

From this relation it follows (see the proof below) that

$$(2) \quad \overline{\overline{EEV}} < \overline{\overline{EV}},$$

so that the sets V, EV, EEV, \dots will possess decreasing cardinal numbers. The existence of such a decreasing sequence of cardinals shows that these cardinals cannot be alephs, whence it follows that not all sets can be well-ordered. Therefore, the axiom of choice cannot be added to the other axioms of Quine's theory without contradictions. We may express this fact by saying that the principle of choice can be proved false in Quine's theory. This was pointed out by Specker.

Proof that (2) follows from (1): Because of (1) there exists a mapping of the set of all unit sets $\{m\}$ on a subset of V . Indeed the identical mapping is of that kind. However, the identical mapping maps the set of all $\{\{m\}\}$ on just this subset of all sets $\{m\}$. Let us on the other hand assume that EV could be mapped onto EEV . The mapping would then consist of mutually disjoint pairs $(\{m\}, \{\{n\}\})$. However, the certainly existing set of pairs $(m, \{n\})$ would then furnish a mapping of V on EV contrary to (1). Hence (2) follows from (1).

The theory of Quine's does not seem to have many adherents among mathematicians. The reason for this is presumably the existence of such sets in it as V which are elements of themselves, pathological sets as they are called. I don't think, however, that this circumstance ought to worry mathematicians, because it is not necessary to take these abnormal sets into account in the development of the ordinary mathematical theories.

14. The ramified theory of types. Predicative set theory

I have already mentioned Poincare's objection to Cantor's set theory, that one makes use of the so-called non-predicative definitions. These definitions collect objects in such a way that the totality of these objects, or objects logically dependent upon that totality, are considered as belonging to the same totality, so that the definition has a circular character. It might perhaps be better to say that a non-predicative definition is the definition of an entity by a logical expression containing a bound variable such that the defined entity is one of the possible values of this variable. However, instead of trying to explain this generally, I think it is better to take a characteristic example.

Let us consider mankind, the domain of all human beings. We have the binary relation "x is a child of y" which I write $Ch(x,y)$. Let us try to define descendant of P , P any given person. If we make use of the notion of finite number we may proceed thus: We define the relation $Ch^n(x,y)$ recursively by letting

$$Ch^1(x,y) \text{ stand for } Ch(x,y)$$

$$Ch^{n+1}(x,y) \text{ stand for } (\exists z)(Ch^n(x,z) \ \& \ Ch(z,y)).$$

Then the proposition "x is a descendant of P" may be written

$$(En)(Ch^n(x,P)).$$

All this is quite clear and simple, but notice that we have to use quantifiers that are logically very different in nature, namely, on the one hand, quantifiers with mankind as range of variation, and, on the other hand, a quantifier extended over natural numbers. What appears most unsatisfactory, however, is the circumstance that the notion natural number itself is of the same kind as the notion descendant of P. Indeed we can say that the numbers are the descendants of 0 by the successor relation +1; therefore the above definition only refers one descendant relation to another. We may therefore ask if we can give a definition of a purely logical character that is independent of the notion of natural number. Following Frege and Dedekind we may do that by letting "z is a descendant of P" stand for

$$(X)\left((X(P) \& (x)(y) (X(y) \& Ch(x,y) \rightarrow X(x))) \rightarrow X(x)\right),$$

where X runs through all classes of human beings. In ordinary language the wording of this is: That z is a descendant of P means that z belongs to every class X with the two properties, 1) P belongs to X, 2) whenever y belongs to X and x is a child of y, then x belongs to X. This is a typical example of a non-predicative definition because the defined class "descendant of P" is itself one of the values which the variable X is assumed to run through. Of course this definition is quite in order in the axiomatic set theory of Zermelo, also in Quine's theory, and in the simple theory of types as well. But in the case of such theories we have the question of consistency. The older and more natural point of view was that we should be able to set up a kind of reasoning which could be considered reliable so that we were assured a priori that contradiction would never arise. If we should try to set up such a logic, then the ramified theory of types, a theory where non-predicative definitions are excluded, might be assumed to be the correct one. It could be reasonable to assume that this theory is really a perfectly reliable one. Then, if we could believe this, a proof of consistency of this theory would be something out of the way, namely unnecessary and without point, because the reasoning yielding this proof could not be considered more reliable than the theory itself.

In the ramified theory of types we have, just as in the simple theory, a distinction of type such that $a \in b$ only has a meaning when the type of b is a unit more than that of a. However we have also a distinction of order between objects of the same type. Thus if a class of objects of type zero is defined in such a way that only quantifiers extended over objects of type zero are used, then this class is of first order. If a class, still of objects of type 0, is defined so that beside eventual quantifiers extended over objects of type 0, there are also quantifiers extended over the just mentioned classes of order 1, then this class is said to be of order 2, and so on. A similar distinction of order must take place for the objects of type 1, 2, But there are even further distinctions, because a class of objects of type 0, say, can also be defined by a logical expression containing quantifiers extended over objects of type 2 or even higher types. I shall, however, not try to go into further detail in this rather complicated affair, but rather give some examples of the kind of reasoning that is possible when we proceed in a predicative manner.

As a first example we may look at the proof of the Bernstein theorem of equivalence. We had sets M, M', M_1 such that

$$M \sim M', \quad M' \subset M_1 \subset M$$

and we proved the existence of a 1 - 1 - correspondence between M and M_1 . In the proof of this which I gave earlier I used, however, at one point a non-predicative definition, namely, reckoning DT as a subset of M in the same meaning as the diverse elements of T . If we assume that the correspondence between M and M' is of 1st order, M, M' and M_1 sets of 1st order and we let T be the set—which of course is of type one unit higher than the type of M, M', M_1 —of all subsets of 1st order A such that for $Q = M_1 - M'$

$$Q \subseteq A, \quad A' \subseteq A,$$

then DT is a subset of 2nd order and the earlier conclusion that $A_0 = DT$ is ϵT is no longer valid. Nevertheless we may prove the identity

$$A_0 = Q + A'_0$$

which we obtained in the earlier proof, but it must now be shown in a different way. Let us here write D instead of A_0 . Then I shall first show that we have

$$D = Q + D'.$$

Let us assume that a d existed such that

$$d \in D, \quad \text{but } d \bar{\epsilon} Q \text{ \& } d \bar{\epsilon} D'.$$

The assumption $d \bar{\epsilon} D'$ means that an $X \in T$ exists such that $d \bar{\epsilon} X'$, because D' is just the intersection of all X' , where $X \in T$. On the other hand we have $d \in X$ and $d \bar{\epsilon} Q$. Now the set

$$Y = X - \{d\}$$

is of order 1 just as X and still Q is $\subseteq Y$. Let y be ϵY . Then $y \in X$, whence $y' \in X$ because $X' \subseteq X$. Hence $y' \in Y$, because y' cannot be $= d$, since $d \bar{\epsilon} Y'$ and $y' \in Y'$. Thus we have proved that

$$Q \subseteq Y \quad \text{and} \quad Y' \subseteq Y$$

so that $Y \in T$. Now we had $d \bar{\epsilon} Y$, whence $d \bar{\epsilon} D$ which is a contradiction. Therefore I have shown that if $d \in D$, then $d \in Q \cdot v \cdot d \in D'$, that is

$$(1) \quad D \subseteq Q \cup D'.$$

Since $Q \subseteq A$ for every $A \in T$ we have $Q \subseteq D$, and since $A' \subseteq A$ for every $A \in T$ we get $D' \subseteq D$. Thus

$$(2) \quad Q \cup D' \subseteq D$$

(1) and (2) then yield as before

$$D = Q + D'$$

and the remaining part of the proof can be carried out just as before.

There are however also theorems in the usual set theory which are no longer provable in predicative set theory. As an example I shall mention Cantor's theorem that UM always possesses higher cardinality than M . We must replace M by EM of course, so that we would have to try to prove the

nonexistence of a 1-1-correspondence between UM and EM. Our earlier proof was essentially due to the possibility of deriving a contradiction by considering the set N of all $m \in M$ such that, if F was the assumed correspondence, $m \in X$ where X was the subset of M corresponding to $\{m\}$ by F, that is, $(X, \{m\}) \in F$. Translating the last phrase into logical symbols we have

$$m \in N \leftrightarrow (X) \{X \in UM \rightarrow ((X, \{m\}) \in F \rightarrow m \in X)\}.$$

Since this expression contains the quantifier X extended over all sets X of order 1 say, the defined set N of elements m is of order 2. But then we cannot substitute N instead of X and the derived contradiction disappears. Then Cantor's theorem is not longer provable as before. One might perhaps think that it could be proved in a quite different way, but that does not seem to be the case. In my opinion one has little reason to be worried because of the necessity to drop this theorem. Indeed the distinction of order compensates for the fact that we don't have the usual distinction of cardinality.

As a further example of predicative reasoning I shall develop elementary arithmetic basing it as before on a definition of the simple infinite sequence, now, however, taking into account the order distinction. I prefer now to talk about classes, relations, etc., instead of sets. Also I think the considerations will be easier, if I use suffixes to denote the different orders. To begin with I assume that we have a class M and a binary relation $f_1(x,y)$ both of order 1. The relation f_1 is supposed to be a 1-1-correspondence. The identity relation $x = y$ is assumed to be a relation of order 1; but for simplicity I assume the axiom

$$(x = y) \rightarrow (\phi(x) \leftrightarrow \phi(y))$$

valid for ϕ of arbitrary orders. Then we assume

$$f_1(x, y) \ \& \ f_1(z, y) \rightarrow (x = z)$$

$$f_1(x, y) \ \& \ f_1(x, u) \rightarrow (y = u)$$

For simplicity I denote y, whenever $f_1(x,y)$ takes place, by x' . The class of 1-st order consisting of all x' , x running through X_1 , I denote by X_1' . Then I assume that $M' \subset M$ and O may denote an element of M not in M' . I denote by N_2 the class defined thus:

$$n \in N_2 \leftrightarrow (X_1)(O \in X_1 \ \& \ (x) (x \in X_1 \rightarrow x' \in X_1) \rightarrow (n \in X_1))$$

or, as I now prefer to write it,

$$N_2(n) \leftrightarrow (X_1) (X_1(O) \ \& \ (x) (X_1(x) \rightarrow X_1(x')) \rightarrow X_1(n)).$$

The class of type 2 whose elements are all X_1 for which $X_1(O) \ \& \ (X_1(x) \rightarrow X_1(x'))$ may be denoted by T. Similarly N_3 is defined thus:

$$N_3(n) \leftrightarrow (X_2) (X_2(O) \ \& \ (x) (X_2(x) \rightarrow X_2(x')) \rightarrow X_2(n)),$$

etc. Corresponding to these definitions we have the following principles of induction. If a class X_r of order r contains O and besides x always contains x' , then X_r contains the whole class N_{r+1} . We may regard N_2, N_3, \dots as successively sharpened determinations of the natural number series.

Now I shall show how we can define a ternary relation of second order, $S_2(x,y,z)$, such that, conceiving $S_2(x,y,z)$ as $x + y = z$, we obtain the ordinary theorems of addition.

Let us consider the ternary relations of first order $X_1(x,y,z)$ with the two properties

$$1) (x) X_1(x, 0, x), \quad 2) (x)(y)(z) (X_1(x, y, z) \rightarrow X_1(x, y', z')).$$

They constitute a class Tr of type 2.

These have an intersection $S_2(x,y,z)$ and trivially we have

$$(x) S_2(x, 0, x) \text{ and } (x)(y)(z) (S_2(x, y, z) \rightarrow S_2(x, y', z')).$$

I shall prove such statements as

$$(x)(N_2(x) \rightarrow (x = 0 \cdot v \cdot N_2'(x))) \text{ or in other words}$$

$$(x)(N_2(x) \rightarrow (x = 0 \cdot v \cdot (Ey)(N_2(y) \& (x = y')))).$$

Further

$$(x)(y)\overline{S}_2(x, y', 0) \quad (x)(z)S_2(x, 0, z) \rightarrow (x = z) \text{ and } (x)(y)(S_2(x, y, 0) \rightarrow x = 0 \& y = 0)$$

Proof of

$$(x)(N_2(x) \rightarrow x = 0 \cdot v \cdot (Ey)(N_2(y) \& (x = y'))).$$

Let us assume the existence of an individual a such that $N_2(a) \& (a \neq 0) \& \overline{N}_2'(a)$. Because of $N_2(a)$ we have for every $X_1 \in \text{Tr}$ that $X_1(a)$. Now let X_1^* be $X_1 - \{a\}$. Then I shall show that for at least one X_1 , X_1^* would still have the properties 1) and 2) so that $X_1^* \in \text{Tr}$, whence $\overline{N}_2'(a)$, a contradiction. Indeed we have $X_1^*(0)$ since $X_1(0)$ and $a \neq 0$. Further, if $X_1^*(a')$, then $X_1(a')$, whence $X_1(a')$, X_1 being $\in \text{Tr}$, whence again $X_1^*(a')$ unless $a = a'$. Now there must be at least one $X_1 \in \text{Tr}$ for which this is not the case, because otherwise we should have $N_2'(a)$ contrary to the assumption concerning a . Since there is an $X_1^* \in \text{Tr}$ such that $\overline{X}_1^*(a)$, we should have $\overline{N}_2'(a)$, which is a contradiction.

Proof of $S_2(a, b', 0)$ for arbitrary a and b . Let us assume $S_2(a, b', 0)$. Then we have $X_1(a, b', 0)$ for every $X_1 \in \text{Tr}$. Let X_1^* be $X_1 - \{(a, b', 0)\}$. Then X_1^* still has the property 1), because $(x, 0, x)$ can never be $= (a, b', 0)$, 0 being \neq every y' . However, X_1^* also possesses the property 2). Indeed if $X_1^*(\alpha, \beta, \gamma)$, then $X_1(\alpha, \beta, \gamma)$, whence $X_1(\alpha, \beta', \gamma')$, whence $X_1^*(\alpha, \beta', \gamma')$, unless $(\alpha, \beta', \gamma')$ were $= (a, b', 0)$ which is impossible because $\gamma' \neq 0$. But $X_1^* \in \text{Tr}$ and $\overline{X}_1^*(a, b', 0)$ yields $S_2(a, b', 0)$.

Proof of $S_2(a, 0, c) \rightarrow (a = c)$. Let us assume $S_2(a, 0, c) \& (a \neq c)$. Then for every $X_1 \in \text{Tr}$ we have $X_1(a, 0, c)$. Let X_1^* be $X_1 - \{(a, 0, c)\}$. Then it is seen again that X_1^* will still possess the two properties, so that $X_1^* \in \text{Tr}$. Since $\overline{X}_1^*(a, 0, c)$, it follows that $\overline{S}_2(a, 0, c)$ which is contrary to supposition.

Then the truth of $S_2(a, b, 0) \rightarrow a = 0 \& b = 0$ follows from the last three statements.

Proof of $S_2(a, b', c') \rightarrow S_2(a, b, c)$. Let us assume for some a, b, c that $S_2(a, b', c') \& \overline{S}_2(a, b, c)$. Then for an arbitrary element X_1 of Tr we have $X_1(a, b', c')$, whereas for a certain X_1 we have $\overline{X}_1(a, b, c)$. Let X_1^* be $X_1 - \{(a, b', c')\}$ for such an X_1 . Then it is seen immediately that X_1^* has the property 1). It has the property 2) as well. Indeed, let $X_1^*(\alpha, \beta, \gamma)$ be true. Then $X_1(\alpha, \beta, \gamma)$ is true, whence $X_1(\alpha, \beta', \gamma')$, whence $X_1^*(\alpha, \beta', \gamma')$, unless $(\alpha, \beta', \gamma') = (a, b', c')$ which however would mean $(\alpha, \beta, \gamma) = (a, b, c)$ but that is impossible because we have $\overline{X}_1(a, b, c)$ but $X_1(\alpha, \beta, \gamma)$. Hence $X_1^* \in \text{Tr}$ so that $\overline{X}_1^*(a, b', c')$ leads to $S_2(a, b', c')$ contrary to supposition.

Proof of $S_2(0,b,c) \rightarrow (b = c)$. Let X_1 be ϵ Tr and X_1^* be what remains of X_1 when all triples $(0,y,z)$ with $y \neq z$ are removed from X_1 . Obviously X_1^* is of order 1 just as X_1 is. I assert that also $X_1^* \in$ Tr. Indeed for every triple $(\alpha,0,\alpha)$ we have $X_1(\alpha,0,\alpha)$ whence also $X_1^*(\alpha,0,\alpha)$. Otherwise $(\alpha,0,\alpha)$ would be of the form $(0,y,z)$ with $y \neq z$, but that is not the case. Thus X_1^* has the property 1). Let us assume $X_1^*(\alpha,\beta,\gamma)$. Then $X_1(\alpha,\beta,\gamma)$, whence $X_1(\alpha,\beta',\gamma')$, whence also $X_1^*(\alpha,\beta',\gamma')$ unless (α,β',γ') is of the form $(0,y,z)$ with $y \neq z$, that is, $\alpha = 0$, $\beta' \neq \gamma'$. But then we should have $\bar{X}_1^*(\alpha,\beta,\gamma)$. Thus $X_1^* \in$ Tr and since $S_2(0,b,c) \rightarrow X_1^*(0,b,c)$ we have $b = c$.

Theorem 58. $(x)(y)(z)(S(x',y,z') \rightarrow S(x,y,z))$.

Proof. For each $X_1 \in$ Tr we let X_1^* be what remains of X_1 when all triples (x',y,z') are removed for which we have $X_1(x',y,z')$ but not $X_1(x,y,z)$, that is, $X_1^*(x',y,z') \leftrightarrow X_1(x',y,z') \& X_1(x,y,z)$. Further all triples $(x,y,0)$ are removed for which x or y is $\neq 0$. Then X_1^* has the property 1). Indeed for all $(\alpha,0,\alpha)$ we have $X_1(\alpha,0,\alpha)$, whence $X_1^*(\alpha,0,\alpha)$, because if $\alpha = \alpha_1'$, we have also $X_1(\alpha_1,0,\alpha_1)$. Now let us assume $X_1^*(\alpha,\beta,\gamma)$. Then $X_1(\alpha,\beta,\gamma)$ whence $X_1(\alpha,\beta',\gamma')$ whence $X_1^*(\alpha,\beta',\gamma')$, unless $(\alpha,\beta',\gamma') =$ a certain (x',y,z') for which $X_1(x',y,z')$ & $\bar{X}_1(x,y,z)$. That would mean $X_1(\alpha,\beta',\gamma') \& \bar{X}_1(\alpha_1,\beta',\gamma')$ with $\alpha = \alpha_1'$. Let us first consider the case $\gamma \neq 0$, that is, $\gamma = \gamma_1'$ for a certain γ_1 . Then because of $X_1^*(\alpha,\beta,\gamma)$ we have $X_1(\alpha,\beta,\gamma) \& X_1(\alpha_1,\beta,\gamma_1)$. But $X_1(\alpha_1,\beta,\gamma_1)$ yields $X_1(\alpha,\beta',\gamma)$ so that we get a contradiction. It remains for us to look at $X^*(\alpha,\beta,0)$. This requires $\alpha = \beta = 0$. But $X_1(0,0',0')$ is true and therefore also $X_1^*(0,0',0')$ because $(0,0',0')$ is not removed from X_1 by the construction of X_1^* . Thus X_1^* has the property 2) as well, so that $X_1^* \in$ Tr. Now let a,b,c be arbitrary. I assert that

$$S_2(a',b,c') \rightarrow S_2(a,b,c).$$

Let us assume $S_2(a',b,c') \& \bar{S}_2(a,b,c)$. Then there exists an $X_1 \in$ Tr such that $X_1(a',b,c') \& \bar{X}_1(a,b,c)$. We build the corresponding X_1^* as above. Then we have

$$X_1^* \in \text{Tr} \quad \text{and} \quad \bar{X}_1^*(a',b,c'),$$

whence

$$\bar{S}_2(a',b,c')$$

which is a contradiction.

Corollary. $(x)(y)(z)(S_2(x',y,z') \rightarrow S_2(x,y',z'))$.

Proof. $S_2(a',b,c') \rightarrow S_2(a,b,c) \rightarrow S_2(a,b',c')$.

I will only mention that such a statement as $(y)(N_2(y) \rightarrow (x)(Ez)X_1(x,y,z))$ is easily proved. I shall not make any use of that, but instead prove the following theorems.

Theorem 59. $(y)(N_3(y) \rightarrow (x)(z)(u)(S_2(x,y,z) \& S_2(x,y,u) \rightarrow (z = u)))$.

Proof. Let C_2 be the class of all y such that $(x)(z)(u)(S_2(x,y,z) \& S_2(x,y,u) \rightarrow (z = u))$. Clearly $C_2(0)$ is true, because $S_2(a,0,c)$ is only true for $a = c$. Now let $C_2(b)$ be true. If, then, for certain a,c,d we have $S_2(a,b',c) \& S_2(a,b',d)$, then according to a remark above, c must be $= c_1'$ for a certain c_1 and $d = d_1'$ likewise, whence $S_2(a,b,c) \& S_2(a,b,d)$, whence, because of

$C_2(b)$, $c_1 = d_1$, whence $c = d$. Thus $C_2(0) \ \& \ (y)(C_2(y) \rightarrow C_2(y'))$ is true, whence the theorem, because of the definition of N_3 .

Theorem 60. $(y)(N_3(y) \rightarrow (x)(\dot{E}z)S_2(x,y,z))$.

Proof. Let C_2 here be the class of all y such that $(x)(\dot{E}z)S_2(x,y,z)$. Obviously $C_2(0)$ is true. Let us assume $C_2(b)$ and let a be arbitrary. Then we have $S_2(a,b,c)$ for a certain c , whence $S_2(a,b',c')$ whence $C_2(b')$. Hence the theorem.

The last two theorems may be combined in the single statement

$$(y) (N_3(y) \rightarrow (x)(\dot{E}z) S_2(x,y,z)),$$

where E means "there exists one and only one". Of course this yields in particular

$$(x)(y)(N_3(x) \ \& \ N_3(y) \rightarrow (\dot{E}z) S_2(x,y,z) ,$$

but the question arises, whether the z here again is an element of N_3 . I shall now show that this is really the case.

Let C_2 denote an arbitrary class of 2. order with the two properties 1) $C_2(0)$ and 2) $(x) (C_2(x) \rightarrow C_2(x'))$.

Then for every such class C_2 I construct another class C_2^* thus:

$$C_2^*(y) \leftrightarrow (x) (C_2(x) \rightarrow (\dot{E}z) (S_2(x,y,z) \ \& \ C_2(z))).$$

Now I assert that C_2^* has again the properties 1) and 2). The truth of $C_2^*(0)$ is immediately seen, because we have $S_2(x,0,x)$ and $C_2(x) \rightarrow C_2(x)$. Let us assume $C_2^*(b)$. Then for an arbitrary a we have a unique c such that $S_2(a,b,c)$ and $C_2(c)$. Hence $S(a,b',c') \ \& \ C(c')$, and according to a theorem above we cannot have $S_2(a,b',d)$ unless $d = c'$. Thus $C_2^*(b')$ follows from $C_2^*(b)$.

Theorem 61. $(x)(y)(N_3(x) \ \& \ N_3(y) \rightarrow (\dot{E}z) S_2(x,y,z) \ \& \ N_3(z))$.

Proof. According to the definition of C_2^* we have for arbitrary C_2 of the supposed kind

$$(x)(y)(C_2(x) \ \& \ C_2^*(y) \rightarrow (\dot{E}z)(S_2(x,y,z) \ \& \ C_2(z))).$$

Now N_3 is $\subseteq C_2$ and C_2^* . Therefore

$$(x)(y) (N_3(x) \ \& \ N_3(y) \rightarrow (\dot{E}z) (S_2(x,y,z) \ \& \ C_2(z))).$$

Here C_2 is an arbitrary chain of 2. order, that is, a class of 2. order with the properties 1) and 2). Therefore we may just as well write

$$(x)(y) (N_3(x) \ \& \ N_3(y) \rightarrow (\dot{E}z) (S_2(x,y,z) \ \& \ (X_2)(X_2(0) \ \& \ (u)(X_2(u) \rightarrow X_2(u')) \rightarrow X_2(z))))),$$

which, by taking into account the definition of N_3 , is just our theorem. In this way we have succeeded in obtaining a ternary relation $S_2(x,y,z)$ which in N_3 will play the role of addition, as I shall show.

Theorem 62. $(z) (N_3(z) \rightarrow (x)(y)(u)(v)(w)(S_2(x,y,v) \ \& \ S_2(v,z,u) \ \& \ S_2(y,z,w) \rightarrow S_2(x,w,u)))$

Proof. Let $C_2(b)$ denote

$$(x)(y)(u)(v)(w)(S_2(x,y,v) \ \& \ S_2(v,b,u) \ \& \ S_2(y,b,w) \rightarrow S_2(x,w,u)).$$

Clearly C_2 is a class of second order. We have that $C_2(0)$ is true, because $S_2(v,0,u) \& S_2(y,0,w) \rightarrow (u = v) \& (y = w)$. Let $C_2(b)$ be true and let us assume $S_2(x,y,v) \& S_2(v,b',u) \& S_2(y,b',w)$. Then we have $u = u_1'$, $w = w_1'$ for some u_1 , w_1 and $S_2(v,b,u_1) \& S_2(y,b,w_1)$ which, together with $S_2(x,y,v)$, because of $C_2(b)$, yields $S_2(x,w_1,u_1)$, whence $S(x,w,u)$. Thus the implication $C_2(b) \rightarrow C_2(b')$ is generally valid. Then the theorem follows from the definition of N_3 . A fortiori we have

$$(x)(y)(z)(u)(v)(w) (N_3(x) \& N_3(y) \& N_3(z) \& N_3(u) \& N_3(v) \& N_3(w) \rightarrow (S_2(x,y,v) \& S_2(v,z,u) \& S_2(y,z,w) \rightarrow S_2(x,w,u))).$$

This is the associative law of addition.

Theorem 63. $(x)(N_3(x) \rightarrow (y)(z)(S_2(x,y,z) \rightarrow S_2(y,x,z)))$.

Proof. Let $C_2(a)$ be an abbreviation for

$$(y)(z)(S_2(a,y,z) \rightarrow S_2(y,a,z)).$$

Then $C_2(0)$ is true because, according to a result above, $S(0,y,z) \rightarrow (y = z)$ and $S_2(y,0,z) \leftrightarrow (y = z)$. Let us assume the truth of $C_2(a)$ and let $S_2(a',b,c)$ be true. Then by some results above we have $c = c_1'$ for a certain c_1 and $S_2(a',b,c') \rightarrow S_2(a,b',c')$ so that because of $C_2(a)$, we also get $S_2(b',a,c)$, whence $S_2(b,a',c)$. Therefore we have

$$(y)(z)(S_2(a',y,z) \rightarrow S_2(y,a',z)),$$

so that

$$C_2(a) \rightarrow C_2(a').$$

According to the definition of N_3 , the theorem must be valid.

A fortiori we have

$$(x)(y)(z) (N_3(x) \& N_3(y) \& N_3(z) \rightarrow (S_2(x,y,z) \rightarrow S_2(y,x,z))).$$

This is the commutative law of addition.

Thus the ternary relation $S_2(x,y,z) \& N_3(x) \& N_3(y) \& N_3(z)$ which we can write $\Sigma_3(x,y,z)$ or $z = x + y$ is a relation of 3. order which has the ordinary properties of addition, in particular,

$$x + (y + z) = (x + y) + z, \quad x + y = y + x.$$

Now let us define a relation "less than or equal to" of second order, namely,

$$M_2(x, y) \leftrightarrow (Ez) S_2(x,z,y).$$

Then inside N_3

Theorem 64. $M_2(a,b) \& M_2(b,c) \rightarrow M_2(a,c)$.

Proof. The hypothesis of the implication amounts to

$$S_2(a,d,b) \& S_2(b,e,c)$$

for some d and e . According to Theorem 59 there is an f such that $S_2(d,e,f)$.

Then theorem 62 furnishes $S_2(a,f,c)$, whence $M_2(a,c)$.

Theorem 65. $(y)(N_3(y) \rightarrow (x)(M_2(x,y) \vee M_2(y,x)))$.

Proof. Let $C_2(b)$ be $(x)(M_2(x,b) \vee M_2(b,x))$. Then $C_2(0)$ is true, because $M_2(0,x)$ is obviously true. Let us assume $C_2(b)$. If $M_2(x,b')$ is true, we have at once $C_2(b')$, and $M_2(x,b')$ is true if $M_2(x,b)$ is. Otherwise we have $M_2(b,x)$ that is $(\exists z) S_2(b,z,x)$. If $z \neq 0$, we have $z = z_1'$ and $S_2(b,z,x) \rightarrow S_2(b',z_1',x)$, that is, $M_2(b',x)$. If $z = 0$, we have $x = b$, whence $M_2(x,b')$. Thus C_2 is a chain of 2. order, and hence $(y)(N_3(y) \rightarrow C_2(y))$, which is the theorem.

It follows that M_2 will have the ordinary properties of the relation \cong in N_3 .

Now in order to develop elementary arithmetic we must introduce multiplication. This can again be done by considering some ternary relations. It must be remarked, however, that these relations ought to be chosen as 1. order relations $Y_1(x,y,z)$. Otherwise we might have to make a transition to unnecessarily high orders of the number series. It would not be advantageous to take, for example, the relations $Z_2(x,y,z)$ which have the properties 1) $(x)Z_2(x,0,0)$ and 2) $(x)(y)(z)(Z_2(x,y,z) \& S_2(z,x,u) \rightarrow Z_2(x,y',u))$. It is better to introduce addition and multiplication simultaneously as follows. Let us consider all quaternary relations $U_1(x,y,z,u)$ such that U_1 is true only for $u = 0$ or 1 and has the properties

- 1) $(x)U_1(x,0,x,0)$, 2) $(x)U_1(x,0,0,1)$, 3) $(x)(y)(z)(U_1(x,y,z,0) \rightarrow U(x,y',z',0))$,
- 4) $(x)(y)(z)(U_1(x,y,z,1) \& U_1(z,x,u,0) \rightarrow U_1(x,y',u,1))$.

Then if $S_2(x,y,z)$ denotes the intersection of all $U_1(x,y,z,0)$ and $P_2(x,y,z)$ the intersection of all $U_1(x,y,z,1)$, one is able to show that in a suitable N_n all of the ordinary principles of addition and multiplication are provable, $x + y = z$ meaning $S_2(x,y,z)$ and $xy = z$ meaning $P_2(x,y,z)$. However, I will not carry out all that here in detail, in particular for the reason that different procedures are possible.

One fact ought to be noticed: The relation $S_2(x,y,z)$, which in N_3 defined addition, does that also in N_n for any $n > 3$, that is, every N_n is closed with regard to this addition. Let us, for example, consider N_4 . If $N_4(a)$ and $N_4(b)$, then $N_3(a)$ and $N_3(b)$ so that a unique c exists such that $S_2(a,b,c) \& N_3(c)$. But how can we conclude $N_4(c)$? This can be seen thus: Let $S_3(x,y,z)$ be the intersection of all $X_2(x,y,z)$ with the properties 1) and 2). Then we can prove in the same way as above that

$$(x)(y)(N_4(x) \& N_4(y) \rightarrow (\exists z) S_3(x,y,z) \& N_4(z)).$$

Furthermore let us write the z for which $S_3(x,y,z) \& N_4(z)$ as $x + 'y$. Now it is obvious that $S_3(x,y,z) \rightarrow S_2(x,y,z)$. Hence, for arbitrary a and b such that $N_4(a)$ and $N_4(b)$, we get that

$$c = a + 'b \rightarrow c = a + b,$$

so that the result of the operation $+'$ is the same as the result of $+$. In the same way the other operations we may introduce, such as multiplication, exponentiation, etc., all will retain their meaning for the natural number sequences of higher orders.

I must confine my remarks to these hints, which I nevertheless hope are sufficient to show that a purely logical development of arithmetic similar to that given by Dedekind in his work "Was sind und was sollen die Zahlen" is possible even in the ramified type theory.

If we turn to analysis it must be remarked that the classical form of it cannot be obtained. Indeed it will be necessary to distinguish between real numbers of different orders. A class of real numbers of 1. order which is bounded above possesses an upper bound, but this bound may then be a real number of order 2. Nevertheless a great part of analysis can be developed as usual, namely, the most useful part of it dealing with continuous functions, closed point-sets, etc. The reason for this is that it is often possible to prove theorems of reducibility, namely, theorems saying that a class (or relation) of a certain order coincides with one of lower order. I will not enter into this but only refer the reader to the book: "Das Kontinuum" by H. Weyl, where he has developed such a kind of predicative analysis.

15. Lorenzen's operative mathematics

In more recent years the German mathematician P. Lorenzen has set forth a system of mathematics which in some respects resembles the ramified theory of types, but it has also one important feature in common with the simple theory of types, namely, that the simple infinite sequence and similar notions are characterized by an induction principle which is assumed valid within all layers of objects. Lorenzen talks namely about layers of objects, not of types or orders. To begin with he takes into account some original objects, say numerals, figures built up in a so-called calculus as follows. We have the rules of production

$$1$$

$$k \rightarrow k1$$

which means that the object or symbol 1 is originally given and whenever we have a symbol or a string of symbols k we may build the string k 1 obtained by placing 1 after k. He introduces the notion "system". A system is a finite set of symbols. The systems are obtained by the rules

$$x$$

$$X \rightarrow X, x$$

The length or cardinal number of a system X is denoted by |X|. He gives the rules

$$|x| = 1$$

$$|X,x| = |X| + 1$$

for these lengths. Now the explanation of the successive layers of language is as follows.

From certain originally given symbols called atoms, say $u_1 \dots u_n$, he constructs strings of symbols by the schema

$$x \rightarrow xu_1$$

$$\dots$$

$$x \rightarrow xu_n$$