1 Introduction

The theory of minimal submanifolds is a fascinating field in differential geometry. The simplest, one-dimensional minimal submanifold, the geodesic, has been studied quite exhaustively, yet there are still a lot of interesting open problems. In general, minimal submanifold theory deeply involves almost all major branches of mathematics; analysis, algebraic and differential topology, geometric measure theory, calculus of variations, and partial differential equations, to name just a few of them.

In these lecture notes our aim is quite modest. We discuss minimal surfaces in $\mathbf{R}^3$, and concentrate on the class of the embedded complete minimal surfaces of finite topological type.

I intend to introduce minimal surfaces with the minimum preliminary requirements. A student who has basic knowledge of differential geometry of curves and surfaces in $\mathbf{R}^3$ and of complex analysis will be able to understand and grasp the material supplied in these notes. I hope these notes will introduce one into a very old but still rapidly growing field of mathematics, and via it to go much further.

We begin with the definition of minimal surfaces in the setting of parametrised surfaces. We define minimal surfaces as conformal harmonic immersions from two dimensional manifolds to $\mathbf{R}^3$. Then we give the proof of the equivalence of this definition to that that the mean curvature of the surface is zero everywhere. After that, we introduce the first variation of surface area, also in the setting of parametrised surfaces, to show that a surface is minimal if and only if it is a stationary point of the area functional. Then we introduce the minimal surface equation and use it to prove several classical theorems of minimal surfaces, such as the maximum principle, the extension theorem, the reflection and rotation theorem, etc. One of the most important features of the theory of minimal surfaces in $\mathbf{R}^3$, which is quite different from the general case of minimal submanifolds in Riemannian manifolds (even in $\mathbf{R}^n$, $n > 3$), is the Enneper-Weierstrass representation. This representation connects minimal surfaces in $\mathbf{R}^3$ to one variable complex analysis. We introduce the Enneper-Weierstrass representation immediately after the necessary preparations and try to use it consistently throughout these notes.

The most interesting minimal surfaces in $\mathbf{R}^3$ are complete and are divided into two groups according to whether the total curvature is finite or infinite. We mainly discuss complete minimal surfaces of finite total curvature. We prove the classical theorem of Osserman (Theorem 10.8) about such surfaces. Then we further discuss the annular ends of such surfaces. After introducing the concept of flux (a formula based on Stokes' theorem), we prove a theorem of López and Ros about uniqueness of the catenoid.

A major part of these notes is devoted to the work of Hoffman and Meeks about global properties of complete minimal surfaces in $\mathbf{R}^3$. In particular, we introduce the Halfspace Theorem, the Cone Lemma, the standard barriers and the Annular End Theorem, and the partial classification of the conformal type of such surfaces.

An annular end of a complete minimal surface is a minimal annulus with compact boundary. In the last part of these notes we discuss minimal annuli. We first introduce results of Osserman and Schiffer, including the isoperimetric inequality for minimal annuli. Then we concentrate on minimal annuli in a slab, proving Shiffman's theorems and some generalisations. For this we first introduce the second variation of area functional and the concept of stability of minimal surfaces. We finish these notes with Nitsche's conjecture and two partial results. Recently, Pascal Collin [6] gives a proof of Nitsche's conjecture, I am regret that I cannot add it to these notes since the proof is quite involved and Collin's paper has not been published yet.

To help readers not familiar with PDE, we include an appendix on the eigenvalue problem of linear second order elliptic differential operators.

In these notes, we emphasize the close relation between minimal surfaces in $\mathbf{R}^3$ and complex analysis. This makes the theory of minimal surfaces in $\mathbf{R}^3$ both much simpler and more beautiful. But the draw back is that the methods are hardly generalisable to the study of general minimal submanifolds in Riemannian manifolds. Nevertheless, by its simplicity and beauty, the complex analysis method, via the Enneper-Weierstrass representation, deserves to be emphasized. Thus we work with isothermal coordinates and whenever possible, we try to express and analyse geometric quantities via the Enneper-Weierstrass representation. Using the Enneper-Weierstrass representation, we are able to give new proofs of the total curvature formula of a complete minimal surface of finite total curvature, and of Shiffman's second theorem and its generalisations.

A very active part of the theory of minimal surfaces in $\mathbf{R}^3$ is the construction of new embedded complete minimal surfaces. Minimal surface theory is among the oldest branches in mathematics. For over two hundred years, the only known embedded complete minimal surfaces of finite topology were the plane, the catenoid, and the helicoid. In 1984, Hoffman and Meeks started a new wave of discovery. Infinite embedded complete minimal surfaces were constructed via the Enneper-Weierstrass representation and with the aid of computer graphics. These discoveries stimulated a new wave of active researches in the theory of minimal surfaces in $\mathbf{R}^3$. It is a regret that we cannot discuss in detail the techniques of construction of minimal surfaces in these notes. The interested reader is recommoned to works such as [26], [27], [31], [39], [40], [41], [80].

Some classical topics such as the Plateau problem are not discussed here since there are already many excellent books available, for example, [9], [46], [77], [61], [37], [12]. We also do not discuss the regularity problem, which requires tools from the theory of partial differential equations, see [12].

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