13 Total Curvature of Branched Complete Minimal Surfaces

Let $X : M \hookrightarrow \mathbf{R}^3$ be a complete minimal surface with finite total curvature. Osserman's theorem says that conformally $M = S_k - \{p_1, \cdots, p_n\}$, $n \geq 1$, where S_k is a closed Riemann surface of genus k. Each p_i corresponds to an end E_i of M. Using Theorem 12.1, we can prove:

Theorem 13.1 *The total curvature of X is*

$$K(M) = 2\pi \left(\chi(M) - \sum_{i=1}^{n} I_i \right), \tag{13.57}$$

where $\chi(M) = 2(1-k) - n$ is the Euler characteristic of M and I_i is the multiplicity of E_i.

Proof. Let $\Gamma_i^r = X^{-1}(rW_i^r)$ be as in the proof of Theorem 12.1. Let $p_i \in D_i^r$ be the disk in S_k such that $\partial D_i^r = \Gamma_i^r$. When r is large enough the D_i^r's are disjoint from each other. Then $M_r := S_k - \bigcup_{i=1}^n \partial D_i^r$ is a Riemann surface with boundary $\bigcup_{i=1}^n \partial D_i^r$ and $\chi(M_r) = \chi(M)$. Now by the Gauss-Bonnet formula we have

$$\int_{M_r} K dA + \sum_{i=1}^{n} \int_{\Gamma_i^r} \kappa_g \, ds = 2\pi\chi(M_r) = 2\pi\chi(M),$$

where κ_g is the geodesic curvature. Since $W_i^r = \frac{1}{r}X(\Gamma_i^r)$ converges in the C^∞ sense to a great circle on S^2 with multiplicity I_i and X is an isometric immersion, we have

$$\lim_{r \to \infty} \int_{\Gamma_i^r} \kappa_g \, ds = 2\pi I_i.$$

Taking limit we have

$$K(M) = \int_M K dA = 2\pi \left(\chi(M) - \sum_{i=1}^{n} I_i \right). \tag{13.58}$$

□

In the remainder of this section, our surfaces will be branched minimal surfaces. Note that the concepts of completeness, properness, etc., can be easily generalised to branched minimal surfaces.

The Enneper-Weierstrass representation of a branched complete minimal surface of finite total curvature $X : M \to \mathbf{R}^3$ is given by

$$X(p) = \Re \int_{p_0}^{p} \left(\frac{1}{2}(1-g^2),\ \frac{i}{2}(1+g^2),\ g \right) \eta + C, \tag{13.59}$$

where $g : M = S_k - \{p_1, \cdots, p_n\} \to \mathbf{C} \cup \{\infty\}$ is a meromorphic function, η is a holomorphic 1-form on M and C is a constant vector. Both g and η can be extended to S_k as a meromorphic function and 1-form respectively. Note that we have proved this for regular minimal surfaces. But since the proof only involves the neighbourhoods of the punctures p_i, it works for branched minimal surfaces as well.

Locally, $\eta = f(z)dz$ where $z = x + iy$. The metric induced by X is given by

$$ds^2 = \Lambda^2(dx^2 + dy^2), \tag{13.60}$$

where

$$\Lambda = \frac{1}{2}|f|(1 + |g|^2). \tag{13.61}$$

From (13.61) it is clear that $q \in M$ is a branch point only if η vanishes at q. Hence all branch points are isolated and if η is a meromorphic 1-form on S_k, there is only a finite number of branch points.

Therefore, given g and η as above, we can define a metric h with isolated degenerate points on $M = S_k - \{p_1, \cdots, p_n\}$ by $h_{ij} = \Lambda^2 \delta_{ij}$, where Λ is defined as in (13.61). We can study the intrinsic geometry of the branched complete Riemannian manifold (M, h) even though the mapping X in (13.59) may not be well defined. When X is well defined, it is a branched complete minimal surface.

Let U_i be a disk coordinate neighbourhood of p_i such that $z(p_i) = 0$. Let J_i be the order of Λ at p_i, i.e., J_i is an integer such that in U_i,

$$\lim_{z \to 0} |z|^{J_i} \Lambda(z) = C_i > 0,$$

for $1 \leq i \leq n$. Since (M, h) is complete, $J_i \geq 1$.

Suppose q_i, $1 \leq i \leq m$, are branch points of M. Let V_i be a disk coordinate neighbourhood of q_i such that $z(q_i) = 0$. Let K_i be the branch order of Λ, i.e.,

$$\lim_{z \to 0} |z|^{-K_i} \Lambda(z) = C_i > 0, \quad \text{in} \quad V_i.$$

There is a generalised version of (13.57) in [16] which allows X to have branch points.

Theorem 13.2 *The total curvature of (M, h) is given by*

$$\int_M K dA = 2\pi \left(\chi(M) - \sum_{i=1}^{n} (J_i - 1) + \sum_{i=1}^{m} K_i \right). \tag{13.62}$$

Proof. Let $R > 0$ be such that $D_R^i := \{|z| < R\} \subset U_i$, $1 \leq i \leq n$ and $D_R^i := \{|z| < R\} \subset V_{i-n}$, $n + 1 \leq i \leq n + m$. When R is small enough, $D_R^i \cap D_R^j = \emptyset$ for $i \neq j$.

Let $M_R = M - \bigcup_{i=1}^{n+m} D_R^i$. By the Gauss-Bonnet formula, we have

$$\int_{M_R} K dA + \sum_{i=1}^{n+m} \int_{\partial D_R^i} \kappa_g \, ds = 2\pi\chi(M_R) = 2\pi(\chi(M) - m). \tag{13.63}$$

If $g(p_i) \neq \infty$, then $\eta = z^{-J_i} f_i(z)dz$ where f_i is a holomorphic function in U_i and $f_i(0) \neq 0$. Write $z = re^{it}$. By Minding's formula, see [12], Volume I, pages 33-34, the geodesic curvature on ∂D_R^i is given by

$$\kappa_g \Lambda = -\frac{1}{R} + \frac{\partial \log \Lambda}{\partial \nu},$$

where ν is the inward unit normal (in the Euclidean metric on D_R^i) of ∂D_R^i. Now $\Lambda = \frac{1}{2}|z|^{-J_i}|f_i|(1+|g|^2)$, so

$$\frac{\partial \log \Lambda}{\partial \nu} = -\frac{\partial \log \Lambda}{\partial r} = \frac{J_i}{r} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1+|g|^2)}{\partial r},$$

and

$$\int_{\partial D_R^i} \kappa_g \, ds = \int_0^{2\pi} \kappa_g \Lambda R \, dt = \int_0^{2\pi} \left(\frac{J_i - 1}{R} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1+|g|^2)}{\partial r} \right) R \, dt.$$

Since

$$\int_0^{2\pi} \frac{\partial \log |f_i|}{\partial r} R \, dt = \int_{D_R^i} \triangle(\log |f_i|) dx \, dy = 0,$$

and $\partial \log(1+|g|^2)/\partial r$ is bounded, we have

$$\lim_{R \to 0} \int_{\partial D_R^i} \kappa_g ds = 2\pi(J_i - 1).$$

If $g(p_i) = \infty$ then $g = z^{-m_i} g_i(z)$, $m_i > 0$, and $\eta = z^{-J_i + 2m_i} f_i(z)dz$, where f_i and g_i are holomorphic functions in U_i and $f_i(0) \neq 0$, $g_i(0) \neq 0$. Then

$$\frac{\partial \log \Lambda}{\partial \nu} = -\frac{\partial \log \Lambda}{\partial r} = \frac{J_i - 2m_i}{r} - \frac{\partial \log |f_i|}{\partial r} - \frac{\partial \log(1+|g|^2)}{\partial r}.$$

Since

$$\frac{\partial \log(1+|g|^2)}{\partial r} = \frac{1}{1+r^{-2m_i}|g_i|^2} \left(-2m_i r^{-2m_i - 1}|g_i|^2 + r^{-2m_i} \frac{\partial |g_i|^2}{\partial r} \right),$$

we have

$$\begin{aligned} -\int_0^{2\pi} \frac{\partial \log(1+|g|^2)}{\partial r} R \, dt &= \int_0^{2\pi} \frac{2m_i R^{-2m_i}|g_i|^2}{1+R^{-2m_i}|g_i|^2} dt - \int_0^{2\pi} \frac{R^{-2m_i} \frac{\partial |g_i|^2}{\partial r}}{1+R^{-2m_i}|g_i|^2} R \, dt \\ &\to 4m_i \pi \quad \text{as} \quad R \to 0. \end{aligned}$$

We have the same limit

$$\lim_{R \to 0} \int_{\partial D_R^i} \kappa_g ds = 2\pi(J_i - 1).$$

Similarly, for the branch ponts q_i, if $g(q_i) \neq \infty$, then $\eta = z^{K_i} f_i(z)dz$ where f_i is a holomorphic function defined in V_i and $f_i(0) \neq 0$. Similar calculation gives

$$\int_{\partial D_R^{i+n}} \kappa_g \, ds = -\int_0^{2\pi} \left(\frac{K_i + 1}{R} + \frac{\partial \log |f_i|}{\partial r} + \frac{\partial \log(1 + |g|^2)}{\partial r} \right) R \, dt.$$

Hence

$$\lim_{R \to 0} \int_{\partial D_R^{i+n}} \kappa_g ds = -2\pi(K_i + 1).$$

If $g(q_i) = \infty$, then $g(z) = z^{-m_i} g_i(z)$ and $\eta = z^{K_i + 2m_i} f_i(z)$, similar calculation still gives us the same limit.

Note that

$$\lim_{R \to 0} \int_{M_R} K dA = \int_M K dA.$$

Letting $R \to 0$ in (13.63), we get (13.62). The proof is complete. $\square$

Remark 13.3 Suppose $X : S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is a regular complete minimal surface, then $h_{ij} = \Lambda^2 \delta_{ij}$ is the pull back metric of X. Comparing the proofs of Theorem 13.1 and Theorem 13.2, we see that $J_i - 1 = I_i$, thus (13.62) is a generalization of (13.57).

The calculation in the proof of Theorem 13.2 also works for boundary branch points. Let M be a compact domain of a Riemann surface with a C^2 boundary $\Gamma = \partial M$. Suppose that g and η are given meromorphic function and 1-form respectively, and h is the Riemannian metric with isolated degenerate points defined by (13.60) and (13.61). Let $q_i \in M$ $(1 \leq i \leq m)$ be the interior branch points with branch order K_i and $s_i \in M$ $(1 \leq i \leq n)$ be the boundary branch points with branch order L_i. Then:

Theorem 13.4 *The total curvature of (M, h) is given by*

$$\int_M K dA = 2\pi \left(\chi(M) + \sum_{i=1}^m K_i \right) + \pi \sum_{i=1}^n L_i - \int_\Gamma \kappa_g \, ds. \tag{13.64}$$

A sketch of the proof of (13.64) is as follows:

Define D_R^i as before and $M_R = M - \bigcup_{i=1}^{n+m} D_R^i$. By the Gauss-Bonnet formula,

$$\int_{M_R} K dA + \int_{\partial M_R} \kappa \, ds + \sum_{i=1}^n (\alpha_R^i + \beta_R^i) = 2\pi(\chi(M) - m),$$

where α_R^i and β_R^i are the exterior angles near the boundary branch points and

$$\lim_{R \to 0} \alpha_R^i = \frac{\pi}{2}, \quad \lim_{R \to 0} \beta_R^i = \frac{\pi}{2}.$$

Then (13.64) follows by

$$\lim_{R \to 0} \int_{\partial D_R^i \cap \partial M_R} \kappa \, ds = \lim_{R \to 0} \int_{\epsilon_R^i}^{\delta_R^i} \left(\frac{-1}{R} - \frac{\partial \log \Lambda}{\partial r} \right) R \, dt = \lim_{R \to 0} (\epsilon_R^i - \delta_R^i)(1 + L_i) = -\pi(1 + L_i),$$

for the boundary branch points.

Remark 13.5 If X in (13.59) is well defined then X is a minimal surface and h is induced by X. In this case, (13.64) is the same as the formula in [12], Volume II, page 128.

Since if $X : S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is a complete minimal immersion, then $J_i \geq 2$ and $J_i = 2$ if and only if the end E_i is embedded, we get a corollary.

Corollary 13.6 *The total curvature of a regular complete minimal surface of genus k with n ends satisfies*

$$K(M) \leq 4\pi(1 - k - n) = 2\pi(\chi(M) - n). \tag{13.65}$$

Moreover,

$$K(M) = 2\pi(\chi(M) - n)$$

if and only if each end of M is embedded.

The inequality (13.65) is a result of Osserman.