INTRODUCING QUATERNIONIC GERBES.

FINLAY THOMPSON

ABSTRACT. The notion of a quaternionic gerbe is presented as a new way of bundling algebraic structures over a four manifold. The structure groupoid of this fibration is described in some detail. The Euclidean conformal group $\mathbb{R}^+ SO(4)$ appears naturally as a (non-commutative) monoidal structure on this groupoid. Using this monoidal structure we indicate the existence of a canonical quaternionic gerbe associated to a conformal structure on a four manifold.

It is natural to think that quaternionic algebra and four dimensional geometry should be closely linked. Certainly complex algebra and analysis provide indispensable tools for exploring two dimensional Riemaniann geometry.

However, despite many attempts, quaternionic algebra has not been usefully applied to the differential geometry of four manifolds.[1] The most commonly held view is that quaternionic algebra is too rigid to be useful in studying four manifolds. It is generally assumed that the natural setting for quaternionic differential geometry is hyperKähler or hypercomplex. [10]

The purpose of this talk/article is to present the notion of a **quaternionic gerbe**, and to demonstrate that they appear naturally as a quaternionic algebraic structure on four manifolds. This work appears as part of an effort to realize the goal of "doing four dimensional geometry and topology with quaternionic algebra."

Although quaternionic structures are defined [8] for all $4n$ dimensional manifolds, the basic structures and difficulties are already present in only four dimensions. The notion of "quaternionic curve" has been equated with that of a "self dual conformal" structure.[2] Note that even this class of manifolds is strictly larger than the hyperKähler manifolds. Here we restrict our attention to smooth oriented four manifolds, including hyperKähler and self dual conformal manifolds.

It is proposed that a "quaternionic structure" on a four manifold is essentially a Euclidean conformal structure. This compares favourably

[1] Except perhaps Atiyah's notes on solutions to the Yang-Mills equations on the four sphere, [1]

with the two dimensional case where fixing a complex structure is equivalent to fixing a conformal structure.

1. The Problem.

The most obvious definition of a quaternionic structure on a four manifold M requires the existence of three integrable complex structures, $I, J, K \in \mathrm{End}(TM)$, such that,

$$I^2 = J^2 = K^2 = IJK = -1.$$

In terms of holonomy, this implies a reduction of the frame bundle's structure group to $\mathbb{H}^*$, the group of unit quaternions. Note that $\mathbb{H}^* = GL(1, \mathbb{H})$, which generalises the complex case in an obvious way.

The problem comes when we consider Berger's list [2] of holonomy groups for Riemannian manifolds:

$$\begin{array}{rl} \text{real} & O(n), \quad SO(n), \\ \text{complex} & U(n), \quad SU(n), \\ \text{quaternionic} & Sp(n) \cdot Sp(1), \quad Sp(n) \\ \text{exceptional} & G_2, \quad Spin(7) \end{array}$$

The **quaternionic-Kähler** series $Sp(n) \cdot Sp(1)$ is clearly related to quaternionic algebraic structures, however it is not contained in $GL(n, \mathbb{H})$. Does this mean that quaternionic-Kähler manifolds are not quaternionic? In reaction to this apparent contradiction, S. Salamon defined quaternionic manifolds (see [8]) as having a holonomy reduction to $GL(n, \mathbb{H}) \cdot Sp(1)$. Then quaternionic-Kähler implies quaternionic, as you might expect.

There are two interesting low dimensional coincidences in Berger's list. The first $U(1) = SO(2)$ tells us the complex Kähler curves are simply Riemannian surfaces. Moreover, because fixing a conformal structure on a Riemannian surface corresponds to a holonomy reduction to $\mathbb{R}^+ SO(2)$, and $\mathbb{R}^+ SO(2) = \mathbb{C}^* = GL(1, \mathbb{C})$, geometrically speaking, fixing a conformal structure is equivalent to fixing a complex structure on two dimensional manifolds.

The second coincidence $Sp(1) \cdot Sp(1) = SO(4)$ seems similar, with "complex" replaced by "quaternionic". We also have,

$$GL(1, \mathbb{H}) \cdot Sp(1) = \mathbb{H}^* \cdot Sp(1) = \mathbb{R}^+ Sp(1) \cdot Sp(1) = \mathbb{R}^+ SO(4).$$

The implication is that fixing a quaternionic structure is equivalent to fixing a conformal structure on four manifolds. But what exactly do we mean by a "quaternionic structure"?

1.1. **The Impasse.** The algebra of quaternions appears naturally as the generator of the Brauer group of the reals, $\mathrm{Br}(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\}$. The group structure is given by the tensor product, moduli "matrix" algebra. It is not necessary to go into the details of the Brauer group here, instead we simply note that $\mathbb{H}$ generates $\mathrm{Br}(\mathbb{R})$ because of the following algebra isomorphism,

$$\phi : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq \mathrm{End}_{\mathbb{R}}(\mathbb{H}),$$

where $\phi(p \otimes q) : v \mapsto p \cdot v \cdot q$ for any $p, q, v \in \mathbb{H}$. Note that we have used both the left and the right module structures in defining ϕ.

The Euclidean conformal group $\mathbb{R}^+ SO(4)$ has a natural quaternionic presentation using the isomorphism ϕ. Let $i : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ be the canonical map associated to the tensor product. The image of the multiplicative group $\mathbb{H}^* \times \mathbb{H}^*$ under these maps is precisely the conformal group. We have the following exact sequence of groups,

$$1 \longrightarrow \mathbb{R}^* \longrightarrow \mathbb{H}^* \times \mathbb{H}^* \xrightarrow{\phi \circ i} \mathbb{R}^+ SO(4) \longrightarrow 1$$

where $\mathbb{R}^* \to \mathbb{H}^* \times \mathbb{H}^*$ acts as $r \mapsto (r, r^{-1})$.

Proposition . *The Euclidean conformal group in four dimensions appears in a natural and quaternionic way as,*

$$\mathbb{R}^+ SO(4) = \{p \otimes q = i(p, q) \mid p, q \in \mathbb{H}^*\}.$$

PROOF: The Euclidean norm of $x \in \mathbb{H}$ is $|x|^2 = x \cdot \bar{x}$. Let $p \otimes q = i(p, q)$. Then,

$$\begin{aligned} |p \otimes q(x)| &= |p \cdot x \cdot q| = \sqrt{p \cdot x \cdot q \cdot \bar{q} \cdot \bar{x} \cdot \bar{p}} \\ &= |p||q||x| = \lambda |x| \end{aligned}$$

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The above presentation of the conformal group, using the isomorphism $\phi : \mathbb{H} \otimes \mathbb{H} \to \mathrm{End}(\mathbb{H})$, places equal emphasis on the left and right module structures of $\mathbb{H}$ on itself. Indeed, the isomorphism ϕ is the $\mathbb{H}$-bimodule structure on $\mathbb{H}$! It is then natural to consider the full bimodule structure as the important structure that we want to integrate over four manifolds. However this way is blocked.

Proposition . *The automorphisms of $\mathbb{H}$ considered as a $\mathbb{H}$-bimodule are all scale multiples of the identity,*

$$\mathrm{Aut}_{\mathbb{H}\text{-}bimodule}(\mathbb{H}) = \mathbb{R}^+ \cdot \mathrm{Id}\,.$$

PROOF: This is simply a consequence of Shur's lemma applied to the representations of $\mathrm{M}_4(\mathbb{R})$.

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Thus a four manifold with an integrable $\mathbb{H}$-bimodule structure defined on the tangent bundle has holonomy contained in $\mathbb{R}^+ \cdot \mathrm{Id}$, which forces the manifold to be affine.

So we reach an impasse:

- The Euclidean conformal group has a natural quaternionic presentation using the $\mathbb{H}$-bimodule structure on $\mathbb{H}$.
- The automorphisms of $\mathbb{H}$ as a $\mathbb{H}$-bimodule are simply scale multiples of the identity.

We show that quaternionic gerbes provide a way of going past this impasse.

1.2. **The Suggested Solution.** The central idea is to use a more sophisticated way of "gluing" local data together.

Although $\mathbb{H}$ has very few automorphisms when considered as an $\mathbb{H}$-bimodule, it does have an interesting group of automorphisms as an $\mathbb{R}$-algebra,

$$\mathrm{Aut}(\mathbb{H}) = \mathrm{Inn}(\mathbb{H}) = SO(3).$$

Note that all the automorphisms are inner. The suggestion is to consider the set of linear maps in $\mathrm{End}(\mathbb{H})$ that commute with the $\mathbb{H}$-bimodule structure, *up to inner automorphisms*. Such a map: $f : \mathbb{H} \to \mathbb{H}$ is required to satisfy the equation,

$$f(p \cdot v \cdot q) = \alpha(p) f(v) \beta(q),$$

where α, β are inner automorphisms associated to f. It turns out that the set of all such generalised automorphisms is precisely the Euclidean conformal group, $i(\mathbb{H}^* \times \mathbb{H}^*) = \mathbb{R}^+ SO(4)$.

The idea of allowing the two actions to commute *up to an automorphism* is natural in category theory. A gerbe is a special kind of sheaf of categories and provides a rich enough language to handle the inner automorphisms coherently.

An excellent presentation of the theory of *Abelian* gerbes has been presented by Jean-Luc Brylinski in "Loop Spaces, Characteristic Classes and Geometric Quantisation." [5].

Nigel Hitchen, studying special Lagrangian sub-manifolds in dimension three, has also made use of Abelian gerbes. Hitchen's approach [6] stresses the idea that Abelian gerbes certain cohomology classes.

Michael Murray has presented [7] Abelian gerbes in a different light as bundle gerbes.

However the theory we are looking for is non-Abelian. L. Breen has defined **non-Abelian** gerbes [3, 4] for arbitrary Lie groups and has developed the theory of 2-gerbes. Breen's work applies quite well to our present situation.

2. The Structure Groupoid

Just as there is a structure group associated to a principal bundle, a gerbe has an associated groupoid. In this section we will describe the structure groupoid associated to a quaternionic gerbe.

Following Breen [3], we can associate to any *crossed module* a groupoid. We start from the crossed module defined by,

$$\delta : \mathbb{H}^* \to SO(3).$$

where δ is the natural map onto the inner automorphisms, $\delta(p) = p \otimes p^{-1}$, and $SO(3)$ acts on $\mathbb{H}^*$ as automorphisms.

Recently R. Brown and collaborators have been relating groupoids and crossed modules to algebraic topology. (see [12])

Definition . *The **quaternionic structure groupoid** $\mathcal{H}$ is defined:*

- *objects of $\mathcal{H}$ are elements of $SO(3)$,*
- *any element $p \in \mathbb{H}^*$ is considered a morphism $p \in \mathcal{H}(\alpha, \beta)$*

$$p : \alpha \to \beta$$

  whenever $\delta(p) \circ \alpha = \beta$.
- *For any two morphisms $p : \alpha \to \beta$ and $q : \beta \to \gamma$, the composition is given by the map,*

$$q \circ p = qp : \alpha \to \gamma = \delta(qp)\alpha = \delta(q)\delta(p)\alpha.$$

It is easy to check that all of the axioms of a small category are satisfied. In addition, because $\mathbb{H}$ is a division algebra, all the morphisms are invertible so that $\mathcal{H}$ is a groupoid.

Note that the set of all morphisms in $\mathcal{H}$ consists of pairs $(p, \alpha) \in \mathbb{H}^* \times SO(3)$. We will abuse notation a little and say that $\mathcal{H} = \mathbb{H}^* \times SO(3)$ as sets.

Although we have used the left $SO(3)$-action on itself, we have not used the group structure on $SO(3)$.

2.1. Tensor product on $\mathcal{H}$.

The small category $\mathcal{H}$ has a monoidal structure coming from the group structure on $SO(3)$,

$$\otimes : \mathcal{H} \times \mathcal{H} \to \mathcal{H}.$$

We use the tensor product symbol because the central example of a monoidal structure on a category is that of the tensor product in vector spaces. This tensor product is not commutative, however it is associative.

For any $\alpha, \beta \in SO(3)$,

$$\alpha \otimes \beta = \alpha\beta$$

For any two maps $p : \alpha \to \delta(p)\alpha$ and $q : \beta \to \delta(q)\beta$ we define $p \otimes q$ as,

$$p \otimes q = p\alpha[q] : \alpha\beta \to \delta(p\alpha[q])\alpha\beta.$$

To see that $\otimes$ is well defined we should check that tensor product of the ranges is the range of the tensor product of the maps,

$$\begin{aligned}\delta(p\alpha[q])\alpha\beta &= \delta(p)\delta(\alpha[q])\alpha\beta = \delta(p)\alpha\delta(q)\alpha^{-1}\alpha\beta \\ &= (\delta(p)\alpha) \otimes (\delta(q)\beta)\end{aligned}$$

Note that $\otimes$ is simply the semi-direct group structure coming from the action of $SO(3)$ on $\mathbb{H}^*$ via inner automorphisms,

$$(\mathcal{H}, \otimes) = \mathbb{H}^* \rtimes SO(3).$$

where $(p, \alpha) \otimes (q, \beta) = (p\alpha[q], \alpha\beta)$.

Moreover, this semi-direct product is isomorphic to the Euclidean conformal group,

$$\mathbb{H}^* \rtimes SO(3) \simeq \mathbb{R}^+ SO(4).$$

2.2. $\mathbb{H}$-**bimodules.** We can also represent the groupoid $\mathcal{H}$ as a category of quaternionic bimodules in such a way that the tensor product is really a tensor product. In order to do this we need to define carefully what we mean by an $\mathbb{H}$-bimodule.

Definition . *An $\mathbb{H}$-**bimodule** is a vector space V with two commuting actions of the quaternions. Or equivalently, a bilinear map,*

$$\rho : \mathbb{H} \times \mathbb{H} \to \mathrm{End}(V).$$

Given an $\mathbb{H}$-bimodule (V, ρ) we can present the action as,

$$\rho : \mathbb{H} \otimes \mathbb{H} \to \mathrm{End}(V),$$

by using the universal property of the tensor product. In this form we see that the $\mathbb{H}$-bimodules are simply the modules over $\mathrm{End}(\mathbb{H}) = \mathrm{M}_4(\mathbb{R})$, the algebra of four by four matrices. Therefore the only simple module is $\mathbb{R}^4$. For our purposes we restrict ourselves to real four dimensional $\mathbb{H}$-bimodules. The objects of $\mathcal{H}$ will be identified with the four dimensional $\mathbb{H}$-bimodules.

Although all such objects are structurally identical, we will distinguish between different quaternionic structures on the same underlying vector space.

Proposition . *Let V be a $\mathbb{H}$-bimodule in $\mathcal{H}$. Then there is an $\mathbb{H}$-bimodule isomorphism $\epsilon : \mathbb{H} \to V$.*

PROOF: V is a simple module over $\mathbb{H}$, in two different ways. Comparing these actions we define for each $v \in V$, $v \neq 0$, a map $\mathbb{H} \to \mathbb{H} : p \mapsto p^v$ by the rule,

$$p \cdot v = v \cdot p^v.$$

The map $p \mapsto p^v$ is an $\mathbb{R}$-algebra automorphism for all v,

$$\begin{aligned} v \cdot (pq)^v &= (pq) \cdot v = p \cdot (q \cdot v) = p \cdot (v \cdot q^v) \\ &= (p \cdot v) \cdot q^v = v \cdot (p^v q^v) \end{aligned}$$

So we have defined a map $V \to \mathrm{Aut}(\mathbb{H}) = SO(3)$. To show surjectivity we start by fixing some v and define $\alpha[p] = p^v$. For any β in $\mathrm{Aut}(\mathbb{H})$ the tansitivity of $SO(3)$ implies that $\beta = \gamma\alpha$ for some $\gamma \in SO(3)$. All automorphisms are inner, so there is some $r \in \mathbb{H}$ such that $\gamma = \delta(r) = r \otimes r^{-1}$. Then we observe,

$$\begin{aligned} p \cdot (v \cdot r^{-1}) &= (v \cdot p^v) \cdot r^{-1} = (v \cdot r^{-1}) \cdot (r p^v r^{-1}) \\ &= (v \cdot r^{-1}) \cdot \gamma[p^v] = (v \cdot r^{-1}) \cdot \gamma\alpha[p] \\ &= (v \cdot r^{-1}) \cdot \beta[p] \end{aligned}$$

and we see that $v \cdot r^{-1}$ maps onto β. It is easy to see that the map $V \to SO(3)$ fibres through the projection $V \to P(V)$, where $P(V)$ is the real projective space of one dimensional subspaces in V. Let $e \in V$ be chosen in the preimage of the identity of $SO(3)$. Then it is clear that $p \cdot e = e \cdot p$ for all $p \in \mathbb{H}$, and,

$$\begin{aligned} \epsilon : \mathbb{H} &\to V \\ p &\to p \cdot e \end{aligned}$$

is the isomorphism of $\mathbb{H}$-bimodules.

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We see from the above proof that each $\mathbb{H}$-bimodule structure creates an identification of $P(V)$ with $SO(3)$. Recall that as a smooth manifold $SO(3) = \mathbb{RP}^3 = P(V)$.

We can go in the other direction as well. Let V be a right $\mathbb{H}$-module and $\alpha : V \to SO(3)$ be a $\mathbb{H}^*$ equivariant map,

$$\alpha(xp) = \delta(p)^{-1}\alpha(x).$$

Then we define a left $\mathbb{H}^*$ action on V as,

$$px = x\alpha(x)[p]$$

The left action commutes with the right action and so V is an $\mathbb{H}$-bimodule,

$$\begin{aligned} p(xq) &= xq\alpha(xq)[p] = xq\delta(q)^{-1}\alpha(x)[p] \\ &= x\alpha(x)[p]q = (px)q \end{aligned}$$

We distinguish the different objects in $\mathcal{H}$ by using the different identifications $P(V) \to SO(3)$ associated $\mathbb{H}$-bimodule structures. Two objects differ by an element of $SO(3)$.

Now the tensor product can actually be represented as a tensor product. If V and W are the $\mathbb{H}$-bimodules associated to objects α and β in $\mathcal{H}$, then the $\mathbb{H}$-bimodule associated to $\alpha \otimes \beta = \alpha \cdot \beta$ is $V \otimes_{\mathbb{H}} W$.

A quaternionic gerbe consists of the structure groupoid fibred over a four manifold. To see how we do that, we need a closer look at the theory of sheaves of categories.

3. Sheaves of Categories or Stacks.

A gerbe is a special kind of sheaf of categories. Our objective in this section is to present enough of the general theory so that we can understand what is the nature of gerbes, and how they can be useful. We will not present a self contained account here, instead we refer the reader to [5].

A presheaf of categories involves the interplay of locally defined "objects" and "morphisms". A **stack**[2] requires that the objects satisfy a descent property, *up to an isomorphism.* The concept is is quite flexible, but still very precise. The isomorphisms that glue together the object data must satisfy additional coherence identities.

3.1. Local Homeomorphisms.

Instead of working with the category of open sets on a manifold X, we work with local homeomorphisms: continuous map $f : Y \to X$ such that,

- any $y \in Y$ has an open neighbourhood U whose image $f(U)$ is open in X, and,
- the restriction of f to U gives a homeomorphism between U and $f(U)$.

Definition . *The category of* ***spaces over*** *X, C_X, has,*

- *objects are local homeomorphisms to X, $f : Y \to X$,*
- *a morphism $g : (f : Y \to X) \to (h : Z \to X)$ is a local homeomorphisms, $g : Y \to Z$ such that $f = h \circ g$.*

[2]For us the terms "stack" and "sheaf of categories" refer to the same concept.

An important example to keep in mind is associated to an open cover $\{U_i\}$ of X. The canonical projection $f : Y = \coprod_i U_i \to X$ from the disjoint union onto X is a local homeomorphism.

3.2. **Presheaves of Categories.** In the same way that a presheaf of sets is simply a functor from C_X to the category of sets, a **presheaf of categories** over X is a functor $\mathcal{C}$ from the category of spaces over X, C_X, to the (bi)-category of small categories, functors and natural transformations. Or, more explicitly,

- to every local homeomorphism $f : Y \to X$ we associate a small category,
$$\mathcal{C}(f : Y \to X)$$
- to every arrow of local homeomorphisms $k : (Z, g) \to (Y, f)$ we associate a functor,
$$\mathcal{C}(k) = k^{-1} : \mathcal{C}(f : Y \to X) \to \mathcal{C}(g : Z \to X),$$
- to every composition $k \circ l : (W, h) \to (Z, g) \to (Y, f)$ we associate an invertible natural transformation,
$$\theta_{k,l} : l^{-1}k^{-1} \Rightarrow (kl)^{-1}$$

This data must satisfy the following coherence condition,

$$\begin{array}{ccc} m^{-1}l^{-1}k^{-1} & \xrightarrow{\theta_{k,l}} & m^{-1}(lk)^{-1} \\ \Big\downarrow{\scriptstyle \theta_{l,m}} & & \Big\downarrow{\scriptstyle \theta_{lk,m}} \\ (lm)^{-1}k^{-1} & \xrightarrow{\theta_{k,lm}} & (lkm)^{-1} \end{array}$$

It would be possible to define a presheaf of categories with the requirement that $l^{-1}k^{-1}$ is strictly identical to $(kl)^{-1}$. However that does not take advantage of the extra flexibility provided. We will see later how quaternionic gerbes make use of this flexibility.

3.3. **Descent for Morphisms.** Let $\mathcal{C}$ be a presheaf of categories. We say that the **morphisms satisfy descent** if for any two objects A, B in $\mathcal{C}(f : Y \to X)$, the presheaf of sets on Y defined by,

$$\mathrm{Hom}(A, B)(k : Z \to Y) = \mathrm{Hom}(k^{-1}(A), k^{-1}(B))$$

is actually a sheaf on Y.

We can explain this in terms of objects and maps more directly. Let $V \subset X$ be a open neighbourhood, and let A, B be objects in $\mathcal{C}(V) = \mathcal{C}(V \hookrightarrow X)$. Now let $\{U_i\}$ be a open cover of V. Take a collection of morphisms $\alpha_i : A|_{U_i} \to B|_{U_i}$, where $\alpha_i \in \mathcal{C}(U_i)$. The morphisms satisfy descent if,

$$\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}} \quad \forall i, j,$$

implies the existence a unique morphism $\alpha : A \to B$ in $\mathcal{C}(V)$ such that $\alpha_i = \alpha|_{U_i}$ for all i.

In the above we have denoted $A|_{U_i}$ for the "restriction" of A to U_i. Of course the restriction is really a functor $\mathcal{C}(V) \to \mathcal{C}(V \cap U_i)$, and that functor is not necessarily trivial or obvious. However the presentation becomes much easier to if we make use of these small abuses of the notation.

3.4. **Descent for Objects.** The objects satisfy a much more complicated descent property, making use of the natural transformations appearing in the definition.

Let $\mathcal{C}$ be a presheaf of categories. Let V be any open set in X and $f : Y \to V$ be any surjective local homeomorphism. The **descent** data for any $A \in \mathcal{C}(Y)$ consists of an isomorphism $\phi : p_2^{-1}(A) \to p_1^{-1}(A)$ in $\mathcal{C}(Y \times_X Y)$ such that,

$$p_{12}^{-1}(\phi) \circ p_{23}^{-1}(\phi) \circ p_{31}^{-1}(\phi) = \mathrm{Id}_{p_1^{-1}(A)}$$

in $\mathcal{H}(Y \times_X Y \times_X Y)$.[3]

We say that the **objects satisfy descent** if every pair (A, ϕ) as above implies the existence of an object $A' \in \mathcal{C}(V)$ and an isomorphism $\psi : f^{-1}(A') \to A$ in $\mathcal{C}(Y)$ such that the following diagram in $\mathcal{C}(Y \times_X Y)$ commutes,

$$\begin{array}{ccc} p_1^{-1} f^{-1}(A') & \xrightarrow{\theta_{f,p_2}^{-1}\theta_{f,p_1}} & p_2^{-1} f^{-1}(A') \\ \psi \big\downarrow & & \psi \big\downarrow \\ p_1^{-1}(A) & \xrightarrow{\phi} & p_2^{-1}(A) \end{array}$$

This rather complicated prescription can also be understood in terms of open sets in the normal sense.

Let $\{U_i\}$ be an open cover of $V \subset X$. The descent data is equivalent to a set of local objects $A_i \in \mathcal{C}(U_i)$, and isomorphisms $\phi_{ij} : A_i|_{U_{ij}} \to A_j|_{U_{ij}}$ in $\mathcal{C}(U_i \cap U_j)$. The isomorphisms are required to satisfy,

$$\phi_{ik}|_{U_{ijk}} = \phi_{ij}|_{U_{ijk}} \circ \phi_{jk}|_{U_{ijk}},$$

in the category over the triple intersection, $\mathcal{C}(U_i \cap U_j \cap U_k)$. Again note that we have implicitly used the natural transformations by glossing over the restrictions.

Definition . *A **stack** (or sheaf of categories) on X is a presheaf of categories where objects and morphisms satisfy the descent conditions above.*

[3]The natural projections $Y \times_X Y \to Y$ are denoted p_1 and p_2, and the three natural projections $Y \times_X Y \times_X Y \to Y \times_X Y$ are denoted by p_{12},p_{23} and p_{13}.

4. Quaternionic Gerbes

Now we refine the notion of a stack to that of a gerbe by imposing three more conditions:

1. gerbes take values in groupoids, the full sub-category of small categories whose morphisms are invertible.
2. gerbes are **locally non-empty**. This means that there exists a surjective local homeomorphism $f : Y \to X$ such that $\mathcal{C}(Y)$ is non-empty. We could also state this by saying that there exists an covering $\{U_i\}$ of X such that the $\mathcal{C}(U_i)$ are all non-empty.
3. gerbes are **locally connected**. This means that for any two objects A, B in $\mathcal{C}(f : Y \to X)$, there exists an surjective local homeomorphism $g : Z \to Y$ such that $g^{-1}(A)$ and $g^{-1}(B)$ are isomorphic. In terms of covers: if A, B are objects in $\mathcal{C}(U)$ for some $U \subset X$, then there exists an open covering $\{U_i\}$ of U such that $A\,|_{U_i}$ is isomorphic to $B\,|_{U_i}$ for all i.

Definition . *A **gerbe** on X is a locally non-empty and locally connected sheaf of groupoids on X.*

For any group G let $\underline{G}_X$ be the sheaf of G-valued functions on X. A gerbe is said to have **band** in G if for any object $A \in \mathcal{C}(f : Y \to X)$, the sheaf $\mathrm{Aut}(A)$ of automorphisms of A on Y is isomorphic to $\underline{G}_Y$, and the isomorphism $\alpha : \mathrm{Aut}(A) \to \underline{G}_Y$ is unique up to an inner automorphism of G.

Definition . ***Quaternionic Gerbe** is a gerbe with band in $\mathbb{H}^*$.*

4.1. Neutral Gerbes.

A gerbe $\mathcal{G}$ is said to be **neutral** if there exists a global object, $A \in \mathcal{G}(X)$. Because the automorphism sheaf of $\mathrm{Aut}(A)$ is isomorphic to the sheaf of $\mathbb{H}^*$-valued functions, we can identify the groupoid $\mathcal{G}(X)$ with the groupoid of principal $\mathbb{H}^*$-bundles,

$$\begin{aligned} \phi : \mathcal{G}(X) &\to \mathrm{Tor}(\mathrm{Aut}(A)) \\ B &\mapsto \mathrm{Isom}(B \to A) \end{aligned}$$

Quaternionic gerbes are locally non-empty so we can always find an object $A_U \in \mathcal{G}(U)$ over the open set U. Using that local object we can identify $\mathcal{G}(U)$ with the groupoid of $\mathbb{H}^*$-bundles over U. Local non-emptiness implies that the gerbe is locally neutral.

In order to understand this local neutrality, it is helpful to consider an analogy with the relation between a principal G-bundle and an associated vector bundle. To any vector bundle we can associate the principal bundle of frames. The local neutralisation associated to a local object A_U is sort of “frame” for $\mathcal{G}$ over U. The set of all frames for $\mathcal{G}$ forms a local groupoid.

Assuming that U is contractable, all the $\mathbb{H}^*$-bundles differ by an automorphisms valued function $\alpha : U \to SO(3)$. We define a $\mathbb{H}^*$-bundle associated to A_U and α by letting $A_U^\alpha = A_U$ as a fibre bundle. The action of $\mathbb{H}^*$ however is twisted by α. Let $a \in A_U$ and let a^α be the same element considered in A_U^α. Then for any quaternion $p \in \mathbb{H}^*$,

$$a^\alpha \cdot p = (a \cdot \alpha[p])^\alpha.$$

If U is not contractable there can be topologically inequivalent $\mathbb{H}^*$-bundles. Then we can replace the function α above with an $\mathbb{H}$-bimodule $M \to U$. If A_U and B_U are two different objects in $\mathcal{G}(U)$, then there is an $\mathbb{H}$-bimodule M such that,

$$A \otimes_{\mathbb{H}} M = B.$$

The local groupoid $\mathcal{H}(U)$ consists of the "frames" of $\mathcal{G}(U)$. Note that because of the local neutrality axiom, all quaternionic gerbes look the same locally.

4.2. **The Local Groupoid.** We describe here the local structure of $\mathcal{H}$.

An **object** of the local groupoid $\mathcal{H}(U)$ is a diagram of the form,

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \mathrm{Aut}(\mathbb{H}) \\ U & \end{tikzcd}$$

where $\pi : A \to U$ is a principle $\mathbb{H}^*$-bundle and α is an $\mathbb{H}^*$-equivariant map $\alpha(xp) = \delta(p)^{-1}\alpha(x)$.

As we have seen, this data can also be presented in terms of $\mathbb{H}$-bitorsors.

An $\mathbb{H}^*$-bitorsor is a principle right $\mathbb{H}^*$-bundle that is also a principle left $\mathbb{H}^*$-bundle for a commuting action of $\mathbb{H}^*$. For any $(A, \alpha) \in \mathcal{H}(U)$, the left $\mathbb{H}^*$ action on A is, $px = x\alpha(x)[p]$.

The morphisms of $\mathbb{H}$-bitorsors are simply bundle maps that commute with both the left and right actions.

4.3. **Tensor Product on $\mathcal{H}(U)$.** In terms of bitorsors we can present the product structure on $\mathcal{H}(U)$ by using the quaternionic tensor product.

For any $A, B \in \mathcal{H}(U)$,

$$A \otimes_{\mathbb{H}} B = A \otimes_{\mathbb{R}} B / \sim$$

where $xp \otimes y \sim x \otimes py$.

Assuming U is contractable and by fixing a coordinate basis, we get a canonical trivialisation of the tangent bundle, $TU = U \times \mathbb{H}$. In this way TU can be considered as an object in $\mathcal{H}(U)$.

Relative to this fixed object, all the others are given by $SO(3)$ valued functions on U, the morphisms are given by $\mathbb{H}^*$ valued functions.

Over $U \subset \mathbb{H}$ the local groupoid consists of sections $C^\infty(U, \mathcal{H})$. However the strength of this approach is in terms of the global structure. A global quaternionic gerbe is given in terms of "transition functions".

4.4. **Transition Functions or Bitorsor Cocyle.** The transition functions for a quaternionic gerbe are given in terms of $\mathbb{H}$-bitorsors. Maybe we should say "transition bitorsors".

Let $\mathcal{G}$ be a quaternionic gerbe on X and let $\{U_i\}$ be a good cover.[4] Choose $A_i \in \mathcal{G}(U_i)$. Then $\mathrm{Aut}(A_i)$ is isomorphic to the sheaf of $\mathbb{H}^*$-valued functions. Using A_i we have the following local neutralisation,

$$\begin{aligned} \Phi_i : \mathcal{G}(U_i) &\to \mathrm{Tor}(\mathrm{Aut}(A_i)) \\ B &\mapsto \mathrm{Isom}(B \to A_i) \end{aligned}$$

On any intersection $U_{ij} = U_i \cap U_j$ we can define,

$$E_{ij} = \mathrm{Isom}(A_j \mid_{U_{ij}}, A_i \mid_{U_{ij}}),$$

The E_{ij} are $\mathbb{H}$-bitorsors and are the transition functions. The two $\mathbb{H}$-actions are given by the composition of an isomorphism with automorphisms of $A_i|_{Uij}$ and $A_j|_{Uij}$, which are each isomorphic to $\mathbb{H}^*$-valued functions. Note that the isomorphisms $\underline{\mathbb{H}} \simeq \mathrm{Aut}(A_i)$ are unique up to an automorphism. To be really careful we should take care of those automorphisms as well, however that will work will be presented in a comprehensive way later.

The $\mathbb{H}$-bitorsors E_{ij} need to be compared over triple intersections. The natural transformations in the definition of a stack give us the following morphisms as extra data:

$$\psi_{ijk} : E_{ij} \otimes_{\mathbb{H}} E_{jk} \to E_{ik},$$

These morphisms live in $\mathcal{H}(U_{ijk})$ and must satisfy the following coherence condition on four intersections,

$$\begin{tikzcd} E_{ij} \otimes_{\mathbb{H}} E_{jk} \otimes_{\mathbb{H}} E_{kl} \arrow[r, "\psi_{ijk}\otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\psi_{jkl}"'] & E_{ik} \otimes_{\mathbb{H}} E_{kl} \arrow[d, "\psi_{ikl}"] \\ E_{ij} \otimes_{\mathbb{H}} E_{jl} \arrow[r, "\psi_{ijl}"] & E_{il} \end{tikzcd}$$

[4] All intersections $U_i \cap U_j$ are contractable.

The pair (E_{ij}, ψ_{ijk}) is called a **quaternionic bitorsor cocycle** on X.

Of course the above description of a particular quaternionic gerbe depends on the choice of $A_i \in \mathcal{G}(U_i)$. We can measure the dependence on those choices with a coboundary.

4.5. **Coboundary.** Let B_i be a different choice of local objects and (F_{ij}, ϕ_{ijk}) be the associated bitorsor cocyle.

Let $M_i \in \mathcal{H}(U_i)$ be defined by,

$$B_i = A_i \otimes_{\mathbb{H}} M_i$$

The pair (M_i, ν_{ij}) is a **coboundary** relating (F_{ij}, ϕ_{ijk}) to (E_{ij}, ψ_{ijk}) if ν_{ij} is a map in $\mathcal{H}(U_{ij})$,

$$\nu_{ij} : F_{ij} \to M_i^\circ \otimes_{\mathbb{H}} E_{ij} \otimes_{\mathbb{H}} M_j$$

such that as morphisms in $\mathcal{H}(U_{ijk})$,

$$\nu_{ik} \circ \phi_{ijk} = \psi_{ijk} \circ (\nu_{ij} \otimes \nu_{jk})$$

We can present this equation with a commutative diagram,

$$\begin{array}{ccc}
F_{ij} \otimes_{\mathbb{H}} F_{jk} & \xrightarrow{\nu_{ij} \otimes \nu_{jk}} & M_i^\circ \otimes_{\mathbb{H}} E_{ij} \otimes_{\mathbb{H}} E_{jk} \otimes_{\mathbb{H}} M_k \\
{\scriptstyle \phi_{ijk}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Id} \cdot \psi_{ijk} \cdot \mathrm{Id}} \\
F_{ik} & \xrightarrow[\nu_{ik}]{} & M_i^\circ \otimes_{\mathbb{H}} E_{ik} \otimes_{\mathbb{H}} M_k
\end{array}$$

It was demonstrated in [11] that coboundaries define an equivalence relation on the set of quaternionic bitorsor cocycles. Moreover, it is possible to construct a quaternionic gerbe from a given cocyle, and that gerbe will be isomorphic to any gerbe constructed from a cocyle from the same equivalence class.

Although we have used the terminology of cohomology at present there is no actual theory of $\mathbb{H}$-valued cohomology. We use the terminology because it is convenient, and perhaps to be a little optimistic.

5. Conformal Four Manifolds

A conformal structure on a four manifold is a reduction of the frame bundle to $\mathbb{R}^+ SO(4)$. As we saw at the beginning, the Euclidean conformal group can be presented using the groupoid $\mathcal{H}$ with its tensor product acting as the group structure.

We will indicate here briefly how to construct a quaternionic bitorsor cocycle from a given a conformal structure on a four manifold X. The presentation here is very sketchy and a more detailed presentation is being prepared.

We can choose charts $\{\psi_i : U_i \to \mathbb{H}\}$ that are compatible with the conformal structure, i.e.,

$$\partial(\psi_i \circ \psi_j^{-1}) \in i(\mathbb{H}^* \times \mathbb{H}^*),$$

where $\partial(f)$ is the Jacobian matrix of f considered as an element of $\mathbb{H} \otimes \mathbb{H}$. Therefore there are $\mathbb{H}^*$-valued functions x_{ij} and y_{ij} on U_{ij} such that,

$$\partial(\psi_i \circ \psi_j^{-1}) = x_{ij} \otimes y_{ji}.$$

Over each chart U_i the tangent bundle has a canonical $\mathbb{H}$-bitorsor structure coming from the coordinate ψ. The **tangent gerbe cocycle** allows us to relate these various $\mathbb{H}$-bitorsor structures.

The $x_{ij} \otimes y_{ij}$ can be used to define E_{ij} by twisting the left and right $\mathbb{H}$-actions by $\delta(x_{ij})$ and $\delta(y_{ij})$ respectively. In terms of an $SO(3)$ valued function, we can define E_{ij} relative to TU_i with the function $\delta(y_{ij}x_{ij})$.

Over the triple intersections U_{ijk} it is possible to construct isomorphisms $\phi_{ijk} : E_{ij} \otimes E_{jk} \to E_{ik}$.

It can also be shown that the ψ_i are coordinate charts compatible with the conformal structure if and only if the (E_{ij}, ϕ_{ijk}) form a quaternionic gerbe cocyle.

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Finlay Thompson, Victoria University of Wellington, Wellington, New Zealand

E-mail address: Finlay.Thompson@mcs.vuw.ac.nz