ON THE BANACH-ISOMORPHIC CLASSIFICATION OF L_p SPACES OF HYPERFINITE SEMIFINITE VON NEUMANN ALGEBRAS

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ABSTRACT. We present a survey of recent results in the Banach space classification of non-commutative L_p-spaces.

An important role in Banach space theory has always been played by the problem of classifying Banach spaces. This problem has many facets. In our present setting we address this problem by looking at a Banach space as a linear topological space. The natural maps then are continuous linear operators and we look for invariants under isomorphism (=bicontinuous one-to-one linear operator).

In general, the development of Banach space theory has clearly shown that there is no hope left for a complete structural theory of Banach spaces, although one can still hope to have such a theory in some special cases. Our objective in the present talk is to describe recent results in this direction in the special case of non-commutative L_p-spaces associated with semifinite von Neumann algebras.

To place this work in its proper context we briefly review its origins, beginning with the work of both the mathematicians mentioned so far: Banach and von Neumann.

Many fruitful directions in Banach space theory emerged from the famous book [B] by Banach, and the date of appearance of the French edition of this book (1932) is usually regarded as the date of birth of the theory itself. The final chapter (XII) of this book discusses in depth the problems of comparison between the elements of the two families of Banach spaces (perhaps, the most important families of classical Banach spaces): the spaces l_p and $L_p = L_p(0,1)$, $1 \leq p < \infty$. Recall that l_p is the space of all infinite complex-valued sequences $a = (a_n)_{n=1}^{\infty}$, such that

$$\|a\|_{l_p} := (\sum_{n=1}^{\infty} |a_n|^p)^{1/p} < \infty,$$

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and $L_p = L_p(0,1)$ is the space of all (equivalence classes of) Lebesgue measurable functions f on $(0,1)$ such that

$$\|f\|_{L_p} := (\int_0^1 |f(t)|^p dt)^{1/p} < \infty.$$

The result which is of direct relevance to our present theme can be formulated as follows.

Theorem 1 [B], (1932) *There exists an isomorphic embedding of the space L_p into l_q if and only if $p = q = 2$.*

In other words, these two families are pairwise non-isomorphic.

In contrast to Banach's book, the paper of J. von Neumann [N] is almost completely unknown, even to experts. It appeared in 1937 (five years after Banach's book), in an obscure Russian journal, which ceased to exist almost immediately after its first volume was printed. From the present point of view, the theory of non-commutative L_p-spaces began from this paper. Let me briefly describe (a version of) von Neumann's construction of L_p-spaces associated with the von Neumann algebra M_n of all $n \times n$ complex matrices. As a linear space it is identified with M_n. Given the matrix $A = (a_{ij})_{i,j=1}^n \in M_n$, let $|A| = (A^*A)^{1/2}$. Fixing the standard trace Tr on M_n, we set

$$\|A\|_p := Tr(|A|^p)^{1/p}, \quad 1 \le p < \infty.$$

It is established in [N] that $\|\cdot\|_p$ is a norm on M_n and it is customary to denote the space $(M_n, \|\cdot\|_p)$ by C_p^n. In the modern terminology, the space C_p^n is a non-commutative L_p-space associated with von Neumann algebra (M_n, Tr).

It seems clear from [N] that von Neumann was well aquainted with Banach's book and after having constructed the n^2-dimensional space matrix space C_p^n, he remarks that another natural way to metrize the n^2-dimensional linear space M_n is to identify standard matrix units e_{ij}, $1 \le i,j \le n$ with the first n^2 coordinate vectors of the space l_p, in other words to convert M_n into the n^2-dimensional space $l_p^{n^2}$. This leads to an immediate problem: whether these two n^2-dimensional spaces, C_p^n (non-commutative) and $l_p^{n^2}$ (commutative), coincide. In [N], von Neumann easily established that the spaces C_p^n and $l_p^{n^2}$ are non-isometric.

From the viewpoint adopted in our present setting, the natural question would be whether the Banach-Mazur distance between C_p^n and $l_p^{n^2}$

is uniformly bounded. Recall that the Banach-Mazur distance $d(X,Y)$ between Banach spaces X and Y is

$$d(X,Y) := \inf\{\|T\| \cdot \|T^{-1}\| \mid T : X \longrightarrow Y \text{ isomorphism}\},$$

where we adopt the convention $\inf\emptyset = +\infty$. This question was answered only 30 years later, in the negative, by McCarthy (see [M] and also comments and additional references in Pisier's paper [P]). Before formulating McCarthy's result, recall first that the infinite-dimensional analogues of the spaces C_p^n are Schatten-von Neumann ideals C_p; we may also refer to them as to non-commutative L_p-spaces associated with von Neumann algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on the Hilbert space $\mathcal{H}$ equipped with the standard trace Tr. Recall that a compact operator $A \in \mathcal{B}(\mathcal{H})$ belongs to C_p if and only if

$$\|A\|_{C_p} := Tr(|A|^p)^{1/p} < \infty.$$

Theorem 2 [M], (1967) *There exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and n^2-dimensional subspace X of L_p we have*

$$d(C_p^n, X) > Cn^{\frac{1}{3}\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

The following consequence is straightforward.

Corollary 3 [M], (1967) *There exists an isomorphic embedding of the space C_p into L_p if and only if $p = 2$.*

The converse to the result of Corollary 3 was obtained by Arazy and Lindenstrauss [AL].

Theorem 4 [AL], (1975) *There exists an isomorphic embedding of the space L_p into C_p if and only if $p = 2$.*

One by-product of the Arazy-Lindenstrauss arguments was an identification of yet another member of the L_p-family, the spaces S_p, $1 \leq p < \infty$. The definition is very simple

$$S_p := (\oplus_{n=1}^{\infty} C_p^n)_p,$$

i.e. each element $x \in S_p$ is represented by an infinite sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in C_p^n$ and

$$\|x\|_{S_p} := (\sum_{n=1}^{\infty} \|x_n\|_{C_p^n})^{1/p}.$$

This space can be easily viewed as a subspace of C_p, in our terminology we may say that S_p is the L_p-space associated with the von Neumann algebra $(\oplus_{n=1}^{\infty} M_n)$ (von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$) equipped with the standard trace Tr.

Theorem 5 [AL], (1975) *There exists an isomorphic embedding of the space C_p into S_p if and only if $p = 2$.*

Thus development from 1932 till 1975 has clearly shown that the following four families of infinite-dimensional separable L_p-spaces are non-isomorphic:

(a) The L_p-spaces associated with the von Neumann algebra $l_\infty = L_\infty(\mathbb{N})$ with the trace given by counting measure on $\mathbb{N}$, i.e. the spaces l_p;

(b) The L_p-spaces associated with the von Neumann algebra $L_\infty = L_\infty(0,1)$ with trace given by Lebesgue measure on $(0,1)$, i.e. the spaces L_p;

(c) The L_p-spaces associated with the von Neumann algebra $(\oplus_{n=1}^{\infty} M_n)$ with trace Tr, i.e. the spaces S_p;

(d) The L_p-spaces associated with the von Neumann algebra $\mathcal{B}(\mathcal{H})$ with trace Tr, i.e. the spaces C_p.

Let us now overview the situation. Let $\mathcal{M}$ be an infinite dimensional semifinite von Neumann algebra acting on a separable Hilbert space, let τ be a normal faithful semifinite trace on $\mathcal{M}$, and let $L_p(\mathcal{M},\tau)$, $1 \leq p < \infty$ be the Banach space of all τ-measuarble operators A affiliated with $\mathcal{M}$ such that $\tau(|A|^p) < \infty$ with the norm $\|A\|_p := (\tau(|A|^p))^{1/p}$, where $|A| = (A^*A)^{1/2}$ (see e.g. [FK]). It is quite natural to ask the following question:

What is the linear-topological classification of the spaces $L_p(\mathcal{M},\tau)$, $p \neq 2$?

It is natural to subdivide the above question to further subcategories accordingly to various classification schemes for von Neumann algebras. Tne following two results (obtained jointly with V. Chilin) are a simple application of Pełczyński's decomposition method.

Proposition 6 [SC1], (1988) *Let $\mathcal{M}$ be a commutative von Neumann algebra with a normal faithful semifinite trace τ. Then $L_p(\mathcal{M},\tau)$, $p \neq 2$ is Banach isomorphic to one of the spaces l_p or L_p.*

Further, recall that a von Neumann algebra $\mathcal{M}$ is called *atomic* if every nonzero projection in $\mathcal{M}$ majorizes a nonzero minimal projection.

Proposition 7 [SC1], (1988) *Let $\mathcal{M}$ be an atomic von Neumann algebra with a normal faithful semifinite trace τ. Then $L_p(\mathcal{M},\tau)$, $p \neq 2$ is Banach isomorphic to one of the spaces l_p, S_p or C_p.*

The next logical step is the description of L_p-spaces associated with von Neumann algebras of type I with separable predual. It is well-known that such an algebra can be represented as a countable l_∞-direct sum of von Neumann algebras of the type $\mathcal{A}_n \bar{\otimes} M_n$ and $\mathcal{A}\bar{\otimes}\mathcal{B}(\mathcal{H})$, where $\mathcal{A}_n$, $\mathcal{A}$ are commutative von Neumann algebras with separable preduals. One can easily see that the following Banach spaces are actually L_p-spaces associated with von Neumann algebras of type I: the direct sums $L_p \oplus S_p$ and $L_p \oplus C_p$, the Lebesgue-Bochner spaces $L_p(S_p)$ and $L_p(C_p)$, as well as the space $C_p \oplus L_p(S_p)$. The following result (announced in [SC2]) actually shows that these examples actually exhaust the list of such spaces.

Theorem 8 [SC2], (1990) *Let $\mathcal{M}$ be a type I von Neumann algebra with a normal faithful semifinite trace τ. Then $L_p(\mathcal{M},\tau),\ p\neq 2$ is Banach isomorphic to one of the following spaces*

$$l_p,\ L_p,\ S_p,\ C_p,\ S_p\oplus L_p,\ L_p(S_p),\ C_p\oplus L_p,\ L_p(C_p),\ C_p\oplus L_p(S_p).$$

Moreover, the spaces

$$l_p,\ L_p,\ S_p,\ C_p,\ S_p\oplus L_p$$

are pairwise Banach non-isomorphic and non-isomorphic to any of the four remaining spaces

$$L_p(S_p),\ C_p\oplus L_p,\ L_p(C_p),\ C_p\oplus L_p(S_p).$$

Thus, the number of distinct L_p-families has been raised from 4 to 5. However the question whether the remaining 4 spaces are pairwise distinct proved to be very hard. The following result proved to be one of the necessary ingredients.

Theorem 9 [S1], (1996) *Let $\mathcal{N}$ be a finite von Neumann algebra with finite, normal, faithful trace τ_1, let $\mathcal{M}$ be an infinite von Neumann algebra with semifinite, normal, faithful trace τ_2. Then for $p>2$ there is no Banach isomorphic embedding of C_p into $L_p(\mathcal{N},\tau_1)$, whence $L_p(\mathcal{N},\tau_1)$ and $L_p(\mathcal{M},\tau_2)$ are non-isomorphic for all $p\in(1,\infty),\ p\neq 2$.*

This result was subsequently used (together with other methods) in the first part of the following theorem, which raises the number of distinct families of reflexive L_p-families to 8. The second part of Theorem 10 below yields a complete linear-topological classification of the preduals to von Neumann algebras of type I. The proof of the second part is based on the study of the Dunford-Pettis property in von Neumann algebras and its preduals.

Theorem 10 [S2], (2000) *Let $\mathcal{M}$ be an infinite-dimensional von Neumann algebra of type I acting in a separable Hilbert space H, let τ be a normal faithful semifinite trace on $\mathcal{M}$, let $L_p(\mathcal{M},\tau),\ p\in[1,\infty),\ p\neq 2$ be the L_p-space associated with $\mathcal{M}$. Then* *(a) the space $L_p(\mathcal{M}, \tau)$ is isomorphic to one of the following nine spaces:*

$$l_p,\ L_p,\ S_p,\ C_p,\ S_p \oplus L_p,\ L_p(S_p),\ C_p \oplus L_p,\ L_p(C_p),\ C_p \oplus L_p(S_p); \quad (\mathbf{L})$$

and if (E, F) is a pair of distinct spaces from $(\mathbf{L})$, which does not coincide with the pair $(L_p(C_p),\ C_p \oplus L_p(S_p))$, then E is not isomorphic to F;

(b) all nine spaces from $(\mathbf{L})$ are pairwise non-isomorphic, provided $p = 1$.

However the question whether the $L_p(C_p)$ and $C_p \oplus L_p(S_p)$ are Banach distinct for $1 < p < \infty$ remained unresolved. The technique of [S1] for dealing with embeddings of C_p for $p > 2$ has not been sufficient. The breakthrough has come with the following joint result of the author with U. Haagerup and H. Rosenthal.

Theorem 11 [HRS1], [HRS2] (2000) *Let $\mathcal{N}$ be a finite von Neumann algebra with finite, normal, faithful trace τ_1, let $\mathcal{M}$ be an infinite von Neumann algebra with semifinite, normal, faithful trace τ_2. Then for $1 \le p < 2$ there is no Banach isomorphic embedding of C_p into $L_p(\mathcal{N}, \tau_1)$, whence $L_p(\mathcal{N}, \tau_1)$ and $L_p(\mathcal{M}, \tau_2)$ are non-isomorphic for all $p \in [1, \infty),\ p \neq 2$.*

Theorem 11 is crucial in the proof of the following theorem which completes the isomorphic classification of separable L_p-spaces associated with von Neumann algebras of type I for $p > 1$; it yields more than just the non-isomorphism of $L_p(C_p)$ and $C_p \oplus L_p(S_p)$ and strengthens the second part of Theorem 10.

Theorem 12 [HRS1], [HRS2] (2000) *Let $\mathcal{N}$ be a finite von Neumann algebra with a fixed faithful normal tracial state τ on $\mathcal{N}$ and $1 \le p < 2$. Then $L_p(C_p)$ is not isomorphic to a subspace of $C_p \oplus L_p(\mathcal{N}, \tau)$.*

Thus, all nine spaces listed in $\mathbf{L}$ (see Theorem 10) are pairwise non-isomorphic also for $1 \le p \neq 2 < \infty$.

Much more follows via application of (a strengthened version of) Theorem 11. Let $\mathcal{M}$ be a hyperfinite (i.e., $\mathcal{M}$ is a weak closure of a union of an increasing sequence of finite dimensional von Neumann algebras) semifinite von Neumann algebra. In this setting, $\mathcal{M}$ can be decomposed as $\mathcal{M}_I \oplus \mathcal{M}_{II_1} \oplus \mathcal{M}_{II_\infty}$, where $\mathcal{M}_I$, $\mathcal{M}_{II_1}$, $\mathcal{M}_{II_\infty}$ are hyperfinite von Neumann algebras of types I, II_1, and II_∞ respectively. Further, using disintegration and deep results of A. Connes [C], the algebras $\mathcal{M}_{II_1}$ (respectively, $\mathcal{M}_{II_\infty}$) can be realized as a countable l_∞-direct sum of von Neumann algebras of the form $\mathcal{A}\bar{\otimes}\mathcal{B}$, where $\mathcal{A}$ is as above and $\mathcal{B}$ the unique hyperfinite factor $\mathcal{R}$ of type II_1 (respectively, the unique hyperfinite factor $\mathcal{R}_{0,1} = \mathcal{R} \otimes \mathcal{B}(\mathcal{H})$ of type II_∞). Again via Pełczyński's decomposition method (and results of A. Connes) we arrive at the following classification result.

Proposition 13 [HRS1], [HRS2] (2000) *If $\mathcal{M}$ is a hyperfinite semifinite von Neumann algebra with a normal faithful semifinite trace τ and $1 \leq p < \infty$, then $L_p(\mathcal{M}, \tau)$ is isomorphic to one of the following thirteen spaces:*

$$l_p,\ L_p,\ S_p,\ C_p,\ S_p \oplus L_p,\ L_p(S_p),\ C_p \oplus L_p,\ L_p(C_p),\ C_p \oplus L_p(S_p)\ ,$$

$$L_p(\mathcal{R}),\ C_p \oplus L_p(\mathcal{R}),\ L_p(\mathcal{R}) \oplus L_p(C_p),\ L_p(\mathcal{R}_{0,1}).$$

However, the question whether all spaces listed above are pairwise non-isomorphic is much harder. It required the full strength of Theorem 11 combined with very recent results of M. Junge [J].

Theorem 14 [HRS1], [HRS2] (2000) *Let $\mathcal{N}$ be a finite von Neumann algebra with a fixed faithful normal tracial state τ and $1 \leq p < 2$. Then $L_p(\mathcal{R}_{0,1})$ is not isomorphic to a subspace of $L_p(\mathcal{N}) \oplus L_p(C_p)$.*

Theorem 15 [HRS1], [HRS2] (2000) *If $\mathcal{M}$ is a hyperfinite semifinite von Neumann algebra with a normal faithful semifinite trace τ and $1 \leq p < \infty$, then $L_p(\mathcal{M}, \tau)$ is Banach isomorphic to precisely one of the spaces listed in Proposition 13.*

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