The atomic decomposition for tent spaces on spaces of homogeneous type

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Abstract

In the Euclidean context, tent spaces, introduced by Coifman, Meyer and Stein, admit an atomic decomposition. We generalize this decomposition to the case of spaces of homogeneous type.
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Contents

1 Introduction

Tent spaces on $\mathbb{R}^n$ ($n \geq 1$) were introduced by Coifman, Meyer and Stein in [3] and this study was pursued and developed in [4]. These spaces naturally arise in harmonic analysis for such questions as nontangential behavior, Carleson measures, duality between $H^1(\mathbb{R}^n)$ (the Hardy space) and $BMO(\mathbb{R}^n)$ and the atomic decomposition in $H^1(\mathbb{R}^n)$. A relevant general setting for these questions is the framework of spaces of homogeneous type, as introduced by Coifman and Weiss in [5] and [6]. In the present note, we consider tent spaces on such spaces, and prove that they admit an atomic decomposition, following the original proof in [4].

We now define precisely our setting. Let (X, d) be a non-empty metric space endowed with a σ-finite Borel measure μ. For all $x \in X$ and all $r > 0$, denote by $B(x, r)$ the open ball centered at x with radius r, and by $V(x, r)$ its measure. We call (X, d, μ) a space of homogeneous type if, for all $x \in X$

and all $r > 0$, $V(x,r) < +\infty$ and there exists $C > 0$ such that, for all $x \in X$ and all $r > 0$,

$$V(x, 2r) \leq CV(x, r). \tag{1.1}$$

An easy consequence of (1.1) is that there exist $C, D > 0$ such that, for all $x \in X$, all $r > 0$ and all $\theta > 1$,

$$V(x, \theta r) \leq C\theta^D V(x, r). \tag{1.2}$$

There are of course many examples of spaces of homogeneous type. The simplest one is $X = \mathbb{R}^n$, $n \geq 1$, endowed with the Euclidean metric and the Lebesgue measure. Let us describe another example. Let G be a real connected Lie group endowed with a system of left-invariant vector fields $\mathbf{X} = \{\mathbf{X_1}, ..., \mathbf{X_k}\}$ satisfying the Hörmander condition. If d is the Carathéodory metric associated to $\mathbf{X}$ and μ the left-invariant Haar measure, and if, for any $r > 0$, $V(r)$ denotes the volume of any ball with radius r, then there exists $d \in \mathbb{N}^*$ such that $V(r) \sim r^d$ for $0 < r < 1$ ([11]). Moreover, either G has polynomial volume growth, *i.e.* there exists $D \in \mathbb{N}^*$ such that, for all $r > 1$, $V(r) \sim r^D$, or G has exponential volume growth, *i.e.* there exists $c_1, C_1, c_2, C_2 > 0$ such that $c_1 e^{c_2 r} \leq V(r) \leq C_1 e^{C_2 r}$ for all $r > 1$ (see [8]). Among the class of Lie groups with polynomial volume growth, there is the strict subclass of nilpotent Lie groups, a strict subclass of which is made of stratified Lie groups. A real connected Lie group with polynomial volume growth is clearly a space of homogeneous type.

Another example of space of homogeneous type is the case of connected Riemannian manifolds with nonnegative Ricci curvature (this follows from the Bishop comparison theorem, see [2]). More generally, Riemannian manifolds which are quasi-isometric to a manifold with nonnegative Ricci curvature, or cocompact covering manifolds whose deck transformation group have polynomial growth, are spaces of homogeneous type ([7]).

In discrete settings, assumption (1.1) also plays a fundamental role in analysis on graphs (see for instance [1] and the references therein), and is satisfied for instance on the Cayley graph of a finitely generated group with polynomial volume growth or on some fractal graphs, as the Sierpinsky carpet.

Let us now define tent spaces on X. For any $\alpha > 0$ and any $x \in X$, denote by $\Gamma_\alpha(x)$ the cone of aperture α with vertex $x \in X$, namely:

$$\Gamma_\alpha(x) = \{(y,t) \in X \times (0, +\infty);\ d(y,x) < \alpha t\}.$$

For any closed subset $F \subset X$, let $\mathcal{R}_\alpha(F)$ be the union of all cones with vertices in F:

$$\mathcal{R}_\alpha(F) = \bigcup_{x\in X} \Gamma_\alpha(x).$$

If $\alpha > 0$ and O is an open subset of X, then the tent over O with aperture α, denoted by $T_\alpha(O)$, is defined by:

$$T_\alpha(O) = \left(\mathcal{R}_\alpha(O^c)\right)^c.$$

Notice that

$$T_\alpha(O) = \left\{(x,t) \in X \times (0,+\infty)\,;\; d(x,O^c) \geq \alpha t\right\}.$$

In the sequel, we write $\Gamma(x)$ (resp. $\mathcal{R}(F)$ and $T(O)$ instead of $\Gamma_1(x)$ (resp. $\mathcal{R}_1(F)$ and $T_1(O)$).
For any measurable function f on $X \times (0,+\infty)$ and any $x \in X$, define

$$\mathcal{S}f(x) = \left(\iint_{\Gamma(x)} \frac{|f(y,t)|^2}{V(x,t)} d\mu(y)\frac{dt}{t}\right)^{\frac{1}{2}},$$

and, for all $p > 0$, say that $f \in T^p(X)$ if

$$\|f\|_{T^p(X)} := \|\mathcal{S}f\|_{L^p(X)} < +\infty.$$

We have the following notion of atom (see [4], p. 312):

Definition 1.1. Let $p \in (0,+\infty)$. A measurable function a on $X \times (0,+\infty)$ is said to be a $T^p(X)$ atom if there exists a ball $B \subset X$ such that a is supported in $T(B)$ and

$$\iint_{X\times(0,+\infty)} |a(y,t)|^2\, d\mu(y)\frac{dt}{t} \leq \frac{1}{V(B)^{\frac{2}{p}-1}}.$$

It is plain to see that a $T^p(X)$-atom belongs to $T^p(X)$ and that its norm is controlled by a constant only depending on X and p. Conversely, when $0 < p \leq 1$, it turns out that any function in $T^p(X)$ has an atomic decomposition, and this is the result we prove in the sequel:

Theorem 1.1. Let $p \in (0,1]$. Then, there exists $C_p > 0$ with the following property: for all $f \in T^p(X)$, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \in l^p$ and a sequence of $T^p(X)$ atoms $(a_n)_{n\in\mathbb{N}}$ such that

$$f = \sum_{n=0}^{\infty} \lambda_n a_n$$

and

$$\sum_{n=0}^{\infty} |\lambda_n|^p \le C_p^p \, \|f\|_{T^p(X)}^p \, .$$

2 Proof of the atomic decomposition

The proof of Theorem 1.1, for which we closely follow [4], requires the notion of γ-density (see [4]). Let F be a closed subset of X, and $O = F^c$. Assume that $\mu(O) < +\infty$. For any fixed $\gamma \in \,]0,1[$, say that $x \in X$ has global γ-density with respect to F if

$$\frac{\mu(B \cap F)}{\mu(B)} \ge \gamma$$

for any ball B centered at x. The set of all such x's is denoted by F^*. It is a closed subset of F. Define also $O^* = (F^*)^c$. It is clear that $O \subset O^*$. Moreover,

$$O^* = \{x; M(\mathbf{1}_O)(x) > 1-\gamma\}$$

where M denotes the Hardy-Littlewood maximal function. As a consequence,

$$\mu(O^*) \le C_\gamma \mu(O). \tag{2.1}$$

The following integration lemma will be used:

Lemma 2.1. *Let $\eta \in (0,1)$. Then, there exists $\gamma \in \,]0,1[$ and $C_{\gamma,\eta} > 0$ such that, for any closed subset F of X whose complement has finite measure and any nonnegative measurable function $H(y,t)$ on $X \times \,]0,+\infty[$,*

$$\iint_{\mathcal{R}_{1-\eta}(F^*)} H(y,t)V(y,t)d\mu(y)dt \le C_{\gamma,\eta} \int_F \left(\iint_{\Gamma(x)} H(y,t)d\mu(y)dt \right) d\mu(x),$$

where F^* denotes the set of points in X with global density γ with respect to F.

Proof: We claim that there exists $c' > 0$ such that, for all $(y,t) \in \mathcal{R}_{1-\eta}(F^*)$,

$$\mu(F \cap B(y,t)) \geq c'V(y,t). \tag{2.2}$$

Assume that this is proved. Write

$$\int_F \left(\iint_{\Gamma(x)} H(y,t) d\mu(y) dt \right) d\mu(x) = \iiint H(y,t) \mathbf{1}_E(x,y,t) d\mu(x) d\mu(y) dt$$

where

$$E = \{(x,y,t) \in F \times X \times (0,+\infty);\ d(y,x) < t\}.$$

One therefore has

$$\begin{aligned}
\int_F \left(\iint_{\Gamma(x)} H(y,t) d\mu(y) dt \right) d\mu(x) &= \iint_{\mathcal{R}(F)} H(y,t) \mu(F \cap B(y,t)) d\mu(y) dt \\
&\geq \iint_{\mathcal{R}_{1-\eta}(F^*)} H(y,t) \mu(F \cap B(y,t)) d\mu(y) dt \\
&\geq c' \iint_{\mathcal{R}_{1-\eta}(F^*)} H(y,t) V(y,t) d\mu(y) dt.
\end{aligned}$$

Let us now prove (2.2). If $(y,t) \in \mathcal{R}_{1-\eta}(F^*)$, then there exists $x \in F^*$ such that $d(y,x) < (1-\eta)t$. One may write

$$\mu(F \cap B(y,t)) \geq \mu(F \cap B(x,t)) - \mu(B(x,t) \cap (B(y,t))^c).$$

But, since $x \in F^*$, $\mu(F \cap B(x,t)) \geq \gamma V(x,t)$. Moreover, since $d(y,x) < (1-\eta)t$, $B(x,\eta t) \subset B(y,t)$, so that

$$\mu(B(x,t) \cap B(y,t)) \geq V(x,\eta t) \geq \delta V(x,t),$$

where $\delta = \frac{1}{C}\eta^D$ and C, D are given by (1.2). It follows that there exists $c \in (0,1)$ only depending on η and the constants in (1.2) such that

$$\mu(B(x,t) \cap B(y,t)^c) \leq cV(x,t).$$

As a consequence, if $1 > \gamma > c$, one obtains, using (1.1) once more,

$$\begin{aligned}
\mu(F \cap B(y,t)) &\geq (\gamma - c)V(x,t) \\
&\geq c'V(y,t).
\end{aligned}$$

This concludes the proof of Lemma 2.1. $\square$

We now turn to the proof of Theorem 1.1. Let $f \in T^p(X)$. For any integer $k \in \mathbb{Z}$, define

$$O_k = \left\{x \in X;\ \mathcal{S}f(x) > 2^k\right\}$$

and let $F_k = O_k^c$. The O_k's are open subsets of X and, since $\mathcal{S}f \in L^1(X)$, $\mu(O_k) < +\infty$ for all $k \in \mathbb{Z}$. Fix $\eta \in (0,1)$ and consider also, for γ given by Lemma 2.1, the set F_k^* of all the points of global γ- density with respect to F_k, and $O_k^* = (F_k^*)^c$.

We claim that

$$\text{supp} f \subset \bigcup_k T_{1-\eta}(O_k^*). \tag{2.3}$$

Indeed, according to Lemma 2.1, for any $k \in \mathbb{Z}$,

$$\begin{aligned}\iint_{\mathcal{R}_{1-\eta}(F_k^*)} |f(y,t)|^2 \, d\mu(y)\frac{dt}{t} &\leq C\int_{F_k}\left(\iint_{\Gamma(x)} \frac{|f(y,t)|^2}{V(y,t)} d\mu(y)\frac{dt}{t}\right) d\mu(x)\\ &\leq C'\int_{F_k} (\mathcal{S}f)^2(x)d\mu(x).\end{aligned}$$

When $k \to -\infty$, the dominated convergence theorem shows that $\displaystyle\int_{F_k} (\mathcal{S}f)^2(x)d\mu(x) \to 0$. It follows that

$$\iint_{\bigcap_j \mathcal{R}_{1-\eta}(F_j^*)} |f(y,t)|^2 \, d\mu(y)\frac{dt}{t} = 0.$$

This shows that f is zero on almost every point of $\bigcap_j \mathcal{R}_{1-\eta}(F_j^*)$. In other words, (2.3) holds.

We make use of the following lemma (see [3], Ch 3, Th 1.3; see also [9]):

Lemma 2.2. *Let Ω be a proper open subset of finite measure of X. For any $x \in X$, define $r(x) = \dfrac{d(x,\Omega^c)}{10}$. Then, there exist an integer M and a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X such that, if $r_n = r(x_n)$,*

$$\Omega = \bigcup_n B(x_n, r_n),$$

$$i \neq j \Longrightarrow \frac{1}{4}B(x_i,r_i) \cap \frac{1}{4}B(x_j,r_j) = \emptyset,$$

$$\forall n,\ \left|\{m; B(x_n,5r_n) \cap B(x_m,5r_m) \neq \emptyset\}\right| \leq M.$$

Moreover, there exists a sequence of nonnegative functions $(\varphi_n)_{n\in\mathbb{N}}$ *on* X *such that*

$$\operatorname{supp} \varphi_n \subset B(x_n, 2r_n),$$

$$\forall x \in B(x_n, r_n),\ \varphi_n(x) \geq \frac{1}{M},$$

$$\sum_n \varphi_n = \mathbf{1}_\Omega.$$

Let $k \in \mathbb{Z}$. If O_k^* is a proper subset of X, apply this lemma with $\Omega = O_k^*$. The points x_n will be denoted by x_n^k, the radii r_n by r_n^k, the balls $B(x_n^k, r_n^k)$ by B_n^k and the functions φ_n by φ_n^k, where $n \in I^k$ and I^k is a denumerable set. If $O_k^* = X$, then $\mu(X) < +\infty$, which forces X to be bounded ([10]). In this situation, set $I^k = \{1\}$, and define $B_1^k = X$ (indeed, X is a ball itself) and $\varphi_1^k(x) = 1$ for all $x \in X$. One has, for any $(x,t) \in X \times \mathbb{R}_+^*$,

$$\left(\mathbf{1}_{T_{1-\eta}(O_k^*)} - \mathbf{1}_{T_{1-\eta}(O_{k+1}^*)}\right)(x,t) = \sum_{j\in I^k} \varphi_j^k(x) \left(\mathbf{1}_{T_{1-\eta}(O_k^*)} - \mathbf{1}_{T_{1-\eta}(O_{k+1}^*)}\right)(x,t).$$

Indeed, if $(x,t) \in T_{1-\eta}(O_k^*) \setminus T_{1-\eta}(O_{k+1}^*)$, then $x \in O_k^*$, and the two sides of the identity are equal to 1. Otherwise, they are both equal to zero. From this and (2.3), it follows that

$$\begin{aligned} f(x,t) &= \sum_{k\in\mathbb{Z}} f(x,t) \left(\mathbf{1}_{T_{1-\eta}(O_k^*)} - \mathbf{1}_{T_{1-\eta}(O_{k+1}^*)}\right)(x,t) \\ &= \sum_{k\in\mathbb{Z}} \sum_{j\in I^k} f(x,t)\varphi_j^k(x) \left(\mathbf{1}_{T_{1-\eta}(O_k^*)} - \mathbf{1}_{T_{1-\eta}(O_{k+1}^*)}\right)(x,t). \end{aligned}$$

Define, for all $k \in \mathbb{Z}$ and all $j \in I^k$,

$$\begin{aligned} \mu_j^k &= \iint |f(y,t)|^2 \varphi_j^k(y)^2 \left(\mathbf{1}_{T_{1-\eta}(O_k^*)} - \mathbf{1}_{T_{1-\eta}(O_{k+1}^*)}\right)(y,t) d\mu(y) \frac{dt}{t}, \\ a_j^k(y,t) &= f(y,t)\varphi_j^k(y) \left(\mathbf{1}_{T_{1-\eta}(O_k^*)} - \mathbf{1}_{T_{1-\eta}(O_{k+1}^*)}\right)(y,t) V(B_j^k)^{\frac{1}{2}-\frac{1}{p}} (\mu_j^k)^{-\frac{1}{2}}, \\ \lambda_j^k &= V(B_j^k)^{\frac{1}{p}-\frac{1}{2}} (\mu_j^k)^{\frac{1}{2}}. \end{aligned}$$

Then

$$f = \sum_{k\in\mathbb{Z}} \sum_{j\in I^k} \lambda_j^k a_j^k.$$

We claim that, up to a multiplicative constant, the a_j^k's are $T^p(X)$ atoms. To begin with, notice that

$$\operatorname{supp} a_j^k \subset T(CB_j^k) \tag{2.4}$$

where $C := 2 + \frac{12}{1-\eta}$. Indeed, this is obvious when $O_k^* = X$, since $B_1^k = X$ in this case. Assume therefore that O_k^* is a proper subset of X and let $(y,t) \in T_{1-\eta}(O_k^*)$ such that $\varphi_j^k(y) > 0$. Then, $d(y,(O_k^*)^c) \geq (1-\eta)t$ and $y \in 2B_j^k$. We intend to prove that $d(y,(CB_j^k)^c) \geq t$. Let $z \in (CB_j^k)^c$. Then

$$\begin{aligned} d(y,z) &\geq d(z,x_j^k) - d(y,x_j^k) \\ &\geq (C-2)r_j^k. \end{aligned}$$

Moreover, by definition of r_j^k, $d(x_j^k,(O_k^*)^c) = 10r_j^k$. Let $\varepsilon > 0$. There exists $u \notin O_k^*$ such that $d(x_j^k,u) < 10r_j^k + \varepsilon$. Since $u \in (O_k^*)^c$ while $d(y,(O_k^*)^c) \geq (1-\eta)t$, one has

$$\begin{aligned} (1-\eta)t &\leq d(y,u) \\ &\leq d(y,x_j^k) + d(x_j^k,u) \\ &\leq 2r_j^k + 10r_j^k + \varepsilon \end{aligned}$$

and, since it is true for every $\varepsilon > 0$, it follows that $(1-\eta)t \leq 12r_j^k$. Finally, by the choice of C, one has $d(y,z) \geq t$. Thus, (2.4) holds.

The very definition of a_j^k implies that

$$\begin{aligned} \iint \left|a_j^k(y,t)\right|^2 d\mu(y)\frac{dt}{t} &= \frac{1}{V(B_j^k)^{\frac{2}{p}-1}} \\ &\leq \frac{C'}{V(CB_j^k)^{\frac{2}{p}-1}}, \end{aligned}$$

where the last line is due to (1.2). What remains to be proved is that

$$\sum_{k\in\mathbb{Z}}\sum_{j\in I^k} \left|\lambda_j^k\right|^p \leq C \left\|\mathcal{S}f\right\|_p^p.$$

To this purpose, write

$$\mu_j^k \leq \iint_{T(CB_j^k)\cap(T_{1-\eta}(O_{k+1}^*))^c} |f(y,t)|^2 \, d\mu(y)\frac{dt}{t}$$

and apply Lemma 2.1 to

$$H(y,t) = \frac{|f(y,t)|^2}{tV(y,t)} \mathbf{1}_{T(CB_j^k)}(y,t)$$

and

$$F = F_{k+1} = O_{k+1}^c.$$

This yields

$$\iint_{T(CB_j^k)\cap(T_{1-\eta}(O_{k+1}^*))^c} |f(y,t)|^2 \, d\mu(y)\frac{dt}{t} \leq C \int_{O_{k+1}^c} \left(\iint_{\Gamma(x)\cap T(CB_j^k)} \frac{|f(y,t)|^2}{V(y,t)} d\mu(y)\frac{dt}{t} \right) d\mu(x).$$

If $(y,t) \in \Gamma(x) \cap T(CB_j^k)$, then $x \in CB_j^k$. It follows that

$$\begin{aligned}
\iint_{T(CB_j^k)\cap(T(O_{k+1}^*))^c} |f(y,t)|^2 \, d\mu(y)\frac{dt}{t} &\leq C \int_{CB_j^k \cap O_{k+1}^c} (\mathcal{S}f)^2 (x) d\mu(x) \\
&\leq C(2^{k+1})^2 V(CB_j^k) \\
&\leq C' 2^{2k} V(B_j^k).
\end{aligned}$$

Thus, $\mu_j^k \leq C2^{2k}V(B_j^k)$, and, by (1.2),

$$\begin{aligned}
\lambda_j^k &= V(B_j^k)^{\frac{1}{p}-\frac{1}{2}} (\mu_j^k)^{\frac{1}{2}} \\
&\leq C2^k V(B_j^k)^{\frac{1}{p}} \\
&\leq C2^k V\left(\frac{1}{4}B_j^k\right)^{\frac{1}{p}}.
\end{aligned}$$

Since, for all $k \in \mathbb{Z}$, the $\frac{1}{4}B_j^k$ are pairwise disjoint for $i \in I^k$ and included in

O_k^*, one has, by (2.1),

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\sum_{j\in I^k}\left|\lambda_j^k\right|^p &\leq C\sum_{k\in\mathbb{Z}}2^{kp}\mu(O_k^*)\\
&\leq C'\sum_{k\in\mathbb{Z}}2^{kp}\mu(O_k)\\
&\leq Cp\sum_{k\in\mathbb{Z}}(2^{k-1})2^{k(p-1)}\mu\left(\{\mathcal{S}f>2^k\}\right)\\
&\leq Cp\sum_{k\in\mathbb{Z}}\int_{2^{k-1}}^{2^k}t^{p-1}\mu\left(\{\mathcal{S}f>t\}\right)dt\\
&= Cp\int_0^{+\infty}t^{p-1}\mu\left(\{\mathcal{S}f>t\}\right)dt\\
&= C\left\|\mathcal{S}f\right\|_p^p.
\end{aligned}$$

The proof of Theorem 1.1 is complete. □

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