ORBITAL CONVOLUTIONS, WRAPPING MAPS AND e-FUNCTIONS

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ABSTRACT. We survey the theory of wrapping maps as applied to compact groups and vector times compact semidirect products, and give an explicit description of the e-function for compact symmetric spaces. The latter is globally defined.

1. INTRODUCTION

Let G be a Lie group. The Kirillov orbit method gives a heuristic method which relates the Euclidean Fourier transforms of coadjoint orbits in $\mathfrak{g}^*$ to the infinitesimal characters of the irreducible representations. At its simplest, it has the following form

$$j(X)\mathrm{tr}\,\sigma(\exp X) = \int_{\mathcal{O}} e^{i\eta(X)} d\mu_{\mathcal{O}}(\eta), \tag{1.1}$$

where σ is an irreducible representation of G related to the coadjoint orbit $\mathcal{O}$, $\mu_{\mathcal{O}}$ is G-invariant Liouville measure on $\mathcal{O}$ and j is the square root of the Jacobian of the exponential map.

In the case of a compact Lie group, this formula is exact — for other groups, where σ is infinite dimensional and $\mathcal{O}$ need not have compact support, the formula needs more careful interpretation — in general, it should be seen as an equality of distributions. The reader should consult Kirillov's recent survey article [11] for a detailed discussion of the orbit method.

In [4], the author and Norman Wildberger remarked that, for compact groups, the Kirillov formula follows in a simple way from the fact that the wrapping map is a homomorphism of Banach algebras between (say) $M^G(\mathfrak{g})$ and $M^G(G)$. Hence M^G is the set of G-invariant measures (Ad-invariant on $\mathfrak{g}$ and central on G). For an Ad-invariant distribution η of compact support on $\mathfrak{g}$, let $\Phi\eta$ be a distribution on G defined for $f \in C^\infty(G)$, by

$$\langle \Phi\eta, f\rangle_G = \langle \eta, j.f \circ \exp\rangle_{\mathfrak{g}}.$$

The wrapping formula then states

$$\Phi\eta *_G \Phi\nu = \Phi(\eta *_{\mathfrak{g}} \nu). \tag{1.2}$$

From this formula, it follows that the adjoint mapping $\Phi' : M^G(G)' \to M^G(\mathfrak{g})'$ is an injection of Gelfand spaces; thus, to each irreducible character of G, one obtains a character of $\mathfrak{g}$ averaged over adjoint orbits, that is, a mapping of the form $\nu \mapsto \int \int_{\mathcal{O}} e^{i\phi(x)} d\mu_{\mathcal{O}}(\phi) d\nu(x)$. From this the Kirillov character (1.1) follows easily.

The above theorem allows us to relate the central harmonic analysis of G to Euclidean harmonic analysis of $\mathfrak{g}$. This becomes particularly interesting when one realizes that the latter can be described explicitly. In fact, in [3], we gave the following formula for Ad-invariant convolution on $\mathfrak{g}$. Recall that each adjoint orbit $\mathcal{O}$ intersects the positive Weyl chamber $\mathfrak{t}^+$ of the Cartan subalgebra $\mathfrak{t}$ in a unique point — λ say. Then we have

$$\mu_\lambda * \mu_\gamma = \int_{\mathfrak{t}^+} N(\lambda, \gamma, \xi) \mu_\xi \, d\xi,$$

where

$$N(\lambda, \gamma, \xi) = \sum_{w \in W} \operatorname{sgn} w \, e_{w\lambda} * T_\gamma(\xi),$$

$T_\lambda(\xi)$ being the projection on $\mathfrak{t}$ of $\mathcal{O}_\lambda$, given by

$$T_\lambda = \frac{1}{|W|} \sum_{w, w' \in W} \operatorname{sgn} ww' e_{w\lambda} \prod_{\alpha \in \Phi^+} F_{w'\alpha},$$

where F_α is the distribution on $\mathfrak{t}$ given by Lebesgue measure on the ray through α.

In work currently in progress, these results are being extended in two directions — to some non-compact groups, and to compact symmetric spaces. I will describe these results in the next section and in section 4.

2. Semi-direct product groups

In [6], we extend the wrapping map formula to $G = V \rtimes K$, V a vector group, K a compact group. Here already, there is a substantial technical hurdle to be overcome, in the sense that formula (1.2) requires the convolution of Ad-invariant distributions (or measures) to be defined, and there are no non-trivial G-invariant distributions of compact support, as the G-orbits in $\mathfrak{g}$ (and in $\mathfrak{g}^*$) are not compact.

This problem can be overcome by the following device. Notice that conjugacy classes, adjoint orbits and coadjoint orbits are all fibred spaces over K-orbits.

I will illustrate this for adjoint orbits only. For each $A \in \mathfrak{k}$, split V into two subspaces, $V_A = \{v \in V : A \cdot v = v\}$ and $V^A = V_A^\perp$ (the orthogonal complement with respect to a K-invariant inner product) — where $A \cdot v$ is the derivative of the K-action. Then for $v \in V$ the orbit $G \cdot A$ is fibred over the compact orbit $K \cdot v$, the fibre at $k \cdot A$ being $V^{k \cdot A}$.

Now we replace the G-invariant distributions of compact support on G with a family of distributions on $\mathfrak{g} = \mathfrak{k} + V$ which are K-invariant, compactly supported in the $\mathfrak{k}$-direction and for each A in this support, are given by an invariant mean (suitably normalized) on V^A. It turns out that such distributions:

(i) wrap to similarly defined distributions on conjugacy classes,
(ii) belong to the dual of a suitable G-stable family of functions on $\mathfrak{g}$ — they are C^∞ in the $\mathfrak{k}$-direction and for each $A \in \mathfrak{k}$, are almost periodic in the V^A direction,
(iii) have a natural notion of convolution (using the above duality in a standard way),
(iv) have as Fourier transforms, similar distributions on $\mathfrak{g}^*$.

The formula (1.2) continues to hold for $V \rtimes K$ with the above definitions. This leads to a new proof of the Lipsman character theorem [13]. The full details are somewhat technical. We may interpret the above results as follows. Each of the conjugacy classes, adjoint orbits and coadjoint orbits possesses a natural convolution on hypergroup structures. Denote the associated hypergroups as Conj, Adj and Coadj respectively. Now Φ provides a homomorphism $\Phi : \mathrm{Adj} \to \mathrm{Conj}$ and we may identify Coadj as the dual hypergroup of Adj. The Lipsman character may thus be interpreted as the mapping Φ' from

$$\mathrm{Conj}^* \to \mathrm{Coadj} = (\mathrm{Adj})^*.$$

It is possible also to give very explicit formulae for the hypergroup structures of Adj and Coadj: they are no longer identical, in contrast to the compact case. The gist of this structure is that the compact orbits in V (or V^*) are convolved as in section 1, and one forms the sums of the fibres. Full details are in [6].

3. Generalizations of the Duflo Isomorphism

The wrapping map formula (1.2) can be considered as a global version of the Duflo isomorphism. To see this, notice that the Ad-invariant distributions of support $\{0\}$ in $\mathfrak{g}$ wrap to central distributions of support $\{e\}$ in G. The latter may be identified with $\mathfrak{z}\mathfrak{u}(\mathfrak{g})$, the centre of the universal enveloping algebra; the former with $S^*_G(\mathfrak{g})$, the centre of the symmetric algebra of $\mathfrak{g}$.

Recently, far reaching generalizations of the Duflo isomorphism have been proved by Maxim Kontseivich [12]. He proves that for a Poisson manifold (X, γ), there is a family $*_r$ of star products which deform products (at $r = 0$) to convolutions (at $r = 1$).

However, this series can only be shown to converge near $\{0\}$, and it is of considerable interest to ask how far they can be extended, and if global versions such as (1.2) are available in special cases. Andler, Dvorsky and Sahi [1] recently showed, using [12], that every bi-invariant differential operator on a Lie group is locally solvable.

Actually, in the case where $X = G/H$, G a Lie group and H the set of fixed points of an involution σ, Kontseivich's construction coincides with that of Rouvière [14].

Rouvière's theory is as follows. We may write $\mathfrak{g} = \mathfrak{h} + \mathfrak{s}$, where $\mathfrak{h}$ and $\mathfrak{s}$ are the eigenspaces of σ (by which I denote also the differential of the involution, by abuse of notation) of eigenvalues $+1$ and -1 respectively. Then $\mathfrak{h}$ is the Lie algebra of H, and $\mathfrak{p}$ may be identified with the tangent space of G/H. There is, furthermore, an exponential map $\mathrm{Exp} : \mathfrak{p} \to G/H$. We take an H-invariant neighbourhood $\mathfrak{s}$ of o in $\mathfrak{p}$ on which Exp is a diffeomorphism. For $X \in \mathfrak{s}$, let $j(X)$ be a suitable square root of $J_{*,0}(\mathrm{Exp})(X)$. (We leave aside temporarily the existence of a smooth real-valued square root — in the case of interest below, it can be explicitly calculated.) Then $\mathrm{Ad}|_H : \mathfrak{p} \to \mathfrak{p}$ and $j(\mathrm{Ad}(h)X) = j(X)$ for all $x \in \mathfrak{p}$, $h \in H$. We may thus define a version of wrapping using the same ideas as above: for an H-invariant distribution η (of compact support) on $\mathfrak{p}$, and for $f \in C_c^\infty(G/H)$, let

$$\langle \eta, f \rangle_{G/H} = \langle \eta, j.f \circ \mathrm{Exp} \rangle_{\mathfrak{p}}.$$

If now ξ, η are supported in $\mathfrak{s}$, and are such that $\xi *_{\mathfrak{p}} \eta$ is also supported in $\mathfrak{s}$, we can ask whether we have a formula such as "$\Phi(\xi *_{\mathfrak{p}} \eta) = \Phi(\xi) *_{G/H} \Phi(\eta)$". It turns out that the formula requires some modification and that it should read

$$\Phi(\xi *_{\mathfrak{p},e} \eta) = \Phi(\xi) *_{G/H} \Phi(\eta). \tag{3.1}$$

In this formula, $e(X, Y)$ is a certain function of two variables on $\mathfrak{s} \times \mathfrak{s}$, and $\xi *_{\mathfrak{p},e} \eta$ is "twisted" convolution given, for ξ, η H-invariant and locally integrable, by

$$(\xi *_{\mathfrak{p},e} \eta)(X) = \int_{\mathfrak{p}} \xi(Y)\eta(X - Y)e(X, Y)dY. \tag{3.2}$$

(This formula needs an obvious adaptation in order for it to work for distributions — see [14].)

It is instructive to understand where the e-function comes from, as we will be calculating it in some special cases in the next section. We write the right-hand side of (3.1), for $u \in C^\infty(G/H)$, as

$$\begin{aligned}\langle \Phi(\xi) *_{G/H} \Phi(\eta), u\rangle &= \int_{G/H}\int_{G/H} \Phi(\xi)(x)\Phi(\eta)(y)u(xy)dxdy \\ &= \int_{\mathfrak{s}}\int_{\mathfrak{s}} \xi(X)\eta(Y)j(X)j(Y)u(\operatorname{Exp} X \operatorname{Exp} Y)dXdY.\end{aligned}$$

We now claim that there exist $h, k \in H$ so that $\operatorname{Exp} X \operatorname{Exp} Y = \operatorname{Exp}(h.X + k.Y)$. Accepting this for the moment, consider the change of variables $(h.X, k.Y) \mapsto (X, Y)$. Denote the Jacobian of this change of variables by $\psi(X, Y)$. Then the integral transforms to

$$\int_{\mathfrak{s}}\int_{\mathfrak{s}} \xi(h^{-1}X)\eta(k^{-1}Y)j(h^{-1}X)j(k^{-1}Y)\psi(X,Y)u(\operatorname{Exp}(X+Y))dXdY.$$

Now j, ξ and η are all H-invariant, so we obtain

$$\int_{\mathfrak{s}}\int_{\mathfrak{s}} \xi(X)\eta(Y)\frac{j(X)j(Y)}{j(X+Y)}\psi(X,Y)(j.u \circ \operatorname{Exp})(X+Y)dXdY.$$

Letting

$$e(X,Y) = \frac{j(X)j(Y)}{j(X+Y)}\psi(X,Y),$$

we obtain the right-hand side.

Rouvière is able to calculate $e(X, Y)$ as an infinite power series, which converges in a neighbourhood of o in $\mathfrak{s}$.

To see why the elements h and k above exist, consider the Campbell-Baker-Hausdorff series for Exp.

$$\operatorname{Exp} X \operatorname{Exp} Y = \operatorname{Exp}(X + Y + \frac{1}{2}[X,Y] + \frac{1}{6}([X[X,Y]] + [Y[Y,X]]) + \dots).$$

Now the hypothesis that σ is an involutive automorphism of $\mathfrak{g}$ implies that $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$. Thus we may rearrange the above series to get

$$\begin{aligned} & X + \frac{1}{6}[[X,Y],X] + \cdots + Y - \frac{1}{6}[[X,Y],Y] + \cdots + \cdots + H(X,Y) \\ = \ & (I + \frac{1}{6}[X,Y] + \cdots +)X + (I - \frac{1}{6}[X,Y] + \cdots +)Y + H(X,Y), \\ = \ & h.X + k.Y + H,\end{aligned}$$

where $H \in \mathfrak{h}$.

The result now follows. (For a fuller proof, and for computations of the series, see [14].)

4. Symmetric spaces of the compact type

The question I would like to address in this section is: can we find a global version of Rouvière's formula in the case of a symmetric space of the compact type?

Let (G, K) be a Riemannian symmetric pair of the compact type. Then K is the set of fixed points of the Cartan involution σ, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition. Choose a maximal abelian subalgebra $\mathfrak{q}$ of $\mathfrak{p}$ and let A be the corresponding subgroup of G. We then have the Cartan decomposition of G, $G = KA_+K$, where $A_+ = \exp(\mathfrak{a}_+)$. Here $\mathfrak{a}_+$ is the positive Weyl chamber for a set of positive restricted roots Φ_r^+. Let m_α denote the multiplicity of $\alpha \in \Phi_r^+$.

We may identify $\mathfrak{p}$ as the tangent space at eK of $X = G/K$, and have the standard exponential map $\mathrm{Exp} : \mathfrak{p} \to X$. The square root of the Jacobian of Exp is Ad_K invariant of $\mathfrak{p}$, and is given by

$$j(H) = \prod_{\alpha \in \Phi_r^+} \left[\frac{\sin \alpha(H)}{\alpha(H)}\right]^{m_{\alpha/2}}, \quad (H) \in \mathfrak{a}_+.$$

For $X, Y \in \mathfrak{p}$, let $X = \mathrm{Ad}\,(k_1)H_1$, $Y = \mathrm{Ad}(k_2)H_2$ and $X + Y = \mathrm{Ad}(k_3)H_3$. We define

$$e(X,Y) = \frac{j(H_1)j(H_2)}{j(H_3)} \prod_{\alpha\in\Phi_r^+} \prod_{w,w'\in W_r} \left[\frac{\cos\frac{1}{2}(\alpha(H_1)+\alpha^w(H_2)+\alpha^{w'}(H_3))}{\alpha(H_1)+\alpha^w(H_2)+\alpha^{w'}(H_3)}\right]^{m_{\alpha/2}},$$

where W_r denotes the restricted Weyl group and $\alpha^w(H)$ is the image of the root α by the W_r-action. The following theorem then holds.

Theorem (i) e is defined on all of $\mathfrak{p}$.
(ii) Let μ and ν be K-invariant distributions of compact support on $\mathfrak{p}$. Then

$$\Phi\mu *_X \Phi\nu = \Phi(\mu *_{\mathfrak{p},e} \nu).$$

Details of the proof of this theorem will appear in [7]. In order to prove it, we need to find explicitly the hypergroup convolution of K-orbits in $\mathfrak{p}$ and of K-orbits in X. The e-function is then found by comparing the two structures.

The gist of the calculation is already present in the case of the sphere $X = SO(3)/SO(2)$, which is discussed in [15]. We briefly describe this calculation here.

Let us discuss the convolution of two circles (radii r_1 and r_2) in the plane. (This corresponds to K-orbits in $\mathfrak{p}$.) One needs to write

$$r_1 + r_2 e^{i\theta} = re^{i\psi}$$

and then compute "dr" in terms of "$d\theta$". (It is obvious that the resulting measure is rotationally invariant.) A little first year calculus yields

$$\frac{2r}{2r_1r_2\sin\theta}\,\frac{dr}{\pi} = \frac{d\theta}{\pi}$$

and one identifies the denominator on the left-hand side as the area of a triangle in the complex plane with vertices at 0, r_1 and $r_2e^{i\theta}$. By Heron's formula, this is given also by $[(r^2-(r_1-r_2)^2)((r_1+r_2)^2-r^2)]^{\frac{1}{2}}$, which we may write as

$$\left[\prod_{\pm}(r \pm r_1 \pm r_2)\right]^{\frac{1}{2}},$$

where the product is over all choices of $\pm$ signs.

Thus, the convolution of two circles of radius r_1, r_2 is a rotationally invariant density given by

$$f_{r_1,r_2}(r) = \frac{2r}{\prod_{\pm}(r \pm r_1 \pm r_2)^{\frac{1}{2}}}\ \chi_{[|r_1-r_2|,r_1+r_2]}(r).$$

If one now carries out the same calculation on the surface of the sphere S^2 — there is a convenient version of Heron's formula for spherical geodesic triangles — one obtains the density function:

$$\begin{aligned} g_{r_1,r_2}(r) &= \frac{\sin r}{\pi}\,\frac{[(\cos(r_1-r_2)-\cos r)(\cos r-\cos(r_1+r_2))]^{\frac{1}{2}}}{\sin r_1\ \sin r_2} \\ &= \frac{1}{\pi}\,\frac{\sin r}{\sin r_1\ \sin r_2}\left[\prod_{\pm}\cos\frac{1}{2}(r \pm r_1 \pm r_2)\right]^{\frac{1}{2}} \end{aligned}$$

using the half-angle formula for cosine.

The e-function is now given by the ratio g/f

$$e = \frac{\sin r}{\sin r_1 \sin r_2}\ \prod_{\pm}\left[\frac{\frac{1}{2}\cos\frac{1}{2}(r \pm r_1 \pm r_2)}{\frac{1}{2}(r \pm r_1 \pm r_2)}\right]^{\frac{1}{2}}.$$

In essence, the proof for the symmetric space case is to reduce everything to two dimensions and to use these elementary ideas.

For the n-dimensional sphere, one obtains

$$e(X,Y) = \left[\prod_{\pm}\frac{2\cos(r \pm r_1 \pm r_2)/2}{(r \pm r_1 \pm r_2)}\right]^{\frac{n-3}{2}}\quad \chi_{(|r_1-r_2|,r_1+r_2)}(r).$$

These formulae will have interesting consequences for harmonic analysis — for example in finding fundamental solutions of K-invariant differential operators on X.

One can also prove a compact version of the Gindikin-Karpilevič formula for the Harish-Chandra c-function.

$$c(\lambda) = \prod_{\alpha \in \Phi_r^+} c_\alpha(\lambda_\alpha),$$

where the right-hand side is the c-function for each of the symmetric spaces G_α/K_α, $\alpha \in \Phi_r^+$. It seems most likely that the proof will also go through in the non-compact case and yield an elementary proof of this formula.

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