Lecture 2: Stratifications in o-minimal structures

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Introduction

This note is devoted to the study of stratifications of definable subjects in o-minimal structures. The main results come from [**L1**]-[**L4**]. For the theory of stratifications, we refer the readers to [**Ma**],[**GPW**], [**T1**] and [**T2**].

In Section 1 we prove that the definable sets admit Verdier Stratification, and that the Verdier condition (w) implies the Whitney condition (b) in o-minimal structures. Note that the theorems were proved for subanalytic sets in [V] and [ŁSW] (see also [DW]), the former based on Hironaka's Desingularization, and the latter on Puiseux's Theorem. But, in general, these tools cannot be applied to sets belonging to o-minimal structures (e.g. to the set $\{(x,y) \in \mathbb{R}^2 : y = \exp(-1/x), x > 0\}$ in the structure generated by the exponential function).

In Section 2 we study the stratifications of definable functions. First we prove the existence of the stratifications of definable maps. Then we come to the existence of stratifications satisfying the Thom condition (a_f) for continuous functions definable in any o-minimal structures. In general, definable functions cannot be stratified to satisfy the strict Thom condition (w_f). However, if the structure is polynomially bounded, then its definable functions admit (w_f)-stratification. Our proof of this assertion is based on piecewise uniform asymptotics for definable functions from [M2], instead of Pawłucki's version of Puiseux's theorem with parameters, which is used in [KP] to prove the assertion for subanalytic functions.

Notations and Conventions. Throughout this note, let $\mathcal{D}$ denote some fixed, but arbitrary, o-minimal structure on $(\mathbb{R}, +, \cdot)$. "*Definable*" means definable in $\mathcal{D}$. Let p be a positive integer. If $\mathbb{R}^k \times \mathbb{R} \ni (y,t) \mapsto f(y,t) \in \mathbb{R}^m$ is a differentiable function, then $D_1 f$ denotes the derivative of f with respect to the first variables y. As usual, $d(\cdot,\cdot), \|\cdot\|$ denote the Euclidean distance and norm respectively. We will often use Cell decomposition theorem and Definable choice (see Lecture 1) in our arguments without citations. Submanifolds will always be embedded submanifolds.

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1. Stratifications of definable sets

DEFINITION 1.1. Let Γ, Γ' be C^1 submanifolds of $\mathbb{R}^n$ such that $\Gamma \subset \overline{\Gamma}' \setminus \Gamma'$. Let $T_y\Gamma$ denotes the tangent space of Γ at y. The distance of vector subspaces T, T' of $\mathbb{R}^n$ is defined by

$$\delta(T,T') = \sup_{v\in T, \|v\|=1} d(v,T').$$

• **Whitney and Verdier conditions.** Let y_0 be a point of Γ. We say that the pair (Γ, Γ') satisfies the Whitney condition (b) at y_0 if the following holds:
(b) For every sequence (x_k) in Γ' and (y_k) in Γ converging to y_0 such that $(T_{x_k}\Gamma')$ converges to a limit τ and $(\mathbb{R}(x_k - y_k))$ converges to a limit l, then $l \subset \tau$.
We say that the pair (Γ, Γ') satisfies the Verdier condition at y_0 if the following holds:
(w) There exists a constant $C > 0$ and a neighborhood U of y_0 in $\mathbb{R}^n$ such that

$$\delta(T_y\Gamma, T_x\Gamma') \ \leq \ C\|x-y\| \quad \text{for all } x \in \Gamma' \cap U, y \in \Gamma \cap U,$$

Note that (w) is invariant under C^2-diffeomorphisms.
• A *definable C^p stratification* of $\mathbb{R}^n$ is a partition $\mathcal{S}$ of $\mathbb{R}^n$ into finitely many subsets, called strata, such that:
(S1) Each stratum is a connected C^p submanifold of $\mathbb{R}^n$ and also definable set.
(S2) For every $\Gamma \in \mathcal{S}$, $\overline{\Gamma} \setminus \Gamma$ is a union of some of the strata.
• A *definable C^p Whitney stratification* (resp. *Verdier stratification*) is a definable C^p stratification $\mathcal{S}$ such that for all $\Gamma, \Gamma' \in \mathcal{S}$, if $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$, then (Γ, Γ') satisfies the condition (b) (resp. (w)) at each point of Γ.
• We say that $\mathcal{S}$ is *compatible with* a class $\mathcal{A}$ of subsets of $\mathbb{R}^n$ if each $A \in \mathcal{A}$ is a finite union of some strata in $\mathcal{S}$.

Main results of this section are the following theorems:

THEOREM 1.2 (Verdier Stratification). *Let $A_1, \cdots, A_k$ be definable sets in $\mathbb{R}^n$. Then there exists a definable C^p Verdier stratification of $\mathbb{R}^n$ compatible with $\{A_1, \cdots, A_k\}$.*

THEOREM 1.3 (Whitney Stratification). *Let $A_1, \cdots, A_k$ be definable sets in $\mathbb{R}^n$. Then there exists a definable C^p Whitney stratification of $\mathbb{R}^n$ compatible with $\{A_1, \cdots, A_k\}$.*

To prove the theorems, we first make an observation similar to that of [ŁSW]. Let (P) be a property of pairs (Γ, Γ') at y in Γ, where Γ, Γ' are subsets of $\mathbb{R}^n$. Put $P(\Gamma, \Gamma') = \{y \in \Gamma : (\Gamma, \Gamma') \text{ satisfies (P) at } y\}$.

PROPOSITION 1.4. *Suppose that for every pair (Γ, Γ') of definable C^p submanifolds of $\mathbb{R}^n$ with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$ and $\Gamma \neq \emptyset$, the set $P(\Gamma, \Gamma')$ is definable and $\dim(\Gamma \setminus P(\Gamma, \Gamma')) < \dim \Gamma$. Then given definable sets $A_1, \cdots, A_k$ contained in $\mathbb{R}^n$, there exists a definable C^p stratification $\mathcal{S}$ of $\mathbb{R}^n$ compatible with $\{A_1, \cdots, A_k\}$ such that*

(P) $\quad P(\Gamma, \Gamma') = \Gamma \quad$ *for all* $\Gamma, \Gamma' \in \mathcal{S}$ *with* $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$ *and* $\Gamma \neq \emptyset$.

PROOF. Similar to the proof of [ŁSW, Prop. 2]. We can construct, by decreasing induction on $d \in \{0, \cdots, n\}$, partitions $\mathcal{S}^d$ of $\mathbb{R}^n$ into C^p-cells compatible with $\{A_1, \cdots, A_k\}$, such that $\mathcal{S}^d$ has properties (S1)(S2) and the following property:

$(P_d) \qquad P(\Gamma, \Gamma') = \Gamma \quad$ for all $\Gamma, \Gamma' \in \mathcal{S}^d$ with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$ and $\dim \Gamma \geq d$.

Indeed, by Cell Decomposition and the fact that $\dim(\overline{A} \setminus A) < \dim A$, for all definable set A, we can construct a C^p cell decomposition of $\mathbb{R}^n$ compatible with $\{A_1, \cdots, A_k\}$ and has (S1)(S2). This cell decomposition can be refined to satisfy (P_d) by the assumption.
Obviously, $\mathcal{S} = \mathcal{S}^0$ is a desired stratification. □

By the proposition, Theorem 1.2 is a consequence of the following.

PROPOSITION 1.5. *Let Γ, Γ' be definable C^p-submanifolds of $\mathbb{R}^n$. Suppose that $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$ and $\Gamma \neq \emptyset$. Then* $W = \{y \in \Gamma : (\Gamma, \Gamma') \text{ satisfies (w) at } y\}$ *is definable, and* $\dim(\Gamma \setminus W) < \dim \Gamma$.

To prove the proposition we prepare some lemmas.

LEMMA 1.6. *Under the notation of Proposition 1.5, W is a definable set.*

PROOF. Note that the Grassmannian $G_k(\mathbb{R}^n)$ of k-dimensional linear subspaces of $\mathbb{R}^n$ is semialgebraic, and hence definable; δ and the tangent map: $\Gamma \ni x \mapsto T_x\Gamma \in G_{\dim \Gamma}(\mathbb{R}^n)$ are also definable. (To see this, first note that

$$\begin{array}{rcl} A & = & \{(\lambda, T) : \lambda \in G_1(\mathbb{R}^n), T \in G_k(\mathbb{R}^n), \lambda \subset T\}, \\ B & = & \{(x, y, \lambda) : x \in \Gamma, y \in \Gamma, x \neq y, \lambda = \mathbb{R}(x - y)\} \end{array}$$

are definable sets. So $C = \overline{B} \cap \Delta_\Gamma \times G_1(\mathbb{R}^n)$, where $\Delta_\Gamma = \{(x, x) : x \in \Gamma\}$, is a definable set. These imply that the graph of the tangent map belongs to $\mathcal{D}$, because

$$\{(x, T_x\Gamma) : x \in \Gamma\} = \{(x, T) : x \in \Gamma, T \in G_{\dim \Gamma}(\mathbb{R}^n), \forall (x, x, \lambda) \in C, (\lambda, T) \in A\}).$$

Therefore,

$$\begin{array}{rl} W = \{y_0 : & y_0 \in \Gamma, \exists C > 0, \exists t > 0, \forall x \in \Gamma', \forall y \in \Gamma \\ & (\|x - y_0\| < t, \|y - y_0\| < t \Rightarrow \delta(T_y\Gamma, T_x\Gamma') \leq C\|x - y|\} \end{array}$$

is a definable set. □

LEMMA 1.7. *(Wing Lemma). Let $V \subset \mathbb{R}^k$ be a nonempty open definable set, and $S \subset \mathbb{R}^k \times \mathbb{R}^l$ be a definable set. Suppose $V \subset \overline{S} \setminus S$. Then there exist a nonempty open subset U of V, $\alpha > 0$, and a definable map $\bar{\rho} : U \times (0, \alpha) \longrightarrow S$, of class C^p, such that $\bar{\rho}(y, t) = (y, \rho(y, t))$ and $\|\rho(y, t)\| = t$, for all $y \in U, t \in (0, \alpha)$.*

PROOF. See Lecture 1. □

To control the tangent spaces we need the following lemma.

LEMMA 1.8. *Let $U \subset \mathbb{R}^k$ be a nonempty open definable set, and $M : U \times (0, \alpha) \longrightarrow \mathbb{R}^l$ be a C^1 definable map. Suppose there exists $K > 0$ such that $\|M(y, t)\| \leq K$, for all $y \in U$ and $t \in (0, \alpha)$. Then there exists a definable set F, closed in U with $\dim F < \dim U$, and continuous definable functions $C, \tau : U \setminus F \longrightarrow \mathbb{R}_+$, such that*

$$\|D_1 M(y, t)\| \leq C(y) \ , \quad \text{for all } y \in U \setminus F \text{ and } t \in (0, \tau(y)).$$

PROOF. It suffices to prove this for $l = 1$. Suppose the assertion of the lemma is false. Since $\{y \in U : \lim_{t \to 0^+} \|D_1 M(y, t)\| = +\infty\}$ is definable, there is an open subset B of U, such that

$$\lim_{t \to 0^+} \|D_1 M(y, t)\| = +\infty, \quad \text{for all } y \text{ in } B.$$

By Monotonicity theorem, for each $y \in B$, there is $s > 0$ such that $t \mapsto \|D_1 M(y,t)\|$ is strictly decreasing on $(0,s)$. Let

$$\tau(y) = \sup\{s : \|D_1 M(y,\cdot)\| \text{ is strictly decreasing on } (0,s)\}.$$

Note that τ is a definable function, and, by Cell Decomposition, τ is continuous on an open subset B' of B, and $\tau > \alpha'$ on B', for some $\alpha' > 0$. Let $\psi(t) = \inf\{\|D_1 M(y,t)\| : y \in B',\ 0 < t < \alpha'\}$. Shrinking B', we can assume that $\lim\limits_{t\to 0^+} \psi(t) = +\infty$. Then, for each $y \in B'$, we have

$$\|D_1 M(y,t)\| > \psi(t)\ , \quad \text{for all } t \in (0,\alpha').$$

This implies $|M(y,t) - M(y',t)| > \psi(t)\|y - y'\|$, for all $y, y' \in B'$, and $t < \alpha'$. Therefore, $\psi(t) \le \dfrac{2K}{\operatorname{diam} B'}$, for all $t \in (0,\alpha')$, a contradiction. □

Proof of Proposition 1.5. The first part of the proposition was proved in Lemma 1.7. To prove the second part we suppose, contrary to the assertion, that $\dim(\Gamma \setminus W) = \dim \Gamma = k$.
Since (w) is a local property and invariant under C^2 local diffeomorphisms, we can suppose Γ is an open subset of $\mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$. In this case $T_y\Gamma = \mathbb{R}^k$, for all $y \in \Gamma$. Then by the assumption, applying Lemma 1.7, we get an open subset U of Γ, a C^p definable map $\bar\rho : U \times (0,\alpha) \longrightarrow \Gamma'$ such that $\bar\rho(y,t) = (y, \rho(y,t))$ and $\|\rho(y,t)\| = t$, and, moreover, for each $y \in U$

$$\frac{\delta(\mathbb{R}^k, T_{(y,\rho(y,t))}\Gamma')}{\|\rho(y,t)\|} \to +\infty\ , \quad \text{when } t \to 0^+.$$

On the other hand, applying Lemma 1.8 to $M(y,t) := \dfrac{\rho(y,t)}{t}$ and shrinking U and α, we have

$$\|D_1\rho(y,t)\| \le Ct, \quad \text{for all } y \in U, t \in (0,\alpha),$$

with some $C > 0$.
Note that $T_{(y,\rho(y,t))}\Gamma' \supset \operatorname{graph} D_1\rho(y,t)$. Therefore,

$$\frac{\delta(\mathbb{R}^k, T_{(y,\rho(y,t))}\Gamma')}{\|\rho(y,t)\|} \le \frac{\|D_1\rho(y,t)\|}{\|\rho(y,t)\|} \le C, \quad \text{for } y \in U, 0 < t < \alpha.$$

This is a contradiction. *Box*

Note that Whitney's condition (b) does not imply condition (w), even for algebraic sets (see [**BT**]). And, in general, we do not have (w) $\Rightarrow$ (b) (e.g. $\Gamma = (0,0)$, $\Gamma' = \{(x,y) \in \mathbb{R}^2 : x = r\cos r, y = r\sin r, r > 0\}$, or $\Gamma' = \{(x,y) \in \mathbb{R}^2 : y = x\sin(1/x), x > 0\}$). In o-minimal structures such spiral phenomena or oscillation cannot occur. The following is a version of Kuo-Verdier's Theorem (see[**K**] and [**V**]).

Proposition 1.9. *Let $\Gamma, \Gamma' \subset \mathbb{R}^n$ be definable C^p-submanifolds ($p \ge 2$), with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma$. If (Γ, Γ') satisfies the condition (w) at $y \in \Gamma$, then it satisfies the Whitney condition (b) at y.*

Proof. Our proof is an adaptation of [V, Theorem 1.5] and based on the following observation:
If $f : (0,\alpha) \longrightarrow \mathbb{R}$ is definable with $f(t) \ne 0$, for all t, and $\lim\limits_{t\to 0^+} f(t) = 0$, then,

by Cell Decomposition and Monotonicity, there is $0 < \alpha' < \alpha$, such that f is of class C^1 and strictly monotone on $(0, \alpha')$. By Mean Value Theorem and Definable Choice, there exists a definable function $\theta : (0, \alpha') \to (0, \alpha')$ with $0 < \theta(t) < t$, such that $f(t) = f'(\theta(t))t$. Since $|f(t)| > |f(\theta(t))|$, by Monotonicity, $\lim_{t\to 0^+} \dfrac{f(t)}{f'(t)} = 0$.
Now we prove the proposition . By a C^2 change of local coordinates, we can suppose Γ is an open subset of $\mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^l$ $(l = n - k)$, and $y = 0$. Let $\pi : \mathbb{R}^k \times \mathbb{R}^l \longrightarrow \mathbb{R}^l$ be the orthogonal projection. Since (Γ, Γ') satisfies (w) at 0, there exists $C > 0$ and a neighborhood U of 0 in $\mathbb{R}^n$, such that

$$(*) \qquad \delta(T_y\Gamma, T_x\Gamma') \leq C\|x - y\| \ , \quad \text{for all } x \in \Gamma' \cap U, y \in \Gamma \cap U.$$

If the condition (b) is not satisfied at 0 for (Γ, Γ'), then there exists $\epsilon > 0$, such that $0 \in \overline{S} \setminus S$, where

$$S = \{x \in \Gamma' : \delta(\mathbb{R}\pi(x), T_x\Gamma') \geq 2\epsilon\}.$$

Since $S \cap \{x : \|x\| \leq t\} \neq \emptyset$, for all $t > 0$, by Curve selection, there exists a definable curve $\varphi : (0, \alpha) \longrightarrow S$, such that $\|\varphi(t)\| \leq t$, for all t. By the above observation, we can assume φ is of class C^1. Write $\varphi(t) = (u(t), v(t)) \in \mathbb{R}^k \times \mathbb{R}^l$. Then $\|u'(t)\|$ is bounded. Since $\varphi((0, \alpha)) \subset \Gamma'$, $v \not\equiv 0$. Shrinking α, we can assume $v'(t) \neq 0$, for all t. Since $\lim_{t\to 0^+} v'(t)$ exists, we have $\delta(\mathbb{R}v'(t), \mathbb{R}v(t)) \to 0$, when $t \to 0$. Therefore

$$(**) \qquad \delta(\mathbb{R}v'(t), T_{\varphi(t)}\Gamma') \geq \epsilon \ , \ \text{ for all } t \text{ sufficiently small.}$$

On the other hand, we have

$$\begin{aligned} \delta(\mathbb{R}v'(t), T_{\varphi(t)}\Gamma') &= \frac{1}{\|v'(t)\|}\delta(v'(t), T_{\varphi(t)}\Gamma') = \frac{1}{\|v'(t)\|}\delta(u'(t), T_{\varphi(t)}\Gamma') \\ &\leq \frac{\|u'(t)\|}{\|v'(t)\|}\delta(\mathbb{R}u'(t), T_{\varphi(t)}\Gamma'). \end{aligned}$$

From $(*)$ and $(**)$, we have $\epsilon \leq C\|v(t)\| \dfrac{\|u'(t)\|}{\|v'(t)\|}$.
By the observation, the right-hand side of the inequality tends to 0 (when $t \to 0$), which is a contradiction. □

Note that Theorem 1.2 and Proposition 1.9 together yield the Whitney Stratification Theorem 1.3 (c.f. [**DM**],[**L1**],[**S**]).

2. Stratifications of definable functions

Definition 2.1. Let $f : X \to Y$ be a definable map. A C^p *stratification* of f is a pair $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are definable C^p Whitney stratifications of X and Y respectively, and for each $\Gamma \in \mathcal{X}$, there exists $\Phi \in \mathcal{Y}$, such that $f(\Gamma) \subset \Phi$ and $f|_\Gamma : \Gamma \to \Phi$ is a C^p submersion.

We provide a proof of the existence of the stratifications of definable maps. The theorem is proved in [**DM**] Th.4.8. with a gap.

Theorem 2.2. *Let $f : X \to Y$ be a continuous definable map. Let $\mathcal{A}$ and $\mathcal{B}$ be finite collections of definable subsets of X and Y respectively. Then there exists a C^p stratification $(\mathcal{X}, \mathcal{Y})$ of f such that $\mathcal{X}$ is compatible with $\mathcal{A}$ and $\mathcal{Y}$ is compatible with $\mathcal{B}$.*

Proof. We follow closely the proof of [S] Th.I.2.6 for subanalytic maps. Let $m = \dim Y$. We will construct a chain of definable sets

$$Y^m \subset Y^{m-1} \subset \cdots \subset Y^0 = Y \ ,$$

and the pairs $(\mathcal{X}^k, \mathcal{Y}^k)$, $k = m, m-1, \cdots, 0$, satisfying the following conditions

(F_k) $Y \setminus Y^k$ is a closed subset of Y and $\dim(Y \setminus Y^k) < k$; $\mathcal{X}^k$ is a definable C^p Whitney stratification of $X^k = f^{-1}(Y^k)$ compatible with $\mathcal{A}$; $\mathcal{Y}^k$ is a definable C^p Whitney stratification of Y^k compatible with $\mathcal{B}$, and $\dim \Phi \geq k, \forall \Phi \in \mathcal{Y}^k$; $\mathcal{X}^{k+1} \subset \mathcal{X}^k$ and $\mathcal{Y}^{k+1} \subset \mathcal{Y}^k$; and $(\mathcal{X}^k, \mathcal{Y}^k)$ is a C^p stratification of $f|_{X^k} : X^k \to Y^k$.

This inductive construction leads to a stratification $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X}^0, \mathcal{Y}^0)$, which satisfies the demands of the theorem.
Suppose $(\mathcal{X}^{k+1}, \mathcal{Y}^{k+1})$ is constructed. By Theorem 1.3, there exists a finite or empty collection $\mathcal{Z}^k$ of disjoint definable submanifolds of dimension k, contained in $Y \setminus Y^{k+1}$ such that: $\mathcal{Z}^k$ is compatible with $\mathcal{B}$; $\dim(Y \setminus Y^{k+1} \setminus |\mathcal{Z}^k|) < k$ (where $|\mathcal{Z}^k| = \cup_{Z \in \mathcal{Z}^k} Z$); and $\mathcal{Y}^{k+1} \cup \mathcal{Z}^k$ is a definable C^p Whitney stratification of a subset of Y.
We will prove that for each $Z \in \mathcal{Z}^k$, there is a definable closed subset Z^0 of Z with $\dim Z^0 < k$, and we will modify $\mathcal{A}|_{f^{-1}(Z \setminus Z^0)}$ to a stratification $\mathcal{W}_Z$ so that the pair $(\mathcal{X}^k = \mathcal{X}^{k+1} \cup \cup_{Z \in \mathcal{Z}^k} \mathcal{W}_Z,\ \mathcal{Y}^k = \mathcal{Y}^{k+1} \cup \{Z \setminus Z^0 : Z \in \mathcal{Z}^k\})$ satisfies (F_k).
For $Z \in \mathcal{Z}^k, f^{-1}(Z) = \emptyset$, let $Z^0 = \emptyset$ and $\mathcal{W}_Z = \emptyset$.
For $Z \in \mathcal{Z}^k, f^{-1}(Z) \neq \emptyset$, by Cell Decomposition, we may assume that $\mathcal{A}$ is compatible with $f^{-1}(Z)$. Moreover, by [DM] Lemma C.2, for each $A \in \mathcal{A}|_{f^{-1}(Z)}$, there is a definable subset B_A of A such that $A \setminus B_A$ is a submanifold and $f|_{A \setminus B_A}$ is submersive into Z (if $A \setminus B_A \neq \emptyset$), and $\dim f(B_A) < k$. Then $Z \cap \cup_{A \in \mathcal{A}|_{f^{-1}(Z)}} \overline{f(B_A)}$ is of dimension $< k$. By deleting a closed subset of dimension $< k$ from Z, we may assume that $f|_A : A \to Z$ is submersive for every $A \in \mathcal{A}|_{f^{-1}(Z)}$. Under the above assumptions, let $n = \dim f^{-1}(Z)$, we now construct chains of definable sets

$$\emptyset = Z^m \subset Z^{m-1} \subset \cdots \subset Z^0 \subset Z \text{ and } W^n \subset W^{n-1} \subset \cdots \subset W^0 \subset f^{-1}(Z),$$

and for $l = n, n-1, \cdots, 0$, partitions $\mathcal{W}_Z^l$ of W^l into definable submanifolds satisfying the following conditions

(G_l) $\dim Z^l < k$; $\dim f^{-1}(Z \setminus Z^l) \setminus W^l < l$; $\mathcal{W}_Z^l$ is compatible with $\mathcal{A}$ and $\dim W \geq l, \forall W \in \mathcal{W}_Z^l$; $\mathcal{W}_Z^{l+1} \subset \mathcal{W}_Z^l$; $\mathcal{X}^{k+1} \cup \mathcal{W}_Z^l$ is a definable C^p Whitney stratification; and for each $W \in \mathcal{W}_Z^l$, $f|_W : W \to Z$ is submersive.

Suppose Z^{l+1} and $\mathcal{W}_Z^{l+1}$ are constructed. For each $A \in \mathcal{A}|_{f^{-1}(Z)}$, let $A' = A \setminus f^{-1}(Z^{l+1}) \setminus W^{l+1}$. By Theorem 1.3 and [DM] Lemma C.2, there exist definable subsets B'_A and B''_A of A' such that $A' \setminus (B'_A \cup B''_A)$ is a submanifold of dimension l (if not empty), $\dim B'_A < l$, $\dim f(B''_A) < k$, $f|_{A' \setminus (B'_A \cup B''_A)}$ is submersive, and $\mathcal{X}^{k+1} \cup \mathcal{W}_Z^{l+1} \cup \{A' \setminus (B'_A \cup B''_A), A \in \mathcal{A}\}$ is a definable C^p Whitney stratification. Let $Z^l = Z^{l+1} \cup \left(Z \cap \cup_{A \in \mathcal{A}|_{f^{-1}(Z)}} \overline{f(B''_A)} \right)$, and $\mathcal{W}_Z^l = \mathcal{W}_Z^{l+1} \cup \{A' \setminus (B'_A \cup B''_A), A \in \mathcal{A}|_{f^{-1}(Z)}\}$. Then Z^l and $\mathcal{W}_Z^l$ satisfy (G_l).
Obviously, Z^0 and $\mathcal{W}_Z = \mathcal{W}_Z^0|_{f^{-1}(Z \setminus Z^0)}$ have the desired properties. □

Definition 2.3. Let $f : X \to \mathbb{R}$ be a continuous definable function, where $X \subset \mathbb{R}^n$. Let $\mathcal{S}$ be a definable C^p stratification of f. For each $\Gamma \in \mathcal{S}$ and $x \in \Gamma$, $T_{x,f}$ denotes the tangent space of the level of $f|_\Gamma$ at x, i.e. $T_{x,f} = \ker D(f|_\Gamma)(x)$.

Let $\Gamma, \Gamma' \in \mathcal{S}$ with $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$. We say that the pair (Γ, Γ') satisfies the *Thom condition* (a_f) at $y_0 \in \Gamma$ if and only if the following holds:

(a_f) for every sequence (x_k) in Γ', converging to y_0, we have

$$\delta(T_{y_0,f}, T_{x_k,f}) \longrightarrow 0 \ .$$

We say that (Γ, Γ') satisfies the *strict Thom condition* (w_f) at y_0 if:

(w_f) there exist a constant $C > 0$ and a neighborhood U of y_0 in $\mathbb{R}^n$, such that

$$\delta(T_{y,f}, T_{x,f}) \leq C\|x - y\| \quad \text{for all } x \in \Gamma' \cap U, y \in \Gamma \cap U.$$

Note that the conditions are C^2-invariant.
The existence of stratifications satisfying (w_f) (and hence (a_f)) for subanalytic functions was proved in [**KP**] (see also [**B**] and [**KR**]). For functions definable in o-minimal structures on the real field we have:

THEOREM 2.4. *There exists a definable C^p stratification of f satisfying the Thom condition (a_f) at every point of the strata.*

PROOF. see [**L2**]. □

REMARK 2.5. In general, definable functions cannot be stratified to satisfy the condition (w_f). The following example is given by Kurdyka.
Let $f : (a,b) \times [0, +\infty) \longrightarrow \mathbb{R}$ be defined by $f(x,y) = y^x$ $(0 < a < b)$. Let $\Gamma = (a,b) \times 0$, and $\Gamma' = (a,b) \times (0, +\infty)$. Then the fiber of $f|_{\Gamma'}$ over $c \in \mathbb{R}_+$ equals

$$\{(x, y(x) = \exp(-\frac{1}{tx})) : x \in (a,b) \ \} \ , \quad t = -\frac{1}{\ln c}.$$

Then $\dfrac{y'(x)}{y(x)} = \dfrac{1}{tx^2} \to +\infty$, when $t \to 0^+$, for all $x \in (a,b)$,
i.e. $\dfrac{\delta(T_{x,f}, T_{(x,y(x)),f})}{\|y(x)\|}$ cannot be locally bounded along Γ.

The remainder of this section is devoted to the proof of the existence of (w_f)-stratification of functions definable in polynomially bounded o-minimal structures.

DEFINITION 2.6. A structure $\mathcal{D}$ on the real field $(\mathbb{R}, +, \cdot)$ is *polynomially bounded* if for every function $f : \mathbb{R} \longrightarrow \mathbb{R}$ definable in $\mathcal{D}$, there exists $N \in \mathbb{N}$, such that

$$|f(t)| \ \leq \ t^N \ , \text{ for all sufficiently large } t.$$

For example, the structure of global subanalytic sets, the structure generated by real power functions [M2], or by functions given by multisummable power series [DS] are polynomially bounded.

THEOREM 2.7. *Suppose that $\mathcal{D}$ is polynomially bounded. Then there exists a definable C^p stratification of f satisfying the condition (w_f) at each point of the strata.*

Note. The converse of the theorem is also true: If $\mathcal{D}$ is not polynomially bounded, then it must contain the exponential function, by [M1]. So the function given in Remark 2.3 is definable in $\mathcal{D}$ and cannot be (w_f)-stratified.

Theorem 2.4 is implied by Theorem 2.2, Proposition 1.4 and the following.

PROPOSITION 2.8. *Suppose that $\mathcal{D}$ is polynomially bounded. Let Γ, Γ' be definable C^p submanifolds of $\mathbb{R}^n$. Suppose $\Gamma \subset \overline{\Gamma'} \setminus \Gamma'$, $\Gamma \neq \emptyset$, and $f : \Gamma \cup \Gamma' \longrightarrow \mathbb{R}$ is a continuous definable function such that $f|_\Gamma$ and $f|_{\Gamma'}$ have constant rank. Then*
(i) $W_f = \{x \in \Gamma : (\mathrm{w}_f)$ *is satisfied at* $x\}$ *is definable, and*
(ii) $\dim(\Gamma \setminus W_f) < \dim \Gamma$.

PROOF. The proof is much the same as that for the condition (a_f) in [L2].
(i) Since $x \mapsto D(f|_\Gamma)$ is a definable map, the kernel bundle of $f|_\Gamma$

$$\ker d(f|_\Gamma) = \{(x,v) : x \in \Gamma, v \in T_x\Gamma, D(f|_\Gamma)(x)v = 0 \ \}$$

is definable. Therefore,

$$\begin{aligned} W_f \quad = \quad & \{y_0 : y_0 \in \Gamma, \exists C > 0, \exists t > 0, \forall x \in \Gamma', \forall y \in \Gamma \\ & \|x - y_0\| < t, \|y - y_0\| < t \Rightarrow \delta(\ker D(f|_\Gamma)(y), \ker D(f|_{\Gamma'})(x) \le C\|x - y\| \ \} \end{aligned}$$

is definable.
(ii) To prove the second assertion there are three cases to consider.
Case 1: $\mathrm{rank} f|_\Gamma = \mathrm{rank} f|_{\Gamma'} = 0$. In this case

$$W_f = \{y \in \Gamma : (\Gamma, \Gamma') \text{ satisfies Verdier condition (w) at } y\}.$$

The assertion follows from Theorem 1.2.
Case 2: $\mathrm{rank} f|_\Gamma = 0$ and $\mathrm{rank} f|_{\Gamma'} = 1$.
Suppose the contrary: $\dim(\Gamma \setminus W_f) < \dim \Gamma$. Since (w_f) is C^2 invariant, by Cell Decomposition, we can assume that Γ is an open subset of $\mathbb{R}^k \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, and $f|_{\Gamma'} > 0$, $f|_\Gamma \equiv 0$. So $T_{y,f} = \mathbb{R}^k$, for all $y \in \Gamma$. Let

$$A = \{(y,s,t) : (y,s) \in \Gamma \cup \Gamma', t > 0, f(y,s) = t \ \}.$$

Then A is a definable set. By Definable Choice and the assumption, there exists an open subset U of Γ, $\alpha > 0$, and a definable map $\theta : U \times [0,\alpha) \longrightarrow \mathbb{R}^{n-k}$, such that θ is C^p on $U \times (0,\alpha)$, $\theta|_\Gamma \equiv 0$, and $f(y, \theta(y,t)) = t$, and, moreover, for all $y \in U$, we have

$$(*) \qquad \frac{\|D_1\theta(y,t)\|}{\|\theta(y,t)\|} \ge \frac{\delta(\mathbb{R}^k, T_{(y,\theta(y,t)),f})}{\|\theta(y,t)\|} \ \to +\infty, \quad \text{when } t \to 0^+.$$

On the other hand, by [M2, Prop. 5.2], there exist a nonempty open subset B of U and $r > 0$, such that

$$(**) \qquad \theta(y,t) = c(y)t^r \ + \ \varphi(y,t)t^{r_1} \ , y \in B, t > 0 \text{ sufficiently small},$$

where c is C^p on B, $c \not\equiv 0$, $r_1 > r$, and φ is C^p with $\lim_{t \to 0^+} \varphi(y,t) = 0$, for all $y \in B$. Moreover, by Lemma 1.8, we can suppose that $D_1\varphi$ is bounded. Substituting $(**)$ to the left-hand side of $(*)$ we get a contradiction.

Case 3: $\mathrm{rank} f|_\Gamma = \mathrm{rank} f|_{\Gamma'} = 1$.
If $\dim(\Gamma \setminus W_f) = \dim \Gamma$, then the condition (w_f) is false for (Γ, Γ') over a nonempty open subset B of Γ. It is easy to see that there is $c \in \mathbb{R}$ such that (w_f) is false for the pair $(\Gamma \cap f^{-1}(c), \Gamma')$ over a nonempty open subset of $B \cap f^{-1}(c)$, and hence open in $\Gamma \cap f^{-1}(c)$. This contradicts Case 2. □

REMARK 2.9. If the structure admits analytic cell decomposition, then the theorems hold true with "analytic" in place of "C^p". Our results can be translated to the setting of analytic-geometric categories in the sense of [**DM**].

Bibliography

[B] K.Bekka, *Regular stratification of subanalytic sets,* Bull.London Math.Soc. **25**(1993), 7-16.

[BT] H.Brodersen and D.Trotman, *Whitney (b)-regularity is weaker than Kuo's ratio test for real algebraic stratifications,* Math. Scand. **45**(1979), 27-34.

[D] L.van den Dries, *Tame Topology and o-minimal Structures,* LMS Lecture Notes, Cambridge University Press, 1997.

[DM] L.van den Dries and C.Miller, *Geometric Categories and o-minimal Structures,* Duke Math. J. **84**, No 2(1996), 497-540.

[DS] L.van den Dries and P.Speissegger, *The field of reals with multisummable series and the exponential function,* preprint (1997).

[DW] Z.Denkowska and K.Wachta, *Une construction de la stratification sous-analytique avec la condition* (w)*,* Bull. Pol. Ac. Math. **35** (1987), 401-405.

[GPW] C.B.Gibson, K.Wirthmuller, A.A.du Plessis & E.J.N. Loojenga, *Topological Stability of Smooth Mappings*, Lecture Notes in Mathematics **552**, Springer-Verlag, 1976.

[K] T.C.Kuo, *The ratio test for analytic Whitney stratifications,* Proceedings of Liverpool Symposium I, Springer Verlag Lecture Notes **192** (1971).

[KP] K.Kurdyka and A.Parusiński, (w_f)*-stratification of subanalytic functions and the Lojasiewicz inequality,* C.R.Acad.Sci. Paris, t.**318**, Série I(1994), 129-133.

[KR] K.Kurdyka and G.Raby, *Densité des ensembles sous-analytiques,* Ann.Inst.Fourier **39** (1989), 735-771.

[L1] T.L.Loi, *Whitney Stratification of Sets Definable in the Structure* $\mathbb{R}_{exp}$*,* Banach Center Publications, Vol **33**(1996), 401-409.

[L2] T.L.Loi, *Thom stratification for functions definable in o-minimal structures on* $(\mathbb{R}, +, \cdot)$*,* C.R.Acad.Sci **324**(1997), 1391-1394.

[L3] T. L. Loi, *Verdier Stratifications and Thom Stratifications in O-minimal Structures,* Illinois J. Math. **42**, No 2 (1998), 347-356.

[L4] T.L.Loi, *Stratifications of families of functions definable in o-minimal structures,* Acta. Math. Vietnam., Vol. **27**, No.2 (2002) , 239-244.

[LR] J.M.Lion and J.P.Rolin, *Théorème de Gabrielov et fonctions log-exp-algebriques,* C.R. Acad.Sci. Paris, t.**324**, Série I(1997), 1027-1030.

[LSW] S.Łojasiewicz, J.Stasica and K.Wachta, *Stratifications sous-analytiques. Condition de Verdier,* Bull. Pol. Ac. Math. **34** (1986),531-539.

[Ma] J.Mather, *Notes on topological stability,* University of Havard (1970)

[M1] C.Miller, *Exponentiation is hard to avoid,* Proc. Amer. Math. Soc **122** (1994), 257-259.

[M2] C.Miller, *Expansions of the real field with power functions,* Ann. Pure Appl. Logic **68** (1994), 79-84.

[S] M.Shiota, *Geometry of Subanalytic and Semialgebraic Sets,* Progress in Math., vol. **150**, Birkhäuser, Boston, 1997.

[T1] D.Trotman, *Espaces stratifiés réels,* Stratifications, Singularities and Differential Equation II, Stratifications and Topology of Singular Spaces (D.Trotman & L.C.Wilson, ed.), Travaux en Cours, vol.**55**, Herman, 1997.

[T2] D.Trotman, *Lectures on Real Stratifications Theory,* Singularity Theory, Dedicated to Jean-Paul Brasselet on His 60th Brithday, Proc. Of the 2005 Marseille Singularity School and Conference (D.Chénoit, N.Dutertre, C.Murolo, D.Trotman & A.Pichon eds.), CIRM, Marseille, 2007, 139-156.

[V] J.L.Verdier, *Stratification de Whitney et theoreme de Bertini-Sard,* Invent. Math. **36** (1976), 295-312.

[W1] A.Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function,* J. Amer. Math. Soc. **9** (1996),1051-1094.

[W2] A.Wilkie, *A general theorem of the complement and some new o-minimal structures,* manuscript (1996).

[Wh] H.Whitney, *Tangents to an analytic variety,* Ann. of Math **81** (1965), 496-549.

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