Appendix F

Bifurcating geodesics in $C^{1,\alpha}_{loc}$, $0 < \alpha < 1$, metrics.

In this Appendix we present $C^{1,\alpha}_{loc}$, $0 < \alpha < 1$, metrics for which there exist bifurcating spacelike, timelike or null geodesics. From the classical theory of ODE's it is well known that the initial value problem (IVP) for the geodesic equation is uniquely solvable when the metric is $C^{1,1}_{loc}$. Due to the variational character of the geodesic equations one could hope, at least in the strictly Riemannian case, to have uniqueness of the IVP for geodesics under some weaker conditions: the examples of this Appendix show that the requirement of $C^{1,1}_{loc}$ differentiability of the metric cannot be relaxed without introducing some supplementary conditions. The examples presented here have been worked out in collaboration with J.Isenberg, following a suggestion by R.Hamilton[1].

Let us start by noting that if we have a metric which is $C^{1,1}_{loc}$ except possibly at an isolated point x_o in a neighborhood of which

$$|g(x)_{ij} - \delta_{ij}| \le Cr(x - x_o)^{\alpha}, \quad |\partial_k g_{ij}(x)| \le Cr(x - x_o)^{\alpha-1},$$

$$|\partial_k\partial_l g_{ij}(x)| \le Cr(x - x_o)^{\alpha-2}, \quad 0 < \alpha < 1, \tag{F.0.1}$$

uniqueness of the initial value problem for geodesics can be established by standard fixed point methods in an appropriately weighted space of functions (recall that existence

[1] It has been pointed out to the author by R. Bartnik, that essentially the same example has been presented in [65] for spacelike geodesics in a Riemannian metric.

follows from the classical theorem of Peano [82] which asserts that continuity of the right hand side of the equation

$$\frac{dx}{dt} = f(x,t)$$

is sufficient for existence of solutions; it is only the uniqueness part which necessitates the Lipschitz continuity of f in x). Thus it seems that for nonuniqueness of the IVP for geodesics the set at which the metric is not $C^{1,1}_{loc}$ "must have dimension greater than zero". For non–unique spacelike or timelike geodesics we show that a singular set of dimension 1 is sufficient, for null geodesics dimension 2 is sufficient and probably also necessary; we have undertaken no attempts to prove this last conjecture.

For $y \in I\!R$ and for $\epsilon|x|^{1+\alpha} > -1/2$ consider the metric

$$ds^2 = dx^2 + f(x)dy^2, \quad f(x) = \epsilon(1 + \epsilon|x|^{1+\alpha}),$$

$$\epsilon = \pm 1, \quad 0 < \alpha < 1. \tag{F.0.2}$$

The equations for a geodesic are easily found from the constants of motion,

$$f(x)\frac{dy}{ds} = p_y = const, \quad (\frac{dx}{ds})^2 + f(x)(\frac{dy}{ds})^2 = \epsilon. \tag{F.0.3}$$

Let $x_o > 0$, consider the geodesic which at $s = 0$ passes through (x_o, y_o) with $(dx/ds)(0) < 0, p_y = 1$; from (F.0.3) we have

$$\frac{dx}{ds} = -\frac{|x|^{(1+\alpha)/2}}{(1-\epsilon|x|^{1+\alpha})^{1/2}}. \tag{F.0.4}$$

For s such that $x(s) > 0$ x monotonically decreases, which gives

$$|x|^{-(1+\alpha)/2}\frac{dx}{ds} \le -c_o, \quad c_o^{-1} = min[1, (1-\epsilon|x_o|^{1+\alpha})^{1/2}] \quad \Rightarrow$$

$$|x|^{(1-\alpha)/2}(s) \le x_o^{(1-\alpha)/2} - c_o s, \tag{F.0.5}$$

thus $x(s)$ reaches 0 in finite time, say s_o. (F.0.4) implies that

$$\frac{dx}{ds}(s_o) = 0.$$

For any $s_1 \geq s_o$ we can extend the above geodesic to two different C^1 geodesics as follows: for $s \in [s_o, s_1]$ set

$$x(s) = 0, \quad y(s) = s$$

and for $s_1 \leq s \leq s_1 + s_o$ set either

$$x(s) = x(s_1 - s + s_o),$$

or

$$x(s) = -x(s_1 - s + s_o).$$

This establishes non–uniqueness of the initial value problem. Uniqueness of the boundary value problem for geodesics for $C^{1,\alpha}_{loc}$ metrics remains an open question[2]. Higher dimensional metrics with bifurcating geodesics can be trivially obtained from (F.0.2) by adding terms $dz_1^2 + dz_2^2 + \cdots + dz_k^2$ to (F.0.2).

Let us finally exhibit a metric on a three dimensional Lorentzian manifold with bifurcating null geodesics. For $|x| < 1, y, z \in I\!R$ let

$$ds^2 = dx^2 - a(x)dy^2 + b(x)dz^2,$$

$$a(x) = 1 - |x|^{1+\alpha}/2, \quad b(x) = 1 + |x|^{1+\alpha}/2, \quad 0 < \alpha < 1. \tag{F.0.6}$$

Arguments similar to the ones presented above show that the null geodesics satisfying

$$a(x)\frac{dy}{ds} = 1, \quad b(x)\frac{dz}{ds} = 1, \quad \frac{dx}{ds} = -\frac{|x|^{(1+\alpha)/2}}{(a(x)b(x))^{1/2}}, \quad x(0) = x_o > 0,$$

can be C^1 continued in a non–unique way at a finite value s_o of the affine parameter, at which they reach the plane $x = 0$.

[2]Let us recall that e.g. one has uniqueness of the boundary value problem for causal geodesics in a Lorentzian manifold for metrics which are not $C^{1,1}_{loc}$ but for which the curvature tensor, defined distributionally, is locally bounded, cf. [41].