29 Shiffman's Theorems

Recall that we defined a CBA as a minimal annulus $A \in S(-1,1)$ such that $A(1) = A \cap P_1$ and $A(-1) = A \cap P_{-1}$ are continuous convex Jordan curves. In the article [76] published in 1956, Max Shiffman proved three elegant theorems about a CBA. They are as follows:

Theorem 29.1 *If A is a CBA, then $A \cap P_t$ is a strictly convex Jordan curve for every $-1 < t < 1$. In particular, $X : A_R \hookrightarrow S(-1,1)$ is an embedding.*

Theorem 29.2 *If A is a CBA and $\Gamma = \partial A$ is a union of circles, then $A \cap P_t$ is a circle for every $-1 \le t \le 1$.*

Theorem 29.3 *If A is a CBA and $\Gamma = \partial A$ is symmetric with respect to a plane perpendicular to xy-plane, then A is symmetric with respect to the same plane.*

We are going to prove the three Shiffman's theorems by means of the Enneper-Weierstrass representation. We have already proved a weaker version of Theorem 29.1, namely Theorem 27.2

Let us first prove Theorem 29.1. We follow the proof of Shiffman. We will write the immersion as $X = (x, y, z)$. For any $\zeta = re^{i\theta} \in A_R$, since X is conformal, by (27.124) we have

$$x_\theta^2 + y_\theta^2 = r^2(x_r^2 + y_r^2) + \frac{1}{(\log R)^2}.$$

The immersion $X : A_R \hookrightarrow S(-1,1)$ satisfies

$$x_\theta^2 + y_\theta^2 \ge \frac{1}{(\log R)^2}. \tag{29.150}$$

Since X is continuous on A_R, $A(1)$ and $A(-1)$ are convex and hence rectifiable. Moreover, $x(R,\theta)$ and $y(R,\theta)$ are functions of bounded variation. Thus $x_\theta(R,\theta)$ and $y_\theta(R,\theta)$ exist almost everywhere. Let I denote the set on which $x_\theta(R,\theta)$ and $y_\theta(R,\theta)$ both exist. We will first prove that:

Lemma 29.4 *For any $\theta \in I$,*

$$\lim_{r\to R} x_\theta(r,\theta) = x_\theta(R,\theta), \quad \lim_{r\to R} y_\theta(r,\theta) = y_\theta(R,\theta), \tag{29.151}$$

and

$$x_\theta^2(R,\theta) + y_\theta^2(R,\theta) \ge \frac{1}{(\log R)^2}. \tag{29.152}$$

Proof. Let $\overline{x}$ and $\overline{y}$ be harmonic functions defined over the disk $D_R := r \leq R$ with boundary values given by $x(R,\theta)$ and $y(R,\theta)$ respectively. The functions $x(r,\theta)-\overline{x}(r,\theta)$ and $y(r,\theta)-\overline{y}(r,\theta)$, being harmonic in A_R and having the boundary value 0 on $r=R$, can be extended across $r=R$ by reflection. Thus

$$x_\theta(r,\theta)-\overline{x}_\theta(r,\theta)\to 0,\quad y_\theta(r,\theta)-\overline{y}_\theta(r,\theta)\to 0,$$

as $r\to R$.

Let P be the Poisson kernel of D_R,

$$P(Re^{i\phi},re^{i\theta})=\frac{1}{2\pi}\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\phi-\theta)}.$$

Then the harmonic function $\overline{x}$ can be expressed as

$$\overline{x}(r,\theta)=\int_0^{2\pi}x(R,\phi)P(Re^{i\phi},re^{i\theta})d\phi.$$

Differentiating, we have

$$\overline{x}_\theta(r,\theta)=\int_0^{2\pi}x(R,\phi)\frac{\partial P}{\partial\theta}d\phi=-\int_0^{2\pi}x(R,\phi)\frac{\partial P}{\partial\phi}d\phi=\int_0^{2\pi}Pdx(R,\phi).$$

It follows, as in the proof of theorem of Fatou (see [59] pages 198 -200) that

$$\lim_{r\to R}\overline{x}_\theta(r,\theta)=x_\theta(R,\theta)$$

on I. Similarly for y_θ. From (29.150) it is obvious that (29.152) is true. □

Consider the harmonic function $\psi(r,\theta)$, the angle of the tangent vector of $A\cap P_t$ with the positive x-axis. We denote the angle defined by the tangent direction at $A(1)$ by $\psi(R,\theta)$ on I. Because of the convexity of $A(1)$, $\psi(R,\theta)$ is a monotonic function of θ on I of period $\pm 2\pi$. We can assume that the period is 2π, and we shall call the orientation described on $A(1)$ as θ varies from 0 to 2π the positive orientation of $A(1)$. The following lemma will be proved.

Lemma 29.5 *The period of $\psi(r,\theta)$ is exactly 2π, and*

$$\lim_{r\to R}\psi(r,\theta)=\psi(R,\theta)\quad\text{for }\theta\in I.$$

The single valued function $\psi(r,\theta)-\theta$ is a bounded harmonic function in A_R.

Proof. Consider the convex curve $A(1)$. Select a point Q_1 on $A(1)$ at which there is a unique supporting line L of $A(1)$, and let Q_3 be a point on $A(1)$ where a line parallel to L, but not coinciding with L, is a supporting line of $A(1)$. Select a direction not included among the directions of all the supporting lines of $A(1)$ at the point Q_3 and let Q_2 and Q_4 be two points of $A(1)$ at which there are supporting lines, distinct from each other, in this direction. The numbering is such that Q_1, Q_2, Q_3, Q_4 occur in the positive orientation around $A(1)$. Consider these four supporting lines as taken in the positive direction in describing $A(1)$, and let angles made by them with the positive x-axis be α_1, α_2, $\alpha_1+\pi$, $\alpha_2+\pi$, respectively, where

$$\alpha_1 < \alpha_2 < \alpha_1 + \pi < \alpha_2 + \pi.$$

Let the points on the circle $r = R$ which are mapped onto Q_1, Q_2, Q_3, Q_4, be denoted by q_1, q_2, q_3, q_4, respectively. On the circle $r = R$ denote the open arc from q_1 to q_3 (taken in the positive orientation and therefore including q_2) by B_1, the open arc from q_2 to q_4 by B_2, from q_3 to q_1 by B_3, and from q_4 to q_2 by B_4. Finally, let C_i be a closed arc on $r = R$ contained in B_i, $i = 1, 2, 3, 4$, such that the C_i together cover $r = R$.

Note that

$$(x_\theta, y_\theta) = (x_\theta^2 + y_\theta^2)^{1/2}(\cos\psi, \sin\psi),$$
$$(y_\theta, -x_\theta) = (x_\theta^2 + y_\theta^2)^{1/2}(\sin\psi, -\cos\psi).$$

Consider first the function

$$Y_1(r,\theta) = y(r,\theta)\cos\alpha_1 - x(r,\theta)\sin\alpha_1, \tag{29.153}$$

which is a harmonic function of (r,θ) in A_R. Then

$$\begin{aligned}\frac{\partial Y_1(r,\theta)}{\partial\theta} &= y_\theta(r,\theta)\cos\alpha_1 - x_\theta(r,\theta)\sin\alpha_1\\ &= (y_\theta, -x_\theta)(r,\theta)\bullet(\cos\alpha_1, \sin\alpha_1)\\ &= (x_\theta^2 + y_\theta^2)^{1/2}(\sin\psi, -\cos\psi)\bullet(\cos\alpha_1, \sin\alpha_1)\\ &= (x_\theta^2 + y_\theta^2)^{1/2}\sin(\psi - \alpha_1).\end{aligned} \tag{29.154}$$

On the arc B_1 of $r = R$, the function $Y_1(R,\theta)$ is a monotonically increasing function of θ, since the arc B_1 corresponds to the portion of $A(1)$ from Q_1 to Q_3; thus $\alpha_1 \le \psi \le \alpha_1 + \pi$. In analogy to the proof of Lemma 29.4, the formula for $\frac{\partial \overline{Y}_1(r,\theta)}{\partial\theta}$ is

$$\frac{\partial \overline{Y}_1(r,\theta)}{\partial\theta} = \left(\int_{B_1} + \int_{CB_1}\right) P dY_1(R,\phi) \tag{29.155}$$

where CB_1 is the complement of B_1. The first integral in (29.155) is ≥ 0 for all (r,θ), since $Y_1(R,\phi)$ is an increasing function of ϕ in B_1; the second integral in (29.155) approaches 0 as (r,θ) approaches an interior point of B_1. Thus

$$\lim\ \inf_{(r,\theta)\to C_1} \frac{\partial \overline{Y}_1(r,\theta)}{\partial\theta} \ge 0.$$

It follows that likewise

$$\lim\ \inf_{(r,\theta)\to C_1} \frac{\partial Y_1(r,\theta)}{\partial\theta} \ge 0. \tag{29.156}$$

Take a positive ϵ and $\epsilon_1 = (\log R)^{-1}\epsilon$ such that

$$\delta = \arcsin \epsilon < \min\left(\frac{\alpha_2 - \alpha_1}{2}, \frac{\alpha_1 + \pi - \alpha_2}{2}\right).$$

By (29.156) there is a region R_1 in A_R, enclosing C_1, for which

$$\frac{\partial Y_1(r,\theta)}{\partial \theta} > -\epsilon_1, \quad (r,\theta) \in R_1.$$

From (29.150) and (29.154) we therefore see that

$$\sin(\psi - \alpha_1) > -\epsilon \quad \text{in} \quad R_1. \tag{29.157}$$

Selecting a determination of ψ at a particular point of R_1, we have

$$-\delta < \psi - \alpha_1 < \pi + \delta \quad \text{in} \quad R_1. \tag{29.158}$$

A similar argument applies to each of the other arcs B_2, B_3, B_4 of the circle $r = R$, with α_2, $\alpha_1 + \pi$, $\alpha_2 + \pi$, respectively, replacing α_1 in (29.153)-(29.158). On B_2 the function $Y_2 = y\cos\alpha_2 - x\sin\alpha_2$ is an increasing function of θ, leading to the result

$$\lim \inf_{(r,\theta)\to C_2} \frac{\partial Y_2(r,\theta)}{\partial \theta} \geq 0.$$

There is, therefore, a region R_2 of A_R, enclosing C_2, for which

$$\frac{\partial Y_2(r,\theta)}{\partial \theta} > -\epsilon_1, \quad (r,\theta) \in R_2.$$

And we have, analogously to (29.157),

$$\sin(\psi - \alpha_2) > -\epsilon \quad \text{in} \quad R_2.$$

But this means, from (29.158), begin with an already determined ψ in the region common to R_1, R_2, that

$$-\delta < \psi - \alpha_2 < \pi + \delta \quad \text{in} \quad R_2.$$

Similar arguments apply successively to the determination of the regions R_3, R_4 and of the corresponding inequalities for ψ:

$$-\delta < \psi - (\alpha_1 + \pi) < \pi + \delta \text{ in } R_3, \quad -\delta < \psi - (\alpha_2 + \pi) < \pi + \delta \text{ in } R_4. \tag{29.159}$$

Therefore, in the portion common to R_4 and R_1, the value of ψ returns to its initial value plus exactly 2π, or the period of ψ is exactly 2π.

The regions R_1, R_2, R_3, R_4, together form a neighbourhood of the circle $r = R$ in A_R.

A similar argument as the above applies to the inner circle $r = 1/R$ and $A(-1)$. By continuity, ψ has period 2π for every $1/R \leq r \leq R$.

Let θ be a value in the set I and take the limit of $\psi(r,\theta)$ as $r \to R$. By (29.151), (29.152) the limit of $\psi(r,\theta)$ is $\psi(R,\theta)$ modulo 2π. But the inequalities (29.157), (29.158), (29.159) show that the limit must be exactly $\psi(R,\theta)$. The lemma is proved. □

We can now establish the inequality

$$\psi_\theta(r,\theta) > 0 \tag{29.160}$$

everywhere in the interior of A_R. Let $G = G(R,\phi,r,\theta)$ be the Green's function for the annular ring A_R, with singularity at (r,θ). In its dependence on ϕ and θ, G is a function of $\phi - \theta$. We have

$$\begin{aligned} \psi(r,\theta) - \theta &= \int_{\partial A_R} [\psi(r,\phi) - \phi] \frac{\partial G}{\partial \nu} ds \qquad (29.161) \\ &= \int_{r=R} [\psi(R,\phi) - \phi] R \frac{\partial G}{\partial \nu} d\phi + \int_{r=R^{-1}} [\psi(R^{-1},\phi) - \phi] R^{-1} \frac{\partial G}{\partial \nu} d\phi, \end{aligned}$$

where ν is the inward normal. This follows by considering the analogous formula for an interior annular ring, and performing the passage to the limit. Differentiating (29.162) with respect to θ, using $\partial(\partial G/\partial\nu)/\partial\theta = -\partial(\partial G/\partial\nu)/\partial\phi$, and intergrating by parts, we find

$$\psi_\theta - 1 = \int_{r=R} R \frac{\partial G}{\partial \nu} d[\psi(R,\phi) - \phi] + \int_{r=R^{-1}} R^{-1} \frac{\partial G}{\partial \nu} d[\psi(R^{-1},\phi) - \phi]$$

or

$$\psi_\theta = \int_{r=R} \frac{\partial G}{\partial \nu} R\, d\psi(R,\phi) + \int_{r=R^{-1}} \frac{\partial G}{\partial \nu} R^{-1} d\psi(R^{-1},\phi),$$

since

$$\int_{r=R} \frac{\partial G}{\partial \nu} R d\phi + \int_{r=R^{-1}} \frac{\partial G}{\partial \nu} R^{-1} d\phi = \int_{\partial A_R} \frac{\partial G}{\partial \nu} ds = 1.$$

Since $\partial G/\partial\nu > 0$ and $\psi(R,\phi)$, $\psi(R^{-1},\phi)$ are monotonic increasing functions of ϕ of period 2π, inequality (29.160) is obtained. Thus each $A(t)$ is a closed strictly convex curve and has total curvature 2π, so it must be a Jordan curve. Therefore, X must be an embedding. Theorem 29.1 is proved. □

Theorem 29.2 is a special case of Theorem 30.1 in the next section, so we will postpone the proof until then. Instead we will prove Theorem 29.3 next.

Proof of Theorem 29.3 : We can assume that ∂A is symmetric with respect to the xz-plane. By Theorem 29.1, each $A(z)$ is a strictly convex Jordan curve for $-1 < z < 1$; hence there are exactly two points on $A \cap P_z$ at which the supporting lines of $A(z)$ are perpendicular to the xz-plane. Varying z we get two curves on A, say α_1 and α_2. Let P be the orthogonal projection on the xz-plane. The A consists of two pieces of graphs on the domain $\Omega = P(A) \subset xz$-plane, thus we have $(x, y_i(x,z), z)$, $i = 1, 2$. Moreover, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the projection of $A(1) \cup A(-1)$ and $\Gamma_2 = P(\alpha_1 \cup \alpha_2)$. It is clear that on Γ_2 the graphs $(x, y_i(x,z), z)$ are perpendicular to the xz-plane.

Now assume that $A(1)$ and $A(-1)$ are strictly convex. Reflecting the graph generated by y_2 about the xz-plane we get a minimal graph generated by $\tilde{y}_2 = -y_2 : \Omega \to \mathbf{R}$. On Γ_1, we have $\tilde{y}_2 = y_1$ by the boundary symmetry. A theorem of Giusti ([22] Lemma 2.2) says that if $(x, y_1(x,z), z)$ and $(x, \tilde{y}_2(x,z), z)$ are perpendicular to the xz-plane on Γ_2 and $y_1 \geq \tilde{y}_2$ on Γ_1, then $y_1 \geq \tilde{y}_2$ on Ω. Since $y_1 = \tilde{y}_2$ on Γ_1, we have $y_1 = \tilde{y}_2$ in Ω.

If $A(1)$ or $A(-1)$ is not strictly convex, then by continuity of the surface, we know that for any $\epsilon > 0$ there is a $\delta > 0$ so small that $y_1(x, 1-t) \geq \tilde{y}_2(x, 1-t) - \epsilon$ and $y_1(x, -1+t) \geq \tilde{y}_2(x, -1+t) - \epsilon$ for $0 < t < \delta$. Thus on $\Omega \cap \{(x,z) \,\big|\, -1+\delta < z < 1-\delta\}$, $y_1 \geq \tilde{y}_2$. Letting $\epsilon \to 0$, we have $y_1 \geq \tilde{y}_2$ in Ω. Changing the role of y_1 and $\tilde{y}_2$, we have $y_1 = \tilde{y}_2$ in Ω.

But $y_1 = \tilde{y}_2$ means that A is symmetric about the xz-plane, the proof is complete. □