

and their proofs from the Zermelo-Fraenkel theory to the simple theory of types. Bernstein's equivalence theorem with its proof remains unchanged. Cantor's theorem that  $\overline{\text{UM}} < \overline{\text{M}}$  is always of higher cardinality than  $\text{M}$  must be expressed thus: Let  $\text{EM}$  be the set of all unit sets  $\{m\}$  contained in  $\text{M}$ .

Then  $\overline{\text{EM}} < \overline{\text{UM}}$ . The previous definition of well-ordering (see § 4) must be slightly changed to this wording: A set  $\text{M}$  is well-ordered, if there is a function  $R$  from  $\text{EM}$  to  $\text{UM}$  such that, for  $0 \subset N \subseteq M$ , there is a unique  $n \in N$  such that  $N \subseteq R(\{n\})$ . The wording of Theorem 10 must now be: Let a function  $\phi$  be given such that  $\phi(A)$ , for every  $A$  such that  $0 \subset A \subseteq M$ , denotes a unit subset of  $A$ . Then there is a subset  $\mathfrak{M}$  of  $\text{UM}$  such that to every  $N \subseteq M$  there is one and only one element  $N_0$  of  $\mathfrak{M}$  such that  $N \subseteq N_0$  and  $\phi(N_0) \subseteq N$ . Such slight changes will be necessary in many of the previous theorems and proofs. If we look at Theorem 6 for example, there can be no meaning in an equivalence between  $M + N$  and  $M \cdot N$  or even  $M \times N$ , because the elements of  $M \cdot N$  are of type  $t + 1$  and those of  $M \times N$  are of type  $t + 2$  when those of  $M$  and  $N$  are of type  $t$ . If, however, we replace  $\text{M}$  by its sets of unit subsets  $\text{EM}$  and  $\text{N}$  by  $\text{EN}$ , then  $\text{EM} + \text{EN}$  and  $\text{M} \cdot \text{N}$  will be of same type, and an equivalence between these two sets will be meaningful. Similarly we can compare  $\text{EEM} + \text{EEN}$  and  $\text{M} \times \text{M}$ . I don't think it is necessary to carry out in detail these small changes in the considerations. By the way, it may be remarked that functions may well be introduced such that arguments and values are not of same type, but if functions should be conceived as special cases of relations, and relations as sets of sequences conceived as sets, such a procedure must be avoided.

### 13. The theory of Quine

There have been many attempts to avoid the introduction of types, which are inconvenient. One of these is the theory of Quine. An exposition of this can be found in the book "Logic for Mathematicians" recently published by B. Rosser. Quine's theory is something intermediate between the axiomatic theory of Zermelo-Fraenkel and Russell's type theory. It has in common with the former the feature that there are no type distinctions. On the other hand it has in common with the latter the feature that only stratified propositional functions are admitted for the definition of new sets. Indeed we have in Quine's theory the following axiom of comprehension:

$$(Ey)(x)(x \epsilon y \leftrightarrow \phi(x))$$

with the whole domain of objects as range of variation of  $x$  and  $y$ . Of course  $y$  must not occur in  $\phi(x)$ .

It is easy to see that here we again get only one null set  $A$  and only one universal set  $V$ . We may for example use these definitions:

$$x \epsilon A \leftrightarrow (y)(x \epsilon y \& x \not\in y), \quad x \epsilon V \leftrightarrow (Ey)(x \epsilon y \cdot v \cdot x \not\in y).$$

Obviously the set  $V$  is  $\epsilon V$ . Nevertheless Russell's antinomy cannot be deduced, because the propositional function  $x \epsilon x$  is not stratified, so that no

set  $M$  can be introduced such that  $x \in M$  should be  $\leftrightarrow x \in \bar{x}$ . The ordinary constructions of new sets are, however, valid. If  $A(x)$  and  $B(x)$  are stratified, say without free variables other than  $x$ , also  $A(x) \& B(x)$  and  $A(x) \vee B(x)$  are stratified, making the definition of an intersection and the union of two sets possible. Further, if  $A(x)$  is stratified, and  $x$  does not occur in  $A(y)$ , then  $(\exists y)(x \in y \& A(y))$  is stratified as well. This shows the existence of the union of all elements of a given set. Further  $(x)(x \in y \cdot v \cdot A(x))$  is stratified so that we can always build the set of all subsets of a given set. Since  $\bar{A}(x)$  is also stratified, there always exists a complementary set to any given set. There is therefore a greater possibility for the introduction of new sets in this theory than in Zermelo's. In spite of this, however, it turns out that the existence of infinite sets is not any more provable in Quine's theory than in Zermelo's, so that an axiom of infinity is just as well needed here. This is due to the fact that the propositional functions needed for the definition of an infinite set are not stratified. In Rosser's book the axiom of infinity is set up thus:

$$(m)(n)(m \in N_n \& n \in N_m \& m + 1 = n + 1 \rightarrow m = n).$$

Here  $N_n$  means the set of natural numbers, where the natural numbers are defined as the cardinals of finite sets. The axiom has the effect that none of these cardinals coincides with the set  $\Lambda$ , or in other words, there exist finite cardinal numbers as large as we please. The sequence of natural numbers is then infinite.

It is interesting to look at Cantor's theorem. In type theory we could not compare  $U_m$  with  $m$ . Here we can do that, but Cantor's theorem is not generally valid. That it cannot be generally valid is clear, because at any rate it cannot be true for  $V$ . However, if we modify the theorem a little, saying that  $U_M$  is of higher cardinality than  $E_M$  (this was also the formulation we could use in type theory) then we get a correct statement. This circumstance shows again that  $M$  and  $E_M$  cannot always be equivalent. This appears very peculiar, but if we try to prove the equivalence between  $M$  and  $E_M$  in general, this turns out to be impossible, because we would have to use propositional functions which are not stratified. Nevertheless, in many particular cases the use of non-stratified formulas can be avoided. We therefore have to distinguish between sets  $M$  for which we can prove the equivalence between  $M$  and  $E_M$  and those for which this is not provable. The former kind of sets are said to be Cantorian and Can  $M$  is written for the statement  $M \sim E_M$ . Rosser mentions in his book that the statement Can  $M$  is provable not only for the natural number series,  $M = N_n$ , but for all the sets which occur in ordinary mathematics.

Since  $UV \subseteq V$ , we have

$$\overline{\overline{U}V} \leq \overline{\overline{V}}.$$

On the other hand

$$\overline{\overline{U}V} > \overline{\overline{E}V}.$$

so that

$$(1) \quad \overline{\overline{E}V} < \overline{\overline{V}}$$

From this relation it follows (see the proof below) that

$$(2) \quad \overline{\overline{EV}} < \overline{\overline{EV}},$$

so that the sets  $V, EV, EEV, \dots$  will possess decreasing cardinal numbers. The existence of such a decreasing sequence of cardinals shows that these cardinals cannot be alephs, whence it follows that not all sets can be well-ordered. Therefore, the axiom of choice cannot be added to the other axioms of Quine's theory without contradictions. We may express this fact by saying that the principle of choice can be proved false in Quine's theory. This was pointed out by Specker.

Proof that (2) follows from (1): Because of (1) there exists a mapping of the set of all unit sets  $\{m\}$  on a subset of  $V$ . Indeed the identical mapping is of that kind. However, the identical mapping maps the set of all  $\{\{m\}\}$  on just this subset of all sets  $\{m\}$ . Let us on the other hand assume that  $EV$  could be mapped onto  $EEV$ . The mapping would then consist of mutually disjoint pairs  $(\{m\}, \{n\})$ . However, the certainly existing set of pairs  $(m, \{n\})$  would then furnish a mapping of  $V$  on  $EV$  contrary to (1). Hence (2) follows from (1).

The theory of Quine's does not seem to have many adherents among mathematicians. The reason for this is presumably the existence of such sets in it as  $V$  which are elements of themselves, pathological sets as they are called. I don't think, however, that this circumstance ought to worry mathematicians, because it is not necessary to take these abnormal sets into account in the development of the ordinary mathematical theories.

## 14. The ramified theory of types. Predicative set theory

I have already mentioned Poincare's objection to Cantor's set theory, that one makes use of the so-called non-predicative definitions. These definitions collect objects in such a way that the totality of these objects, or objects logically dependent upon that totality, are considered as belonging to the same totality, so that the definition has a circular character. It might perhaps be better to say that a non-predicative definition is the definition of an entity by a logical expression containing a bound variable such that the defined entity is one of the possible values of this variable. However, instead of trying to explain this generally, I think it is better to take a characteristic example.

Let us consider mankind, the domain of all human beings. We have the binary relation "x is a child of y" which I write  $Ch(x,y)$ . Let us try to define descendant of  $P$ ,  $P$  any given person. If we make use of the notion of finite number we may proceed thus: We define the relation  $Ch^n(x,y)$  recursively by letting

$$Ch^1(x,y) \text{ stand for } Ch(x,y)$$

$$Ch^{n+1}(x,y) \text{ stand for } (\exists z)(Ch^n(x,z) \& Ch(z,y)).$$