Lu + \lambda u = 0$ and $u|_{\partial\Omega'} = 0$, then $u \in W_0^{1,2}(\Omega') \subset H$. See [5], page 21.

Let $\Sigma = S^2$ and $L = \Delta_\Sigma$ be the sphere Laplacian, $\Omega = \Sigma$ and $\partial\Omega = \emptyset$. Then it is well known that $\lambda_1 = 0$ and $\lambda_2 = 2$. Hence we have

Corollary 32.7 *Let $\Omega \subset \Sigma$ be a proper domain, then the second eigenvalue of the sphere Laplacian on Ω is larger than* 2.

Let u_m be the m-th eigenfunction corresponding to the m-th eigenvalue λ_m. Define the *nodal set* of u_m as $Z_m = \{x \in \Omega : u_m(x) = 0\}$.

Theorem 32.8 ([10], p 452) *Z_m divides the domain Ω into no more than m subdomains.*

Proof. Suppose Z_m divides Ω into more than m subdomains; label them as Ω_1, Ω_2, $\cdots$, Ω_k, $k > m$, and let $Z_m \cup \bigcup_{i=1}^k \Omega_i = \Omega$.

Since u_m does not change sign on each Ω_i, $1 \le i \le k$, Harnack's inequality tells us that $u_m \not\equiv 0$ on Ω_i (in fact, the nodal set has measure zero). Hence for each Ω_i, $1 \le i \le m$, we can define a $v_i \in H$ by $v_i = u_m$ on Ω_i, and $v_i = 0$ elsewhere. Define $w_i = \|v_i\|_{L^2}^{-1} v_i$, then $(w_i, w_i) = 1$. We see that w_i satisfies $Lw_i + \lambda_m w_i = 0$. Since $\int_\Omega w_i w_j dx = \delta_j^i$, $\{w_i\}_{i=1}^m$ is linearly independent.

For the $m-1$ eigenfunctions u_1, $\cdots$, u_{m-1} in H corresponding to the first $m-1$ eigenvalues, as in the proof of Theorem 32.3, we can select m constants c_1, $\cdots$, c_m, not all zero, such that

$$\sum_{i=1}^m c_i \int_\Omega w_i u_j dx = 0, \quad 1 \le j \le m-1,$$

and $\sum_{j=1}^m c_j^2 = 1$. Define $\phi = \sum_{i=1}^m c_i w_i$; then $(\phi, \phi) = \sum_{i=1}^m c_i^2 = 1$ and $(\phi, u_i) = 0$ for $1 \le i \le m-1$. Let $\Omega' = \operatorname{Int} \bigcup_{i=1}^m \overline{\Omega}_i$; then $\phi \in H(\Omega') \subset H(\Omega)$. Notice that Ω' is a proper subdomain of Ω, since the Ω_i are nonempty subdomains of Ω for $m+1 \le i \le k$.

By (32.190) we have

$$\begin{aligned} \lambda_m \le J(\phi) &= \mathcal{L}(\phi, \phi) = -\int_\Omega \phi L\phi dx = -\sum c_i c_j \int_\Omega w_i L w_j dx \\ &= -\sum_{i=1}^m c_i^2 \lambda_m \int_\Omega w_i^2 dx = \sum_{i=1}^m c_i^2 \lambda_m = \lambda_m. \end{aligned}$$

Hence ϕ is an eigenfunction corresponding to the m-th eigenvalue, but $\phi|(\Omega - \Omega') \equiv 0$ contradicts Harnack's inequality. This contradiction proves the theorem. □

Corollary 32.9 *The first eigenfunction ϕ_1 corresponding to the first eigenvalue does not change sign in Ω. All other eigenfunctions must change sign in Ω. Moreover,* $\dim V_{\lambda_1} = 1$.

Proof. ϕ_1 does not change sign by Theorem 32.8. This also shows that the eigenfunctions corresponding to the first eigenvalue must be either positive or negative, but two of them cannot orthogonal to each other, thus $\dim V_{\lambda_1} = 1$. Let ϕ_i be the i-th eigenfunction where $i > 1$, then by $(\phi_1, \phi_i) = 0$ we know that ϕ_i has to change sign in Ω. □