27 Minimal Annuli in a Slab

Recall that a catenoid is a rotation surface, hence is foliated by circles in parallel planes. A good question to ask is whether there are other minimal annuli which can be foliated by circles. It was B. Riemann [72] and Enneper [14] who solved this problem very satisfactorily. The answer is that there is only one one-parameter family of such surfaces up to a homothety. Each minimal annuli in this one-parameter family is contained in a slab and foliated by circles, and its boundary is a pair of parallel straight lines. Rotating repeatly about these boundary straight lines gives a one-parameter family of singly periodic minimal surface; these surfaces are called *Riemann's examples.*

For the details of the proof of existence and other properties of Riemann's examples, see [61], section 5.4, Cyclic minimal surfaces. For constructions of Riemann's examples using the Weierstrass functions please see [25]. It is also known that a pair of parallel straight lines can only bound a piece of Riemann's example, if they bound any minimal annulus at all, see for example, [17].

Now we are going to study minimal annuli in a slab. Let $P_t = \{(x,y,z) \in \mathbf{R}^3 | z = t\}$ and $S(t_1,t_2) = \{(x,y,z) \in \mathbf{R}^3 | t_1 \leq z \leq t_2,\ t_1 < t_2\}$. Consider a minimal annulus $X : A_R \hookrightarrow S(t_1,t_2)$ such that $X(\{|z| = 1/R\}) \subset P_{t_1}$, $X(\{|z| = R\}) \subset P_{t_2}$ and X is continuous on A_R. We will call such a minimal annulus a *minimal annulus in a slab*. By a homothety we can normalize t_1 and t_2 such that $t_1 = -1$ and $t_2 = 1$. We will denote the image $X(A_R) \subset S(-1,1)$ by A and let $A(t) = A \cap P_t$ for $-1 \leq t \leq 1$. When discussing a minimal annulus in a slab, we often just refer to it by the image $A = X(A_R)$.

We want to derive the Enneper-Weierstrass representation of a minimal annulus in a slab. Let A be a minimal annulus in a slab. The third coordinate function X^3 is harmonic, $X^3|_{\{|z|=1/R\}} = -1$, and $X^3|_{\{|z|=R\}} = 1$. By uniqueness of solutions to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathrm{Int}(A_R) \\ u|_{\{|z|=1/R\}} = -1, & u_{|\{|z|=R\}} = 1, \end{cases}$$

where $\mathrm{Int}(A_R)$ is the interior of A_R, we have $X^3 = \frac{1}{\log R}\log|z|$, and

$$\omega_3 = f(z)g(z)dz = 2\frac{\partial}{\partial z}X^3\,dz = \frac{d}{dz}\left(\frac{1}{\log R}\log z\right)dz = \frac{1}{\log R}\frac{1}{z}\,dz.$$

Hence $f(z) = \frac{1}{\log R}\frac{1}{zg(z)}$. Here of course g is the Gauss map in the Enneper-Weierstrass representation and $f(z)dz = \eta$. Thus by (6.26) we have

$$\begin{cases} \omega_1 &= \frac{1}{\log R}\frac{1}{2z}(\frac{1}{g} - g)dz \\ \omega_2 &= \frac{1}{\log R}\frac{i}{2z}(\frac{1}{g} + g)dz \\ \omega_3 &= \frac{1}{\log R}\frac{1}{z}dz. \end{cases} \tag{27.124}$$

The immersion is given by

$$X(p) = \frac{1}{\log R}\Re\int_1^p \left(\frac{1}{2z}(\frac{1}{g} - g),\ \frac{i}{2z}(\frac{1}{g} + g),\ \frac{1}{z}\right) dz + C, \tag{27.125}$$

where $C = (a, b, 0) \in \mathbf{R}^3$. Since X is well defined, for $\gamma = \{|z| = 1\} \subset A_R$,

$$\Re\int_\gamma \left(\frac{1}{2z}(\frac{1}{g} - g),\ \frac{i}{2z}(\frac{1}{g} + g),\ \frac{1}{z}\right) dz = \vec{0}. \tag{27.126}$$

On the other hand, if g and f are meromorphic and holomorphic functions in A_R, such that (27.124) defines three holomorphic 1-forms and (27.126) is satisfied, then (27.125) defines a minimal annulus in the slab $S(-1, 1)$.

The conformal factor of a minimal annulus in a slab is

$$\Lambda^2 = \frac{1}{4(\log R)^2|z|^2}\left(\frac{1}{|g|} + |g|\right)^2, \tag{27.127}$$

and the Gauss curvature is

$$K = -\left[\frac{4\log R|z||g||g'|}{(1 + |g|^2)^2}\right]^2. \tag{27.128}$$

One observation about the Gauss map of a minimal annulus in a slab is:

Proposition 27.1 *Let A be a minimal annulus in a slab such that X is smooth up to the boundary (in fact, C^2 will be enough), then the Gauss map g of A has no zeros or poles on A_R. Furthermore, $|g|$ and $|g|^{-1}$ are both bounded.*

Proof. From (27.125) we see that for any $-1 \le t \le 1, A(t) = A \cap P_t$ is the image $X(\{|z| = R^t\})$. From Corollary 4.5 we get immediately that g has no zeros or poles in $\mathrm{Int}(A_R)$, because otherwise the preimage of $A(t)$ will have an equiangular system of at least order 4 at the pole or zero points.

Since X is continuous on A_R, A is compact. It remains only to prove that on the boundary of A, the Gauss map N is not perpendicular to the xy-plane. Since our boundary is smooth, the projection of the boundary into the xy plane satisfies the sphere condition, inner or outer. By boundary regularity theory, X is $C^{1,\alpha}, \alpha \in (0, 1)$, up to the boundary (see [12], Vol. 2, Chapter 7), hence at every boundary point there is a well defined normal direction.

Near any boundary point p that has a vertical normal, the surface is a graph over a small open disk $D \subset P_1$ with p on ∂D, assuming that $p \in A(1)$. Then we can use the minimal surface equation (4.8). We write $(x, y, z) \in A$, where $z = z(x, y)$ satisfies

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy} = 0.$$

Since X^3, the third coordinate function of A, is harmonic, by the maximum principle we have for any $(x,y) \in D$ that $z(x,y) < 1 = z(p)$. Define a uniformly elliptic operator on a smaller domain if necessary,

$$Lu = (1+z_y^2)u_{xx} - 2z_x z_y u_{xy} + (1+z_x^2)u_{yy}.$$

Then z satisfies $Lz = 0$. It is well known that

$$\frac{\partial z}{\partial \nu}(p) > 0,$$

where ν is the outward normal to ∂D at p (see [21], Lemma 4, page 34). But this means that the normal is not vertical. This contradiction proves that N is never vertical on the boundary of A. Since X is smooth, g and g^{-1} are continuous up to boundary (we can see this by $g = \tau \circ N$); hence by the maximum principle both $|g|$ and $|g|^{-1}$ are bounded. $\square$

If a minimal annulus A in a slab satisfies that $A(-1)$ and $A(1)$ are continuous convex Jordan curves, we will call A a *convex boundary minimal annulus* or *CBA*.

Theorem 27.2 *If A is a CBA and $\Gamma = \partial A$ is smooth, then $A \cap P_t$ is a strictly convex Jordan curve for every $-1 < t < 1$. In particular, $X : A_R \hookrightarrow S(-1,1)$ is an embedding.*

Proof. By Proposition 9.2 we have $A = X(A_R)$ for some $R > 1$. And by regularity theory, X is smooth up to the boundary. At any point of $A(t) = A \cap P_t$, $-1 \le t \le 1$, draw a tangent vector to the curve $A(t)$, and let ψ be the angle made by this tangent vector with the positive x-axis. The ψ may be a multivalued function, but we will see that it is harmonic. To see this, consider the unit normal vector $\vec{n}$ of the curve $A(t)$, and its angle with the positive x-axis ϕ. If we orient the surface such that the normal is inward to the unbounded component of $S(-1,1) - A$, then we have $\psi = \phi + \pi/2$. By Proposition 27.1, $g \neq 0$ or ∞ on A_R, hence the unit normal vector $\vec{n}$ must be $\frac{g}{|g|} \in \mathbf{C} \cong \mathbf{R}^2$ in complex form. Because $\phi = \arg g = \Im \log g$, ϕ is harmonic and so is ψ.

Now suppose that s is the arc length parameter of the curve $A(t)$ and notice that by (27.125) $X^{-1}(A(t)) = \{z : |z| = r = R^t\}$. Writing $z = re^{i\theta}$, we can calculate the curvature of A_t as follows:

$$\begin{aligned}\kappa &= \psi_s = \phi_s = \frac{d}{ds}(\Im \log g) = \Im\left(\frac{d}{ds}\log g\right) = \Im\left(\frac{d}{dz}\log g \frac{dz}{ds}\right)\\ &= \Im\left(\frac{g'}{g}\frac{dz}{d\theta}\frac{d\theta}{ds}\right) = \Im\left(\frac{g'}{g} izr^{-1}\Lambda^{-1}\right) = r^{-1}\Lambda^{-1}\Re\left(z\frac{g'}{g}\right).\end{aligned}$$

Here we have used the facts that on the curve $\{|z| = r = R^t\}$,

$$\frac{dz}{d\theta} = ire^{i\theta} = iz, \quad \text{and} \quad ds = \Lambda|dz| = \Lambda r d\theta.$$

Since $h = \Re\left(z\frac{g'}{g}\right) = r\Lambda\kappa$ is harmonic and $r\Lambda > 0$, we see that if Γ is smooth (in fact $C^{2,\alpha}$ is enough) convex then $h \geq 0$ on ∂A_R, and hence by the maximum principle, $h > 0$ in $\text{Int}(A_R)$ and so $\kappa = r^{-1}\Lambda^{-1}h$ is also positive. Thus $A(t)$ is locally strictly convex. Since $\Gamma = A(1) \cup A(-1)$ consists of two Jordan curves, we have

$$\int_{|z|=R^{\pm 1}} \kappa ds = 2\pi.$$

By continuity it must be that

$$\int_{|z|=R^t} \kappa ds = 2\pi \quad \text{for} \quad -1 \leq t \leq 1.$$

This proves that $A(t)$ must be simple. Since $\kappa > 0$ on $A(t)$, we conclude that $A(t)$ is a strictly convex Jordan curve for $-1 < t < 1$. □

Remark 27.3 We have used the non-trivial regularity theorem which says that if ∂A is $C^{2,\alpha}$ then $X : A_R \hookrightarrow S(-1,1)$ is also $C^{2,\alpha}$. See [12] II, Theorem 1, page 33.

Theorem 27.4 *Let A be a CBA and ∂A be smooth. Then there is a $\rho > 1$ such that the Gauss map $g : A_R \to \mathbf{C}$ is a conformal diffeomorphism to $\overline{\Omega} \subset A_\rho = \{z \in \mathbf{C} : 1/\rho \leq |z| \leq \rho\}$.*

Proof. By Proposition 27.1, $|g|$ and $|g|^{-1}$ are both bounded, and so we need only prove that g is a diffeomorphism. Indeed, by Theorem 27.2,

$$r^{-1}\Lambda^{-1}\Re(zg'/g) = \kappa > 0,$$

and so $g' \neq 0$ in $\text{Int}(A_R)$ and hence g is a local diffeomorphism.

Consider the set $\gamma = \{z : \phi(z) = \text{const}\}$. Since $\arg g = \phi = \psi - \pi/2$ is strictly increasing on each $\{|z| = r\} \subset \text{Int}(A_R)$ (remember that $\kappa = \phi_s > 0$, in fact ϕ takes every value between 0 and 2π on $\{|z| = r\}$ exactly once), we see that γ is a smooth Jordan arc connecting $\{|z| = 1/R\}$ and $\{|z| = R\}$. Let $\vec{t}$ be the unit tangent vector of γ and $\vec{n}$ its unit normal vector, such that $(\vec{t}, \vec{n})$ has positive orientation. Then since $\log g = \log|g| + i \arg g$ is holomorphic, we have $-\vec{n}\log|g| = \vec{t}\phi = 0$ and so $\vec{t}\log|g| \neq 0$ on γ, as otherwise we would have $g' = 0$. Thus whenever $\arg g(z_1) = \arg g(z_2)$ and $z_1 \neq z_2$, then $\log|g(z_1)| \neq \log|g(z_2)|$, so $g(z_1) \neq g(z_2)$. The holomorphic function g is a one-to-one local diffeomorphism, hence is a conformal diffeomorphism. □

Corollary 27.5 *The total Gauss curvature of a CBA is larger than -4π.*

One interesting corollary of Theorem 27.4 is that

Corollary 27.6 *If A is a CBA with smooth boundary then the second eigenvalue of L_A is positive.*

Proof. By Theorem 27.4, N is an anti-conformal diffeomorphism. By Corollary 32.7 of Appendix, the second eigenvalue of Δ_S on $(N(A_R))$ is larger than 2, thus $\lambda_2(A) > 0$. □

Remember that the index of A is

$$\textbf{Index}(A) = \sum_{\lambda<0} \dim V_\lambda(A),$$

where V_λ is the eigenspace corresponding to the eigenvalue λ.

Corollary 27.7 *Let A be a CBA, then*

$$\text{Index}(A) = \begin{cases} 0, & \text{if } A \text{ is stable or almost stable;} \\ 1, & \text{if } A \text{ is unstable.} \end{cases} \tag{27.129}$$

Proof. We need only prove the unstable case. First assume that ∂A is smooth. By Corollary 32.9 of Appendix and Corollary 27.6, $\dim V_{\lambda_1} = 1$ and $\lambda_2(A) > 0$, hence $\text{Index}(A) \leq 1$. But if A is unstable, $\text{Index}(A) \geq 1$, thus $\text{Index}(A) = 1$.

If ∂A is only continuous, we define a family of diffeomorphisms of A_R into itself by

$$f_t(z) = f_t(re^{i\theta}) = r^{1-t}e^{i\theta}, \quad 0 \leq t < 1.$$

Then $f_0 = \text{Id}_{A_R}$, $f_t(A_R) \subset f_s(A_R)$ for $0 \leq s < t < 1$, and $\lim_{t\to1} f_t(A_R) = \{z : |z| = 1\}$; thus $\lim_{t\to1}\text{Vol}(f_t(A_R)) = 0$.

Using the embedding X, we get a family of diffeomorphisms of A into A, $c_t = X \circ f_t \circ X^{-1}$, $0 \leq t < 1$, satisfying $c_t(A) = A \cap S(t-1, 1-t)$. Note that by Theorem 29.1 of Section 29, each $c_t(A)$, $0 < t < 1$ is a CBA and has smooth boundary, we can use Theorem 27.4 and Corollary 27.6. Moreover, we have

1. $c_0 =$ identity;
2. $c_t(A) \subset c_s(A)$, for $0 \leq s < t < 1$;
3. $\lim_{t\to1}\text{Vol}(c_t(A)) = 0$.

Recall that $\text{nullity}(c_t(A)) = \dim V_0(c_t(A))$. By a theorem of Morse, Simons, and Smale (see [46], p 52) we have that

$$\text{Index}(A) = \sum_{t>0} \text{nullity}(c_t(A)).$$

If $c_t(A)$ is almost stable then 0 is the first eigenvalue of $c_t(A)$, so by Corollary 32.9 of Appendix, $\text{nullity}(c_t(A)) = \dim V_0(c_t(A)) = 1$. For any $s > t$, $c_s(A) \subset c_t(A)$ is a proper subdomain, so $\lambda_1(c_s(A)) > \lambda_1(c_t(A)) = 0$ and $\text{nullity}(c_s(A)) = 0$. If $c_t(A)$ is unstable and $\text{nullity}(c_t(A)) > 0$, then 0 is at least the second eigenvalue of $c_t(A)$, contradicts Corollary 27.6. Hence we have proved that at most one $t \in (0,1)$ can be such that $\text{nullity}(c_t(A)) = 1$ and for the other t we must have $\text{nullity}(c_t(A)) = 0$. We conclude that $\text{Index}(A) \leq 1$. But if A is unstable, $\text{Index}(A) \geq 1$, thus $\text{Index}(A) = 1$. □

Theorem 27.8 *The index of the catenoid is 1.*

Proof. Let C be the catenoid given by Example 14.2. $C(t) := C \cap S(-t, t)$ is a CBA for $t > 0$. Thus index$(C(t)) \leq 1$. Since any precompact domain B in $\mathbf{C} - \{0\}$ is contained in some A_R, it follows $X(B) \subset X(A_R) = C(\log R)$. By the definition of index of C, see (20.85), we have index$(C) \leq 1$.

Since $g(z) = z$ is one-to-one we know by Section 20 that any precompact domain $\Omega \subset S^2 - \{(0,0,1), (0,0,-1)\}$ such that the first eigenvalue of $\triangle_S$, $\lambda_1(\Omega) < 2$, corresponding to an unstable precompact domain on C. Since the first eigenvalue of $\triangle_S$ on S^2 is 0, there are plenty precompact domains in $S^2 - \{(0,0,1), (0,0,-1)\}$ with the first eigenvalue less than 2, a consequence of the fact that λ_1 is continuously dependent on domains. Thus C is not stable and index$(C) \geq 1$. □