NORMS OF 0–1 MATRICES IN C_P

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To Derek Robinson on his 65th birthday

Abstract. We announce a new result (proved in collaboration with T.A. Gillespie) on the boundedness of a class of Schur multiplier projections on the von Neumann-Schatten ideals C_p. We also show that for $1 \le p \le 2$ the average C_p norm of a 0–1 matrix grows just as quickly as the largest norm of such a matrix.

1. Introduction

Calculating the norm of an operator which acts on one of the von Neumann Schatten ideals C_p is often rather difficult — even for rather simple operators — because of the scarcity of elements $T \in C_p$ for which one can easily calculate (or even approximate) $\|T\|_p = \|T\|_{C_p}$. Even for the algebraically simple Schur projections which act by replacing certain fixed entries of the matrices of elements of C_p by 0, proving boundedness results is typically quite hard.

To fix some notation, for $1 \le p < \infty$, let C_p denote the von Neumann Schatten ideal of compact operators on ℓ^2, with norm $\|T\|_p = \mathrm{trace}((T^*T)^{p/2})^{1/p}$. We take C_∞ to be the set of all compact operators on ℓ^2 with the usual operator norm. We will let C_p^n denote the $n \times n$ matrices equipped with the corresponding norm. We will, as usual, think of elements of C_p as being infinite matrices.

Let $\mathcal{Z}$ denote the set of all zero-one arrays $[a_{i,j}]_{i,j=1}^{\infty}$. We are interested in the norms of projections defined by Schur multiplication of such arrays. If $A = [a_{i,j}] \in \mathcal{Z}$, define the **Schur projection** corresponding to A to be the map $P_A : T \mapsto A \circ T$, where $\circ$ denotes Schur or elementwise multiplication of matrices. Let

$$\mathcal{B}_p = \{A \in \mathcal{Z} \,:\, P_A \in B(C_p)\}\,.$$

The set of $n \times n$ matrices with zero-one entries will be denoted by $\mathcal{Z}^n$. By requiring that each entry in an array is either zero or one with equal

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probability we may regard $\mathcal{Z}$ and $\mathcal{Z}^n$ as being probability spaces with the appropriate product measures.

If $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $C_p^* = C_q$, under the natural pairing $\langle S, T\rangle = \mathrm{trace}(ST)$. If $A \in \mathcal{B}_p$ then $\mathrm{trace}(P_A(S)^*T) = \mathrm{trace}(S^* P_A(T))$ for all $S \in C_p$, $T \in C_q$ and so it follows that $\|P_A\|_p = \|P_A\|_q$. In particular note that $\mathcal{B}_2 = \mathcal{Z}$ with $\|P_A\|_p = 1$ for all $A \in Z$ and that $\mathcal{B}_p = \mathcal{B}_q$.

It is a trivial consequence of the ideal inequalities for C_p that if $A \in \mathcal{Z}$ is a nonzero array which is constant on each row (or on each column) then $\|P_A\|_p = 1$ for all p. Proving boundedness for other types of arrays has been significantly more difficult. The first major result was due to Macaev (see [7]) who showed that the upper triangular truncation map is bounded on C_p for $1 < p < \infty$ (but not for $p = 1$ or $p = \infty$). In the 1980s Bourgain [3] showed boundedness results for a class of 'Toeplitz' arrays which are analogous to multiplier results from Fourier analysis. In particular, for $1 < p < \infty$ there is a constant K_p such that if $A \in \mathcal{Z}$ is any array which is constant on diadic blocks of (long) diagonals, then $\|P_A\|_p \le K_p$.

It is still an open question as to whether if $2 < r < p < \infty$ there is always a Schur multiplier projection which is bounded on C_r, but not on C_p. An important special case when r and p are even integers has recently been solved (in the affirmative) by A. Harcharras [8].

Recently, Alastair Gillespie and I have proved boundedness for a new class of such projections.

Definition 1.1. A zero-one array $[a_{ij}]$ will be said to be **obtainable** if for all indices i_1, i_2, j_1, j_2,

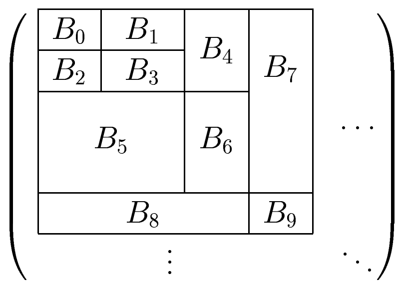
$$\begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} \\ a_{i_2 j_1} & a_{i_2 j_2} \end{pmatrix} \ne \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

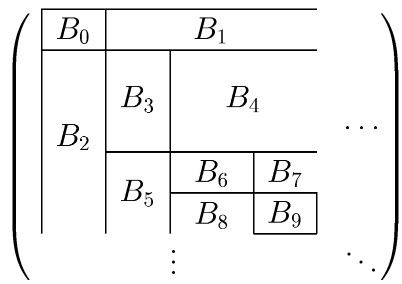
Theorem 1.2. *For all $p \in (1, \infty)$ there exists a constant K_p such that for every obtainable array A, $\|P_A\|_p \le K_p$.*

The proof, which depends on the fact that for $1 < p < \infty$, C_p is a UMD space, uses a new result from spectral theory proved in [5]; the sum of two commuting real scalar-type spectral operators on a UMD space is a well-bounded operator. (Since completing this work we have been made aware of some closely related work by Clément, de Pagter, Sukochev and Witvliet [4] [11] [12].)

As we show in [6], it is relatively easy to use this result to recover Bourgain's theorem (at least as it applies to 0–1 arrays). One can also prove Littlewood-Paley type decomposition results in C_p, of which we shall just give two simple examples here.

Split $\mathbb{Z}^+ \times \mathbb{Z}^+$ into rectangular subarrays as in either of the diagrams below.

(i) 

(ii) 

The actual sizes of the subarrays is not important.

Let P_k denote the projection given by Schur multiplication by the characteristic function of B_k.

Theorem 1.3. *Suppose that $1 < p < \infty$ and that $\emptyset \neq J \subset \mathbb{N}$. Then $\sum_{k \in J} P_k$ converges in the strong operator topology and*

$$\Big\| \sum_{k \in J} P_k \Big\|_p \leq 2K_p + 1$$

where K_p is the constant from Theorem 1.2.

Actually, if all the 'diagonal' subarrays B_{3k} are just 1×1 subarrays, then this result is an easy consequence of Macaev's result. Even for the more general splittings of this form described above, it is probably not too hard to deduce this bound from known estimates (see, for example, Section 4 of [1]). Our techniques however cover a rather wider range of decompositions including some which are not at all related to triangular trunctions. Details will appear in [6].

2. Norms of 0–1 matrices

In proving the above results we were lead to considering what one can say about the bounds of Schur multiplier projections on C_p^n. The following example is the standard way of showing that these projections can have bad norms.

Example 2.1. Let $B_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. For $m \geq 2$, define $B_m = B_{m-1} \otimes B_1$. For example

$$B_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Thus B_m is a $2^m \times 2^m$ orthogonal matrix. Let $n = 2^m$. Then $B_m^* B$ is just nI. Clearly then $\|B_m\|_p = (n\, n^{p/2})^{1/p} = n^{\frac{1}{2}+\frac{1}{p}}$. Let O_m be the $n \times n$ matrix all of whose entries are 1. This is just n times a projection, so $\|O_m\|_p = n$ for all p. Let A_m be the $n \times n$ array formed by replacing the -1's in the matrix B_m by zeros. Thus $A_m = \frac{1}{2}(B_m + O_m)$. Note that $A_m = P_{A_m} O_m$. If $1 \le p < 2$ then

$$\|P_{A_m}\|_p \ge \frac{\|A_m\|_p}{\|O_m\|_p} \ge \frac{\frac{1}{2}\left(n^{\left(\frac{1}{2}+\frac{1}{p}\right)} - n\right)}{n} = \frac{1}{2} n^{\left(\frac{1}{p}-\frac{1}{2}\right)} - \frac{1}{2}.$$

Using the earlier remarks about duality we see that there exists $c > 0$ such that for $1 \le p \le \infty$ and large n,

$$\|P_{A_m}\|_p \ge c n^{\left|\frac{1}{2}-\frac{1}{p}\right|}.$$

□

This order of growth is the worst that one can get from $n \times n$ matrices. Theorem 6.2 of [2] shows that for any $A \in Z_n$, $\|P_A\|_\infty \le n^{1/2}$. Using interpolation and duality gives that $\|P_A\|_p \le n^{\left|\frac{1}{2}-\frac{1}{p}\right|}$.This upper bound also follows from the following upper bound due to Ong [10]:

$$\|P_A\|_\infty \le \min\left\{\max\ \ell^2 \text{ norm of a column},\ \max\ \ell^2 \text{ norm of a row}\right\}.$$

We shall say that $A_1 \in \mathcal{Z}^n$ is a subarray of $A \in \mathcal{Z}$ if A_1 is formed by deleting all but n of the rows and all but n of the columns of A. If A_1 is a subarray of A then $\|P_{A_1}\|_p \le \|P_A\|_p$. With probability one, a randomly chosen infinite string of binary digits contains all finite strings as substrings. In other words, with probability one, an element of $\mathcal{Z}$ contains every array A_m (from Example 2.1 above) as a subarray. It follows immediately that if $p \ne 2$ then $\mathrm{Prob}(\|P_A\| < \infty) = 0$.

A more delicate question concerns whether this 'bad' array is typical of $n \times n$ arrays. The answer is Yes. This is undoubtedly known to the experts, although as far as we are aware, this does not appear in the literature. (Indeed, throughout this area, much more seems to be known than is written down!)

To simplify notation we shall write $g(n) \sim f(n)$ if there exist constants $0 < c_1 < c_2 < \infty$ such that $c_1 f(n) < g(n) < c_2 f(n)$ for all (sufficiently large) n. Throughout we shall use c as a generic absolute constant whose value may change from one line to the next.

Theorem 2.2. $\mathbb{E}(\|P_A\|_p : A \in \mathcal{Z}^n) \sim n^{\left|\frac{1}{2}-\frac{1}{p}\right|}$.

Proof. For each n let X_n denote the matrix-valued random variable where each entry is an independent Gaussian variable with mean 0 and variance 1. That is, $X_n = \sum_{i,j=1}^n g_{ij} E_{ij}$ where each g_{ij} is an independent $N(0,1)$ random variable and $\{E_{ij}\}_{i,j=1}^n$ are the standard matrix units. Let $\{r_{ij}\}_{i,j=1}^n$ be a family of independent identically distributed random variables which take the values -1 and 1 with equal probability.

Suppose first that $1 \le p \le 2$. Some rather deep estimates of Szarek [14] give that

$$\mathbb{E}(\|X_n\|_p) \sim n^{\frac{1}{2}+\frac{1}{p}}.$$

Then, since C_p has cotype 2 for p in this range, [13, Theorem 3.9] implies that

$$\begin{aligned} cn^{\frac{1}{2}+\frac{1}{p}} &\le \mathbb{E}\Big(\Big\|\sum_{i,j=1}^n g_{ij}E_{ij}\Big\|\Big) \\ &\le \Big(\mathbb{E}\Big(\Big\|\sum_{i,j=1}^n g_{ij}E_{ij}\Big\|^2\Big)\Big)^{1/2} \\ &\le c\,\Big(\mathbb{E}\Big(\Big\|\sum_{i,j=1}^n r_{ij}E_{ij}\Big\|^2\Big)\Big)^{1/2} \end{aligned}$$

But by Kahane's inequality [9, Theorem 1.e.13]

$$\Big(\mathbb{E}\Big(\Big\|\sum_{i,j=1}^n r_{ij}E_{ij}\Big\|^2\Big)\Big)^{1/2} \le c\,\mathbb{E}\Big(\Big\|\sum_{i,j=1}^n r_{ij}E_{ij}\Big\|\Big).$$

Following the same argument as in Example 2.1,

$$\mathbb{E}(\|A\|_p \,:\, A \in \mathcal{Z}^n) \ge \frac{1}{2}\Big(\mathbb{E}\Big(\Big\|\sum_{i,j=1}^n r_{ij}E_{ij}\Big\|\Big) - n\Big) \ge c\,n^{\frac{1}{2}+\frac{1}{p}},$$

(for large n) and so

$$\mathbb{E}(\|P_A\|_p \,:\, A \in \mathcal{Z}^n) \ge cn^{\frac{1}{p}-\frac{1}{2}}.$$

Since for any $A \in \mathcal{Z}^n$, $\|P_A\|_p \le n^{\frac{1}{p}-\frac{1}{2}}$ the result is proved. The case when $p > 2$ follows from duality. $\square$

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