2 Definition of Minimal Surfaces

Definition 2.1 A *minimal surface* in $\mathbf{R}^3$ is a *conformal harmonic immersion* $X : M \hookrightarrow \mathbf{R}^3$, where M is a 2-dimensional smooth manifold, with or without boundary. Here *conformal* means that for any point $p \in M$ there is a local coordinate neighbourhood $(U, (u, v))$ on M, such that in U the vectors

$$X_1 := X_u = \frac{\partial X}{\partial u} = \left(\frac{\partial X^1}{\partial u}, \frac{\partial X^2}{\partial u}, \frac{\partial X^3}{\partial u}\right) = (X_u^1, X_u^2, X_u^3) = (X_1^1, X_1^2, X_1^3)$$

and

$$X_2 := X_v = \frac{\partial X}{\partial v} = \left(\frac{\partial X^1}{\partial v}, \frac{\partial X^2}{\partial v}, \frac{\partial X^3}{\partial v}\right) = (X_v^1, X_v^2, X_v^3) = (X_2^1, X_2^2, X_2^3)$$

are perpendicular to each other and have the same length. Thus

$$\Lambda^2 := |X_u|^2 = |X_v|^2 > 0, \quad X_u \bullet X_v \equiv 0.$$

Here $\bullet$ is the Euclidean inner product. Such a coordinate neighbourhood $(U, (u, v))$ is called an *isothermal neighbourhood,* its coordinates (u, v) are called *isothermal coordinates.*

The word *immersion* means that for any $p \in M$, $X_* := dX : T_pM \to T_{X(p)}\mathbf{R}^3$ is a linear embedding. In the case X is conformal, it means simply that $\Lambda > 0$ on M.

The word *harmonic* means that

$$\triangle X = \frac{\partial^2 X}{\partial u^2} + \frac{\partial^2 X}{\partial v^2} = X_{uu} + X_{vv} = X_{11} + X_{22} \equiv \vec{0}.$$

If M is connected, then we say that the surface X is *connected.* We will only consider connected surfaces. Furthermore, since any non-orientable surface has an orientable double covering, we will only consider *oriented minimal surfaces.*

A homothety of $\mathbf{R}^3$ is the composition of a rigid motion and a dilation or a shrinking. Let T be a homothety of $\mathbf{R}^3$, $X : M \hookrightarrow \mathbf{R}^3$ be a surface. It is easy to see that X is a conformal harmonic immersion if and only $T \circ X$ is. Thus we consider all surfaces in $\mathbf{R}^3$ up to a homothety. That is, we do not distinguish the surfaces $X : M \hookrightarrow \mathbf{R}^3$ and $T \circ X : M \hookrightarrow \mathbf{R}^3$.

A classical theorem says that any C^k immersion, $2 \leq k \leq \infty$, can have an atlas of isothermal coordinate charts, so that X being conformal is not a special property of minimal surfaces. The important fact which distinguishes minimal surfaces is that under these isothermal charts, X is harmonic.

For an orientable surface $X : M \hookrightarrow \mathbf{R}^3$, let $\{(U_\alpha, z_\alpha = u_\alpha + iv_\alpha)\}_{\alpha \in A}$ be an atlas of isothermal coordinates of the same orientation, then $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ defines a *complex (conformal) structure* on M.

Precisely, we will prove that if V is any isothermal coordinate neighborhood, with the coordinates $w = x + iy$ having the same orientation as $z = u + iv$ on $U \bigcap V$, then $z \circ w^{-1} : w(U \cap V) \to z(U \cap V)$ is a holomorphic function. Which is equivalent to saying that the functions $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

To see this, compute

$$\frac{\partial X}{\partial x} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial X}{\partial y} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial y}.$$

Since both coordinates are conformal, we get that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2, \quad \text{and} \quad \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}.$$

Thus we have that

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 + 2i\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$= \left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2 - 2i\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = \left(\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right)^2.$$

Hence

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = \pm\left(\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right).$$

But if

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = -\left(\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right),$$

then

$$\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -\left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2 < 0,$$

contradicting the fact that U and V have the same orientation. So

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = \left(\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right),$$

which is the complex form of the Cauchy-Riemann equations.

Since M is orientable, we get a complex analytic atlas $\{(U_\alpha, z_\alpha)\}_{\alpha \in A}$ on M, and M is diffeomorphic to a one-dimensional complex manifold. A one-dimensional complex manifold is usually called a *Riemann surface.*

Since any smooth orientable 2-dimensional manifold can be conformly embedded in $\mathbf{R}^3$, we see that any 2-dimensional smooth orientable surface M is diffeomorphic to a Riemann surface.

Moreover, if X is minimal, under this complex structure on M, X is harmonic, hence locally is the real part of a holomorphic mapping. It is here that complex function theory enters and plays an important role in the study of minimal surfaces.

Thus when we consider a minimal surface $X : M \hookrightarrow \mathbf{R}^3$, we can always assume that M is a Riemann surface with a *conformal structure* given as above.

The easiest global property of minimal surfaces is that if M is a *closed Riemann surface* (compact manifold without boundary), then there is no minimal immersion $X : M \to \mathbf{R}^3$. In fact, since M is compact, each component of X is a bounded harmonic function, and hence must have a maximum value on M. Thus X is a constant by the maximum principle, since M has no boundary. But then X is not an immersion.

Another definition of minimal surfaces is that the *mean curvature* of $X : M \hookrightarrow \mathbf{R}^3$ vanishes.

Remember that the mean curvature H of X is defined by

$$2H = g^{11}h_{11} + 2g^{12}h_{12} + g^{22}h_{22},$$

where $g_{ij} = X_i \bullet X_j$, $h_{ij} = X_{ij} \bullet N$ (N is the Gauss map, i.e., the unit normal vector $X_1 \wedge X_2 / |X_1 \wedge X_2|$, where $\wedge$ is the cross product in $\mathbf{R}^3$), $(g^{ij}) = (g_{ij})^{-1}$, see any differential geometry textbook.

In case X is conformal, $g_{11} = g_{22} = \Lambda^2$, $g^{11} = g^{22} = \Lambda^{-2}$, $g_{12} = g^{12} = 0$. Thus

$$H = \frac{\triangle X \bullet N}{2\Lambda^2} = \frac{1}{2} \triangle_X X \bullet N,$$

where $\triangle_X$ is the Laplace-Beltrami operator under the metric (g_{ij}). Remember that $\triangle_X$ is given by

$$\triangle_X := \sum_{i=1}^{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{2} \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right) = \frac{1}{\Lambda^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{4}{\Lambda^2} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}},$$

where $g = \det(g_{ij})$, $(x^1, x^2) = (x, y)$, $z = x + iy$, and

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Thus in our case (conformal immersion), X is minimal (hence harmonic) implies that $H \equiv 0$, which is essentially an equivalent definition of minimal surface. In fact, this definition is easier to generalise to define minimal submanifolds in arbitrary Riemannian manifolds.

More precisely, $H \equiv 0$ implies that X is conformal harmonic under a certain complex structure. To see this, let us recall that for any immersion $X : M \hookrightarrow \mathbf{R}^3$,

$$\triangle_X X = 2HN. \tag{2.1}$$

Since we can always make X conformal, (2.1) shows that X is a minimal surface if and only if the mean curvature is zero.

Let us give the proof of (2.1) as a short review of differential geometry. Let us first recall that from the Gauss equation we have

$$X_{ij} = \sum_{k=1}^{2} \Gamma_{ij}^{k} X_k + h_{ij} N,$$

where

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right).$$

We calculate

$$\begin{aligned}
\triangle_X X &= \frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial u^i}\left(g^{ij}\sqrt{g}X_j\right) \\
&= \sum_{i,j} g^{ij}X_{ij} + \sum_{i,j}\frac{\partial g^{ij}}{\partial u^i}X_j + \frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial\sqrt{g}}{\partial u^i}g^{ij}X_j \\
&= \sum_{i,j} g^{ij}X_{ij} + \sum_{i,j}\frac{\partial g^{ij}}{\partial u^i}X_j + \frac{1}{2g}\sum_{i,j}\frac{\partial g}{\partial u^i}g^{ij}X_j.
\end{aligned}$$

Now we have an identity

$$\frac{1}{g}\frac{\partial g}{\partial u^i} = \textbf{Trace}\left(\left(g^{kl}\right)\left(\frac{\partial g_{kl}}{\partial u^i}\right)\right) = \sum_{k,l} g^{kl}\frac{\partial g_{kl}}{\partial u^i},$$

see the proof in the next section. Thus we have

$$\triangle_X X = \sum_{i,j} g^{ij}X_{ij} + \sum_{i,j}\frac{\partial g^{ij}}{\partial u^i}X_j + \frac{1}{2}\sum_{i,j,k,l} g^{ij}g^{kl}\frac{\partial g_{kl}}{\partial u^i}X_j.$$

We claim that $\triangle_X X$ is perpendicular to the *tagent planes*, i.e, planes generated by (X_1, X_2). In fact, since $\sum_j g_{ij}g^{jk} = \delta_{ik}$, we have

$$\begin{aligned}
\triangle_X X \bullet X_m &= \sum_{i,j} g^{ij}X_{ij}\bullet X_m + \sum_{i,j}\frac{\partial g^{ij}}{\partial u^i}X_j\bullet X_m + \frac{1}{2}\sum_{i,j,k,l} g^{ij}g^{kl}\frac{\partial g_{kl}}{\partial u^i}X_j\bullet X_m \\
&= \sum_{i,j,k} g^{ij}\Gamma_{ij}^{k}g_{km} + \sum_{i,j}\frac{\partial g^{ij}}{\partial u^i}g_{jm} + \frac{1}{2}\sum_{i,j,k,l} g^{ij}g^{kl}\frac{\partial g_{kl}}{\partial u^i}g_{jm} \\
&= \frac{1}{2}\sum_{i,j,k,l} g^{ij}g_{km}g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right) - \sum_{i,j} g^{ij}\frac{\partial g_{jm}}{\partial u^i} + \frac{1}{2}\sum_{k,l} g^{kl}\frac{\partial g_{kl}}{\partial u^m} \\
&= \frac{1}{2}\sum_{i,j} g^{ij}\left(\frac{\partial g_{im}}{\partial u^j} + \frac{\partial g_{jm}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^m}\right) - \sum_{i,j} g^{ij}\frac{\partial g_{jm}}{\partial u^i} + \frac{1}{2}\sum_{i,j} g^{ij}\frac{\partial g_{ij}}{\partial u^m} \\
&= 0.
\end{aligned}$$

Thus $\triangle_X X$ is in the direction of N, and

$$\triangle_X X = (\triangle_X X \bullet N)N = \left(\sum_{i,j} g^{ij} X_{ij} \bullet N\right) N = \left(\sum_{i,j} g^{ij} h_{ij}\right) N = 2HN.$$

Equation (2.1) also tells us that if X is conformal, then $\triangle X$ is always perpendicular to the corresponding tangent plane of X.

Note that equation (2.1) holds for hypersurfaces in $\mathbf{R}^n$, $n \geq 3$, our proof is valid in the general case.