20 The Second Variation and Stability

We now introduce the concept of *stability* of minimal surfaces which will play an important role in the proof of several theorems in the remainder of these notes.

Let Ω be a precompact domain in a Riemann surface M, $X : \Omega \to \mathbf{R}^3$ a minimal surface. From the calculus of variations definition of a minimal surface, we know that X is a minimal surface if and only if the area A of X is a stationary point of the area functional $A(t)$ for any variation $X(t)$. Note that being stationary does not mean that X has minimum area among all surfaces with the same boundary.

To study when X has locally minimum area, naturally we study the second variation, namely the second derivative $A''(0)$ of the area functional for any variation family $X(t)$. From calculus we know that if $A''(0) > 0$ then $A(0)$ is a local minimum. Note that the word *local* is significant, there are minimal surfaces such that $A''(0) > 0$ for any variation family, yet those surfaces do not have minimum area. Hence we define that X is *stable* if $A''(0) > 0$ for all possible variation families $X(t)$, otherwise X is *unstable*. Sometimes one says X is *almost stable* if $A''(0) \geq 0$.

It is important to express the formula for the second variation of X via the geometric quantities of X. Let (u^1, u^2) be the local coordinates of Ω. We use the fact that X is conformal harmonic, and write $\Lambda^2 = |X_1|^2 = |X_2|^2$, $\triangle = D_{11} + D_{22}$.

From (3.4),

$$\frac{dA(t)}{dt} = -2\int_\Omega H(t)(E(t)\bullet N(t))\,dA_t,$$

where $E(t) = \partial X(t)/\partial t$, $H(t)$ is the mean curvature of $X(t)$, and $N(t)$ is the Gauss map of $X(t)$. Let $E = \alpha X_1 + \beta X_2 + \gamma N$. Since $H(0) = 0$ we have

$$\frac{d^2A(t)}{dt^2}\Big|_{t=0} = -2\int_\Omega \frac{dH(t)}{dt}\Big|_{t=0}(E\bullet N)\,dA_0,$$

where we write $E = E(0)$, etc. Now suppose that each $X(t)$ is a C^2 surface, and the first and second fundamental forms are given on an isothermal coordinate chart U by

$$g_{ij}(t) = X_i(t)\bullet X_j(t), \quad (g^{ij}(t)) = (g_{ij}(t))^{-1}, \quad h_{ij}(t) = X_{ij}(t)\bullet N(t).$$

Then

$$H(t) = \frac{1}{2}\sum_{i,j} g^{ij}(t)h_{ij}(t),$$

hence

$$\frac{dH(t)}{dt}\Big|_{t=0} = \frac{1}{2}\sum_{i,j}\frac{dg^{ij}(t)}{dt}\Big|_{t=0}h_{ij} + \frac{1}{2}\sum_{i,j} g^{ij}\frac{dh_{ij}(t)}{dt}\Big|_{t=0},$$

where we write $g^{ij}(0) = g^{ij}$, etc. From

$$\sum_j g^{ij}(t)g_{jk}(t) = \delta_{ik}, \quad g^{ij} = \Lambda^{-2}\delta_{ij},$$

we see that

$$\frac{dg^{ij}(t)}{dt}\Big|_{t=0} = -\Lambda^{-4}\frac{dg_{ij}(t)}{dt}\Big|_{t=0} = -\Lambda^{-4}(E_i \bullet X_j + E_j \bullet X_i).$$

Using $h_{11} = -h_{22}$ and $X_{11}\bullet X_1 = \frac{1}{2}\Lambda_1^2$, $X_{11}\bullet X_2 = -\frac{1}{2}\Lambda_2^2$, etc., we have

$$\frac{1}{2}\sum_{i,j}\frac{dg^{ij}(t)}{dt}\Big|_{t=0} h_{ij} = \gamma\Lambda^{-4}\sum_{i,j} h_{ij}^2 - \Lambda^{-2}[\alpha_1 h_{11} + (\alpha_2+\beta_1)h_{12} + \beta_2 h_{22}].$$

One calculates that

$$\begin{aligned}\frac{1}{2}\sum_{ij} g^{ij}\frac{dh_{ij}(t)}{dt}\Big|_{t=0} &= \frac{1}{2}\sum_{ij} g^{ij}\frac{dX_{ij}(t)}{dt}\Big|_{t=0}\bullet N + \frac{1}{2}\sum_{ij} g^{ij}X_{ij}\bullet\frac{dN(t)}{dt}\Big|_{t=0} \\ &= \frac{1}{2}\sum_i \Lambda^{-2}E_{ii}\bullet N + \frac{1}{2}\sum_i \Lambda^{-2}X_{ii}\bullet\frac{dN(t)}{dt}\Big|_{t=0} \\ &= \frac{1}{2}\Lambda^{-2}\triangle E\bullet N,\end{aligned}$$

since $\triangle X = 0$. Using $\triangle X_i = 0$ and $N_i \bullet N = 0$, we have

$$\triangle E\bullet N = \triangle\gamma + \gamma\,\triangle N\bullet N + 2[\alpha_1 h_{11} + (\alpha_2+\beta_1)h_{12} + \beta_2 h_{22}].$$

Hence

$$\frac{dH(t)}{dt}\Big|_{t=0} = \gamma\Lambda^{-4}\sum_{ij} h_{ij}^2 + \frac{1}{2}\Lambda^{-2}(\triangle\gamma + \gamma\triangle N\bullet N).$$

Since $h_{11} = -h_{22}$, $\sum_{ij} h_{ij}^2 = -2\det(h_{ij}) = -2\Lambda^4 K$, where K is the Gauss curvature. By (8.36), $\triangle N = 2K\Lambda^2 N$, thus

$$\frac{dH(t)}{dt}\Big|_{t=0} = \frac{1}{2}\Lambda^{-2}(\triangle\gamma - 2K\Lambda^2\gamma) = \frac{1}{2}(\triangle_X\gamma - 2K\gamma).$$

Since the above formula does not depend on the local coordinates, we have the second variation formula for any variation vector field $E = \alpha X_1 + \beta X_2 + \gamma N$, that is

$$A''(0) = -\int_\Omega \gamma(\triangle_X\gamma - 2K\gamma)dA_0. \tag{20.83}$$

We see from (20.83), as in the first variation, that the second variation does not depend on the tangential part of the variation field E.

Let Ω be a plane domain, consider the Dirichlet eigenvalue problem for the second order elliptic operator $L = \triangle - 2K\Lambda^2$,

$$\begin{cases} Lu + \lambda u = 0, & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega \end{cases} \tag{20.84}$$

The classical theory of eigenvalues (see Appendix) says that there is a sequence

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \le \cdots,$$

$\lambda_n \to \infty$ as $n \to \infty$, such that (20.84) has solution if and only if $\lambda = \lambda_n$ for some $n \ge 1$. Moreover, we can select smooth ϕ_n as the solution of (20.84) when $\lambda = \lambda_n$ (ϕ_n is called the *eigenfunction* corresponding to λ_n), such that $\{\phi_n\}$ is orthonormal in $L^2(\Omega)$ and spans $W_0^{1,2}(\Omega)$. Thus if $\gamma \in W_0^{1,2}(\Omega) \subset L^2(\Omega)$ it can be decomposed as

$$\gamma = \sum_{n=1}^{\infty} a_n \phi_n.$$

and if γ is also smooth, then

$$L\gamma = \sum_{n=1}^{\infty} a_n L\phi_n = -\sum_{n=1}^{\infty} a_n \lambda_n \phi_n.$$

We have that

$$A''(0) = -\int_\Omega \gamma L\gamma \, du^1 \wedge du^2 = \int_\Omega \left(\sum_{n=1}^{\infty} a_n \phi_n\right)\left(\sum_{m=1}^{\infty} a_m \lambda_m \phi_m\right) du^1 \wedge du^2 = \sum_{n=1}^{\infty} a_n^2 \lambda_n.$$

Hence if $\lambda_n > 0$, we will have for any variation vector field $E = \alpha X_1 + \beta X_2 + \gamma N$ with smooth $\gamma \in W_0^{1,2}(\Omega)$, that $A''(0) > 0$, and hence locally X has minimum surface area.

Of course, if L has a negative eigenvalue, say $\lambda_1 < 0$, taking $\gamma = \phi_1$, we have

$$A''(0) = \lambda_1 < 0,$$

and so X cannot have minimum area.

Note that $\Delta_X = \Lambda^{-2}\Delta$ is intrinsically defined on the surface X. Based on the discussion above, we have definition equivalent to that given in the beginning of this section:

Definition 20.1 A minimal surface $X : \Omega \hookrightarrow \mathbf{R}^3$ is *stable* on a precompact domain $U \subset \Omega$ if the first eigenvalue of $L_X = \Delta_X - 2K$ in U is positive. That is, if

$$\begin{cases} L_X u + \lambda u = 0, & \text{in} \quad U \\ u = 0, & \text{on} \quad \partial U \end{cases}$$

has a non-trivial solution, then $\lambda > 0$.

In general, if $\overline{\Omega}$ is not compact, we say that X is stable on Ω if it is stable on any precompact subdomain of Ω.

For a minimal surface $X : \Omega \hookrightarrow \mathbf{R}^3$, the Gauss map $N \to S^2$ is anti-conformal. We can consider N as a surface though it may have finite branch points. The first fundamental form induced by N is

$$|N_1 \wedge N_2|\delta_{ij} = -K\Lambda^2\delta_{ij}.$$

Hence the S^2 Laplacian $\triangle_S$ induced by N on Ω is

$$\triangle_S = -K^{-1}\Lambda^{-2}\triangle = -K^{-1}\,\triangle_X\,.$$

The sphere metric induced by N then is $dS = -KdA_0$ on Ω. Suppose $K < 0$ on Ω, then since N is anti-holomorphic, by the area formula,

$$A''(0) = -\int_\Omega \gamma(\triangle_X\gamma - 2K\gamma)dA_0. = -\int_{N(\Omega)} \#(N^{-1}(x))\gamma(\triangle_S\gamma + 2\gamma)(x)dS(x).$$

Thus the corresponding operator L_S on $N(\Omega)$ is

$$L_S = -K^{-1}L_X = \triangle_S + 2.$$

If $N : U \subset \Omega \to S^2$ is one to one, then clearly $A''(0) > 0$ if and only if all eigenvalues of $\triangle_S$ on $N(\Omega)$ are larger than 2. And the eigenvalue problem becomes

$$\begin{cases} \triangle_S u + (2+\lambda)u = 0, & \text{in} \quad N(U) \\ u = 0, & \text{on} \quad \partial N(U) \end{cases}$$

It is well known that if the area of $N(U)$ is less than 2π, then the first eigenvalue of $\triangle_S$ is larger than 2, thus have proved:

Theorem 20.2 *Let $X : \Omega \hookrightarrow \mathbf{R}^3$ be a minimal surface and $U \subset \Omega$ be such that $N : U \to S^2$ is one to one and the area of $N(U)$ is less than 2π. Then $X : U \hookrightarrow \mathbf{R}^3$ is stable.*

Since N is locally one to one except at points p such that $K(p) = 0$, we see that at any point $p \in \Omega$ such that $K(p) \neq 0$, there is a neighbourhood $U \ni p$, such that $X : U \hookrightarrow \mathbf{R}^3$ is stable.

Note that if N is one to one, then

$$\mathbf{Area}(N(U)) = -\int_U KdA,$$

so if N is one to one on U and the area of $N(U)$ is less than 2π, then $-\int_U KdA < 2\pi$. Barbosa and do Carmo [2] proved:

Theorem 20.3 *If $-\int_U KdA < 2\pi$, then X is stable on U.*

In fact, Barbosa and do Carmo proved a stronger version of Theorem 20.3 in [2]:

Theorem 20.4 *If* **Area**$(N(U)) < 2\pi$, *then* X *is stable on* U.

Theorem 20.3 is stronger than Theorem 20.2 since N is not assumed to be one to one on U. Note that the converse of Theorem 20.3 is not true, there are stable minimal surfaces whose total curvature is less then -2π. See, for example, [61], page 99.

Let $X : M \hookrightarrow \mathbf{R}^3$ be a minimal surface. A *Jacobi field* is a function u defineded on M such that

$$L_X u = 0.$$

Note that each component of N is a Jacobi field. Whenever we have a Jacobi field u on M, we are interested in the *nodal set* $Z := u^{-1}(0) \subset M$ of u. The reason is that each component of $M - Z$ is a domain (*nodal domain*) $\Omega \subset M$ such that on Ω the u does not change sign and it vanishes on $\partial\Omega$. If u is continuous on $\overline{\Omega}$, then by the properties of eigenvalues (see Appendix) the first eigenvalue of L_X on Ω is zero, and any domain $\Omega' \supset \overline{\Omega}$ will have negative first eigenvalue. Thus such Ω and $\Omega' \supset \Omega$ are unstable. By Theorem 20.3, the total curvature of X on Ω is less than or equal to -2π. Similarly, any domain $\Omega' \subset \Omega$ such that $\Omega - \Omega'$ has positive area, will have positive first eigenvalue, and therefore is stable. We will apply these comments in the proof of Shiffman's theorems.

In [4], do Carmo and Peng proved that the only stable complete minimal surface in $\mathbf{R}^3$ is plane. This is a generalized version of Bernstein's theorem, which says that a complete minimal graph (which is stable by Theorem 20.4) must be a plane.

Thus all complete non-planar minimal surfaces $X : M \hookrightarrow \mathbf{R}^3$ are unstable. A measure of how unstable is a surface, is the *index*. If $\Omega \subset M$ is precompact, then index(Ω) is the number of negative eigenvalues of L_X on Ω, counting the multiplicity. Hence the index is the dimension of the subspace of $L^2(\Omega)$ spanned by the eigenfunctions corresponding to negative eigenvalues. The index of M then is defined as

$$\text{index}(M) = \text{lub}_{\Omega \subset \text{M}} \text{index}(\Omega), \tag{20.85}$$

where lub means the least upper bound and Ω is taken over all precompact domains in M.

A theorem of Fischer-Colbrie [19] says that a complete minimal surface $X : M \hookrightarrow \mathbf{R}^3$ has finite index if and only if it has finite total curvature.

Let g and η be the Weierstrass data of a complete minimal surface of finite total curvature $X : M \hookrightarrow \mathbf{R}^3$ and $k = \deg g$. A theorem of Tysk [79] says that

$$\text{index of } M \leq C \cdot k.$$

for some constant C. Tysk [79] proved that C can be taken as $C = 7.68183$. The number 7.68183 is certainly not optimal, since for a catenoid $k = 1$ and the index is also 1, see Theorem 27.8. A good problem then is what is the optimal value of C? A guess is that $C = 1$.