Chapter 6

THE APPROXIMATION POLYNOMIAL

1. The aim.

In the present chapter we shall construct a polynomial

$$A(x_1,\dots,x_m) = \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} a_{i_1\dots i_m} x_1^{i_1}\dots x_m^{i_m} \neq 0$$

which has integral coefficients that are not "not large" and which vanishes to a "very high" order at the point

$$x_1 = \xi,\dots,x_m = \xi;$$

here ξ is a given algebraic number. The importance of this approximation polynomial will become clear in the next chapters.

The construction does not involve valuation theory, but it is convenient to admit finite extensions of the rational number field.

2. The powers of an algebraic number.

Here and further on,

$$F(x) = F_0x^f + F_1x^{f-1} + \dots + F_f$$

denotes a fixed polynomial with integral coefficients such that

$$f \geq 1,\quad F_0 \neq 0,\quad F_f \neq 0$$

and therefore $F(0) \neq 0$. We impose the additional condition that $F(x)$ *has no multiple factor,* hence that $F(x)$ *and its derivative* $F'(x)$ *are relatively prime.*

Let Ω be an arbitrary (abstract) extension field of the rational field Γ in which $F(x)$ splits into a product of linear factors

$$F(x) = F_0(x-\xi_1)\dots(x-\xi_f).$$

The f zeros

$$\xi = \xi_1,\dots,\xi_f$$

of $F(x)$ are thus all *distinct* and *different from zero.*

We use the abbreviation

$$c = 2\max(|F_0|,|F_1|,\dots,|F_f|)$$

so that $c \geq 2$ is an integer.

Lemma 1: *For every exponent* $l=0, 1, 2,\dots$ *there exist unique integers* $g_0^{(l)}, g_1^{(l)},\dots,g_{f-1}^{(l)}$ *such that*

(1): $$F_0^l \xi_\psi^l = g_0^{(l)} + g_1^{(l)}\xi_\psi + \dots + g_{f-1}^{(l)}\xi_\psi^{f-1} \quad (\psi = 1,2,\dots,f),$$

(2): $$\max(|g_0^{(l)}|, |g_1^{(l)}|,\dots, |g_{f-1}^{(l)}|) \le c^l.$$

Proof: First, the coefficients g are unique because the Vandermonde determinant

$$\begin{vmatrix} 1 & \xi_1 & \xi_1^2 \dots & \xi_1^{f-1} \\ 1 & \xi_2 & \xi_2^2 \dots & \xi_2^{f-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_f & \xi_f^2 \dots & \xi_f^{f-1} \end{vmatrix}$$

does not vanish. Secondly, the equations (1) hold trivially for $l \le f-1$ with $g_l^{(l)} = F_0^l$ and the other coefficients equal to zero. Third, for $l \ge f-1$,

$$F_0^{l+1}\xi_\psi^{l+1} = F_0\xi_\psi(g_0^{(l)} + g_1^{(l)}\xi_\psi + \dots + g_{f-1}^{(l)}\xi_\psi^{f-1}) =$$

$$= (F_0 g_0^{(l)}\xi_\psi + F_0 g_1^{(l)}\xi_\psi^2 + \dots + F_0 g_{f-2}^{(l)}\xi_\psi^{f-1}) - g_{f-1}^{(l)}(F_f + F_{f-1}\xi_\psi + \dots + F_1\xi_\psi^{f-1})$$

and therefore

$$g_\phi^{(l+1)} = \begin{cases} -F_f g_{f-1}^{(l)} & \text{if } \phi = 0, \\ F_0 g_{\phi-1}^{(l)} - F_{f-\phi} g_{f-1}^{(l)} & \text{if } \phi = 1,2,\dots,f-1, \end{cases}$$

so that the coefficients are integers. Finally,

$$\max(|g_0^{(l)}|, |g_1^{(l)}|,\dots, |g_{f-1}^{(l)}|) = |F_0^l| \le c^l \text{ if } l \le f-1,$$

$$\max(|g_0^{(l+1)}|, |g_1^{(l+1)}|,\dots, |g_{f-1}^{(l+1)}|) \le c\max(|g_0^{(l)}|, |g_1^{(l)}|,\dots, |g_{f-1}^{(l)}|) \text{ if } l \ge f-1,$$

whence the inequalities (2).

3. A lemma by Schneider.

The following lemma is essentially due to Th. Schneider[1]. The proof is taken from Cassels' book on Diophantine Approximation. In the appendix, an entirely different proof is used to prove a stronger result.

Lemma 2: *Let* $r_1,\dots,r_m$ *be positive integers, and let* s *be a positive number. Each of the two systems of inequalities*

$$0 \le i_1 \le r_1,\dots, 0 \le i_m \le r_m, \quad \sum_{h=1}^m \frac{i_h}{r_h} \le \frac{1}{2}(m-s)$$

1. J. reine angew. Math. 175 (1936), 182-192.

and

$$0 \leqslant i_1 \leqslant r_1, \ldots, 0 \leqslant i_m \leqslant r_m, \sum_{h=1}^{m} \frac{i_h}{r_h} \geqslant \frac{1}{2}(m+s)$$

has at most

$$\frac{\sqrt{2m}}{s}\,(r_1+1)\ldots(r_m+1)$$

solutions in sets of integers $(i_1, \ldots, i_m)$.

Proof: The two systems of inequalities are changed into one-another by the transformation

$$(i_1, \ldots, i_m) \rightarrow (r_1 - i_1, \ldots, r_m - i_m)$$

and so have the same number of solutions. It suffices therefore to consider the first system. The proof is by induction for m.

First let $m=1$. The system

$$0 \leqslant i_1 \leqslant r_1, \ \frac{i_1}{r_1} \leqslant \frac{1}{2}(1-s)$$

has no integral solution if $s > 1$, and it has not more than

$$r_1 + 1 < \frac{\sqrt{2}}{s}(r_1+1)$$

such solutions if $s \leqslant 1$; hence the assertion holds in this case.

Secondly let $m \geqslant 2$, and assume the lemma has already been proved for inequalities in $m-1$ unknowns. We may assume that

$$s > \sqrt{2m} > 1$$

because the assertion is trivial otherwise.

For fixed $i = i_m$, where $0 \leqslant i \leqslant r_m$, the system $(i_1, \ldots, i_{m-1})$ satisfies the inequalities

$$0 \leqslant i_1 \leqslant r_1, \ldots, 0 \leqslant i_{m-1} \leqslant r_{m-1}, \sum_{h=1}^{m-1} \frac{i_h}{r_h} \leqslant \frac{1}{2}\left\{(m-1) - (s - 1 + \frac{2i}{r_m})\right\},$$

and so, by the induction hypothesis, has not more than

$$\frac{\sqrt{2(m-1)}}{s - 1 + \frac{2i}{r_m}}\,(r_1+1)\ldots(r_{m-1}+1)$$

possibilities. Hence, on putting

$$\sigma = \frac{\sqrt{\frac{m-1}{m}}}{r_m + 1} \sum_{i=0}^{r_m} \frac{s}{s - 1 + \frac{2i}{r_m}},$$

the original system has at most

$$\sigma \frac{\sqrt{2m}}{s}\,(r_1+1)\ldots(r_m+1)$$

integral solutions $(i_1,\dots,i_{m-1},i)$. The assertion is therefore proved if it can be shown that

$$\sigma \leqslant 1.$$

Now

$$\sum_{i=0}^{r_m} \frac{s}{s-1+\frac{2i}{r_m}} = \frac{1}{2}\sum_{i=0}^{r_m}\left\{\frac{s}{s-1+\frac{2i}{r_m}} + \frac{s}{s-1+\frac{2(r_m-i)}{r_m}}\right\} =$$

$$= \sum_{i=0}^{r_m} \frac{s^2}{s^2-(1-\frac{2i}{r_m})^2} \leqslant \sum_{i=0}^{r_m} \frac{s^2}{s^2-1} = \frac{s^2}{s^2-1}(r_m+1),$$

whence, by $s > \sqrt{2m}$,

$$\sigma \leqslant \sqrt{\frac{m-1}{m}}\cdot\frac{s^2}{s^2-1} < \sqrt{\frac{m-1}{m}}\cdot\frac{2m}{2m-1} = \sqrt{\frac{4m^2-4m}{4m^2-4m+1}} < 1.$$

4. The construction of $A(x_1,\dots,x_m)$. I.

As before, let $r_1,\dots,r_m$ be positive integers. Let further a and s be two positive numbers such that

(3): $$a \geqslant 1, \quad s \geqslant 4f\sqrt{2m},$$

where f is the degree of $F(x)$.

A polynomial of the form

$$B(x_1,\dots,x_m) = \sum_{i_1=0}^{r_1}\dots\sum_{i_m=0}^{r_m} b_{i_1\dots i_m} x_1^{i_1}\dots x_m^{i_m}$$

is said to be *admissible* if, (i) its coefficients $b_{i_1\dots i_m}$ may assume only the $[a]+1$ values

$$0, 1, 2, \dots, [a],$$

and further, (ii)

$$b_{i_1\dots i_m} = 0 \quad \text{unless} \quad \frac{1}{2}(m-s) < \sum_{h=1}^{m}\frac{i_h}{r_h} < \frac{1}{2}(m+s).$$

From Lemma 2, it follows immediately that the condition (ii) demands the vanishing of not more than

$$2\frac{\sqrt{2m}}{s}(r_1+1)\dots(r_m+1) \leqslant \frac{1}{2f}(r_1+1)\dots(r_m+1) \leqslant \frac{1}{2}(r_1+1)\dots(r_m+1)$$

of the $(r_1+1)\dots(r_m+1)$ coefficients of B. Hence not less than

$$\frac{1}{2}(r_1+1)\dots(r_m+1)$$

of the remaining coefficients of B may still run independently over $[a]+1$

distinct values. It follows then that *there are not less than*

$$M = ([a]+1)^{\frac{1}{2}(r_1+1)\dots(r_m+1)}$$

admissible polynomials.

5. The construction of $A(x_1,\dots,x_m)$. II.

As in the last chapter, put

$$B_{j_1\dots j_m}(x_1,\dots,x_m) = \frac{\partial^{j_1+\dots+j_m}B(x_1,\dots,x_m)}{j_1!\dots j_m!\,\partial x_1^{j_1}\dots\partial x_m^{j_m}}.$$

Then

$$B_{j_1\dots j_m}(x_1,\dots,x_m) = \sum_{i_1=0}^{r_1}\dots\sum_{i_m=0}^{r_m} b_{i_1\dots i_m}\binom{i_1}{j_1}\dots\binom{i_m}{j_m}x_1^{i_1-j_1}\dots x_m^{i_m-j_m}$$

has non-negative integral coefficients if B is admissible. The same estimate as in §7 of last chapter leads to the majorant

$$B_{j_1\dots j_m}(x_1,\dots,x_m) << a\cdot 2^{r_1+\dots+r_m}(1+x_1)^{r_1}\dots(1+x_m)^{r_m}$$

and hence to

$$B_{j_1\dots j_m}(x,\dots,x) << a\cdot 2^{r_1+\dots+r_m}\,(1+x)^{r_1+\dots+r_m}.$$

Here

$$(1+x)^{r_1+\dots+r_m} << 2^{r_1+\dots+r_m}(1+x+x^2+\dots+x^{r_1+\dots+r_m}),$$

so that

$$B_{j_1\dots j_m}(x,\dots,x) << a\cdot 2^{2(r_1+\dots+r_m)}(1+x+x^2+\dots+x^{r_1+\dots+r_m}).$$

Thus, for all non-negative suffixes $j_1,\dots,j_m$,

$$B_{j_1\dots j_m}(x,\dots,x), = \sum_{l=0}^{r_1+\dots+r_m}\beta_l^{(j)}x^l \qquad \text{say,}$$

is a polynomial in one variable x with non-negative integral coefficients $\beta_l^{(j)}$ not greater than

$$a\cdot 2^{2(r_1+\dots+r_m)}$$

and of degree not exceeding

$$r_1+\dots+r_m.$$

By Lemma 1, it follows now that

$$F_0^{r_1+\dots+r_m} B_{j_1\dots j_m}(\xi_\psi,\dots,\xi_\psi) = \sum_{l=0}^{r_1+\dots+r_m} \sum_{\phi=0}^{f-1} \beta_l^{(j)} F_0^{r_1+\dots+r_m-l} g_\phi^{(l)} \xi_\psi^\phi =$$

$$= \sum_{\phi=0}^{f-1} B_\phi^{(j)} \xi_\psi^\phi \qquad (\psi = 1,2,\dots,f),$$

where

$$B_\phi^{(j)} = \sum_{l=0}^{r_1+\dots+r_m} \beta_l^{(j)} F_0^{r_1+\dots+r_m-l} g_\phi^{(l)}.$$

Hence

$$|B_\phi^{(j)}| \leq \sum_{l=0}^{r_1+\dots+r_m} a\cdot 2^{2(r_1+\dots+r_m)} \left(\frac{c}{2}\right)^{r_1+\dots+r_m-l} \cdot c^l \leq$$

$$\leq a\cdot 2^{2(r_1+\dots+r_m)} c^{r_1+\dots+r_m} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

because $|F_0| \leq \frac{1}{2}c$. Therefore, for all suffixes $j_1,\dots,j_m$ and ϕ,

$$|B_\phi^{(j)}| \leq 2a(4c)^{r_1+\dots+r_m}.$$

Here $B_\phi^{(j)}$ is an integer since $\beta_l^{(j)}$ and $g_\phi^{(l)}$ are integers. Each number $B_\phi^{(j)}$ has then at most

$$2[2a(4c)^{r_1+\dots+r_m}] + 1 \leq 5a(4c)^{r_1+\dots+r_m}$$

possible values, and the set of all f coefficients

$$B_\phi^{(j)} \qquad (\phi = 0,1,\dots,f-1)$$

of $B_{j_1\dots j_m}(\xi_\psi,\dots,\xi_\psi)$ has at most

$$\{5a(4c)^{r_1+\dots+r_m}\}^f$$

possibilities.

Let $(j_1,\dots,j_m)$ run over all systems of integers satisfying

$$0 \leq j_1 \leq r_1,\dots,0 \leq j_m \leq r_m, \quad \sum_{h=1}^{m} \frac{j_h}{r_h} \leq \frac{1}{2}(m-s);$$

by Lemma 2, there are not more than

$$\frac{\sqrt{2m}}{s}(r_1+1)\dots(r_m+1) \leq \frac{1}{4f}(r_1+1)\dots(r_m+1)$$

such systems. The corresponding set of integral coefficients

$$B_\phi^{(j)}, \text{ where } \phi = 0,1,\ldots,f-1;\ 0 \leq j_1 \leq r_1,\ldots,0 \leq j_m \leq r_m;\ \sum_{h=1}^{m} \frac{j_h}{r_h} \leq \frac{1}{2}(m-s),$$

has then at most

$$M^* = \{5a(4c)^{r_1+\ldots+r_m}\}^{\frac{1}{4}(r_1+1)\ldots(r_m+1)}$$

possibilities.

6. The construction of $A(x_1,\ldots,x_m)$. III.

There are not less than

$$M = ([a]+1)^{\frac{1}{2}(r_1+1)\ldots(r_m+1)} > a^{\frac{1}{2}(r_1+1)\ldots(r_m+1)}$$

admissible polynomials B. We therefore choose

$$a = 5(4c)^{r_1+\ldots+r_m}$$

so that

$$M > M^*.$$

There are then *more* admissible polynomials B than corresponding sets of coefficients $B_\phi^{(j)}$. Hence there exist two distinct admissible polynomials

$$\overline{B}(x_1,\ldots,x_m) = \sum_{i_1=0}^{r_1} \ldots \sum_{i_m=0}^{r_m} \overline{b}_{i_1\ldots i_m} x_1^{i_1}\ldots x_m^{i_m}$$

and

$$\overline{\overline{B}}(x_1,\ldots,x_m) = \sum_{i_1=0}^{r_1} \ldots \sum_{i_m=0}^{r_m} \overline{\overline{b}}_{i_1\ldots i_m} x_1^{i_1}\ldots x_m^{i_m}$$

with the following property: *Define integers* $\overline{B}_\phi^{(j)}$ *and* $\overline{\overline{B}}_\phi^{(j)}$ *such that*

$$F_0^{r_1+\ldots+r_m}\overline{B}_{j_1\ldots j_m}(\xi_\psi,\ldots,\xi_\psi) = \sum_{\phi=0}^{f-1} \overline{B}_\phi^{(j)} \xi_\psi^\phi \qquad (\psi = 1,2,\ldots,f)$$

and

$$F_0^{r_1+\ldots+r_m}\overline{\overline{B}}_{j_1\ldots j_m}(\xi_\psi,\ldots,\xi_\psi) = \sum_{\phi=0}^{f-1} \overline{\overline{B}}_\phi^{(j)} \xi_\psi^\phi \qquad (\psi = 1,2,\ldots,f).$$

Then

$$\overline{B}_\phi^{(j)} = \overline{\overline{B}}_\phi^{(j)} \text{ if } \phi = 0,1,\ldots,f-1;\ 0 \leq j_1 \leq r_1,\ldots,0 \leq j_m \leq r_m;\ \sum_{h=1}^{m} \frac{j_h}{r_h} \leq \frac{1}{2}(m-s).$$

Put

$$A(x_1,\dots,x_m) = \overline{B}(x_1,\dots,x_m) - \overline{\overline{B}}(x_1,\dots,x_m) =$$
$$= \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} a_{i_1\dots i_m} x_1^{i_1} \dots x_m^{i_m}.$$

Since $\overline{B}$ and $\overline{\overline{B}}$ are distinct,

$$A(x_1,\dots,x_m) \not\equiv 0.$$

From the construction, the coefficients $a_{i_1\dots i_m}$ of A are integers of absolute values not exceeding a, thus satisfying

$$|a_{i_1\dots i_m}| \leq 5(4c)^{r_1+\dots+r_m}.$$

Moreover,

$$a_{i_1\dots i_m} = 0 \quad \text{unless} \quad \frac{1}{2}(m-s) < \sum_{h=1}^{m} \frac{i_h}{r_h} < \frac{1}{2}(m+s),$$

and furthermore,

$$A_{j_1\dots j_m}(\xi_\psi,\dots,\xi_\psi) = 0 \text{ if } \psi = 1,2,\dots,f;\ 0 \leq j_1 \leq r_1,\dots,0 \leq j_m \leq r_m;\ \sum_{h=1}^{m} \frac{j_h}{r_h} \leq \frac{1}{2}(m-s).$$

Instead, we may also say that $A_{j_1\dots j_m}(x,\dots,x)$ is divisible by $F(x)$ whenever

$$0 \leq j_1 \leq r_1,\dots,0 \leq j_m \leq r_m,\ \sum_{h=1}^{m} \frac{j_h}{r_h} \leq \frac{1}{2}(m-s);$$

for the zeros $\xi_1,\dots,\xi_f$ of $F(x)$ are all distinct. We note that the upper bound for the coefficients of A implies again majorants analogous to those found for B.

The following result has thus been proved.

Theorem 2: *Let*

$$F(x) = F_0x^f + F_1x^{f-1} + \dots + F_f, \quad \text{where} \quad f \geq 1,\ F_0 \neq 0,\ F_f \neq 0,$$

be a polynomial with integral coefficients which has no multiple factors and does not vanish for $x=0$. *Put*

$$c = 2\max(|F_0|, |F_1|,\dots, |F_f|).$$

Let $r_1,\dots,r_m$ *be positive integers, and let* s *be a real number not less than* $4f\sqrt{2m}$. *There exists a polynomial*

$$A(x_1,\dots,x_m) = \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} a_{i_1\dots i_m} x_1^{i_1} \dots x_m^{i_m} \not\equiv 0$$

with the following properties.

(1): *Its coefficients* $a_{i_1\dots i_m}$ *are integers satisfying*

$$|a_{i_1\ldots i_m}| \leqslant 5(4c)^{r_1+\ldots+r_m};$$

and they vanish unless

$$\frac{1}{2}(m-s) < \sum_{h=1}^{m} \frac{i_h}{r_h} < \frac{1}{2}(m+s) .$$

(2): $A_{j_1\ldots j_m}(x,\ldots,x)$ *is divisible by* $F(x)$ *whenever*

$$0 \leqslant j_1 \leqslant r_1,\ldots,0 \leqslant j_m \leqslant r_m, \sum_{h=1}^{m} \frac{j_h}{r_h} \leqslant \frac{1}{2}(m-s).$$

(3): *The following majorants hold,*

$$A_{j_1\ldots j_m}(x_1,\ldots,x_m) \ll 5(8c)^{r_1+\ldots+r_m}(1+x_1)^{r_1}\ldots(1+x_m)^{r_m},$$

$$A_{j_1\ldots j_m}(x,\ldots,x) \ll 5(8c)^{r_1+\ldots+r_m}(1+x)^{r_1+\ldots+r_m} .$$

This theorem will be applied only for large values of m, and s will always be small compared with m. The last two majorants hold, of course, by the formula

$$A_{j_1\ldots j_m}(x_1,\ldots,x_m) \ll 2^{r_1+\ldots+r_m}a(1+x_1)^{r_1}\ldots(1+x_m)^{r_m}$$

proved in Chapter 5, §7, since in the present case,

$$a \leqslant 5(4c)^{r_1+\ldots+r_m} .$$