Chapter V

MODALITIES

Although the notions of modality in logic are as old as Aristotle, the first attempt to incorporate them in a modern system of symbolic logic is that of C. I. Lewis.[1] He, like many others following him, was concerned with an analysis of logical implication in terms of various combinations of necessity and negation. From the standpoint of formal deducibility, however, this is putting the cart before the horse. Indeed the implication which we defined in the previous chapters was, under certain assumptions, a strict implication; and we have seen that if one wishes to adopt a more or less classical standpoint with respect to negation, the natural system of strict implication is LD. But the notions of necessity and possibility are interesting on their own account. In this chapter we shall study definitions of formal concepts analogous to them by the same methods which have been used in the previous chapters.

The notion of necessity will be discussed first; then the notions of possibility and various associated notions. The treatment is necessarily incomplete; but it is hoped it will be suggestive to those who make a special study of modal systems.

We shall use the notations

$$\#A \qquad\qquad \Diamond A$$

to indicate the necessity and the possibility of A respectively. They may be read as "necessarily A," and "possibly A."

1. **Analysis of Necessity.** To begin with let it be emphasized that we are not attempting to define something in the inner nature of propositions which characterizes some propositions as necessary and others as contingent. On the contrary we are defining, in terms of the notion of formal system, an objective concept, to which we can, with some justice, attach the name "necessity."

We can get such a concept by considering, so to speak, a formal system within a formal system. That is, given a formal system $\mathfrak{S}$, we may distinguish those theorems which are derivable from only a part of its rules. The part in question will then constitute a system $\mathfrak{S}^{\#}$ within $\mathfrak{S}$. I shall call $\mathfrak{S}^{\#}$ the <u>inner system</u> while $\mathfrak{S}$ itself is the <u>outer</u> system. Then we can interpret $\#A$ to mean that A is derivable in the inner system.

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1. See references in the Introduction.

For example, let $\mathfrak{S}$ be the system considered in IV, §2, with Rules 1 and 3 and the axioms $0 \leqq 0$, $0 \leqq 1$. Here I have written the primitive relator as " $\leqq$ " rather than " = " because it now has the meaning usually associated with " $\leqq$ ". As inner system we take that based on the first axiom and Rule 1. Then the relation

$$\#(r \leqq s)$$

is precisely the relation of equality. (If we were to admit both rules and only the second axiom in the inner system, this relation would be the same as $r < s$.)

As noted in the preceding chapter we do not exclude the application, in principle, to the case where $\mathfrak{S}$ is a system of ordinary discourse, while the inner system is some restricted part of it. Thus we may take $\mathfrak{S}$ to be a formulation for physics, with axioms which are established by experiment, while $\mathfrak{S}^{\#}$ is a system of logic and mathematics. Or we may take the outer system to be a formulation of biology, the inner system one of physics. Again we may consider a whole family of outer systems, each constituting what Carnap [6] would call a "state description," or a family derived by permuting the primitive terms as suggested by McKinsey [60].

In the present theory we shall suppose that we have a single outer and inner system, and that both are completely formalized. But the outer system is any system which includes the inner system, and may be the inner system itself; so that theorems stated about the outer system can be specialized to apply to the inner system.[2]

2. **The L-System for Necessity.** We consider now the formalization of these rules.

We suppose the inner and outer systems both formulated as one of the systems LA, LC, LM, LJ, LD, LK, with or without quantifiers.[3] The kind of system so involved will be called the underlying system and designated LX. We consider the changes to be made to adjoin the necessity connective. The new system formed will be called LXY.

On the level of <u>morphology</u> we should make the following changes. We extend the definition of $\mathfrak{P}(\mathfrak{S})$ by adding the rule.

<u>If $A \, \varepsilon \, \mathfrak{P}(\mathfrak{S})$, then $\# A \, \varepsilon \, \mathfrak{P}(\mathfrak{S})$</u>,

it being understood that <u>the inner and outer systems both have</u>

2. An example of use of such an inner system in practice is in [19] 2.74. Here if we call the inner system that formed by the rules 2.71-2.73, the premise of 2.74 is that $\varphi = \Psi$ be derivable from certain premises relative to the inner system.

3. In regard to systems with quantifiers the same remarks as in footnote[1] of Chapter III apply here.

the same $\mathfrak{P}(\mathfrak{G})$. The definition of a prosequence holds without change. We shall use the notation $\#\mathfrak{X}$ to denote a prosequence all of whose constituents have the form $\#A$.[4]

We consider two kinds of elementary statements, viz.,

(1) $$\mathfrak{X} \Vvdash \mathfrak{Y},$$

(2) $$\mathfrak{X} \Vdash \mathfrak{Y}.$$

Here (1) refers to the inner system, (2) to the outer. The rules of LX($\mathfrak{G}\#$) shall be valid for (1), those of LX($\mathfrak{G}$) for (2). (These rules differ only in the rules p2 and Er, which are those valid in the inner and outer systems respectively.)

According to the methods of the previous chapters, we should arrive at theoretical rules by asking under what circumstances we can introduce a constituent $\#A$ on the left and right of an elementary statement. As for introduction on the left it is clear that $\#A$ is to be stronger than A; therefore we can allow A to be replaced by $\#A$. On the right, however, we say $\#A$ is a consequence of the necessity of certain premises when A is itself a consequence of the necessity of those premises in the inner system.

The above rule on the right needs some explaining. It might be thought we should say that $\#A$ is a consequence of certain premises when A is a consequence of those premises in the inner system. But since A is always a consequence of itself, we should end up with $\#A$ equivalent to A. Now of course if A is actually true in the inner system, so is $\#A$ (according to our intentions). But we do not want $\#A$ to follow simply from the assumption of A. So we state that if A is valid in the inner system on the basis of certain assumptions $\mathfrak{X}$, then $\#A$ is valid (in either system) on the basis of the assumptions $\#\mathfrak{X}$. (Of course it is valid on the assumptions $\#\mathfrak{X}$ in the inner system by the first rule, if it is valid on the assumptions $\mathfrak{X}$, so that the statement in the preceding paragraph is sufficient.)

One further restriction comes in the extension to cases LC and LK where there may be more than one constituent on the right The analogy which we have followed so far, where we have simply added a $\mathfrak{Z}$ on the right, would lead to paradoxical results if extended to Yr. For we should get

(3) $$\#(A \vee B) \supset . (\#A) \vee B,$$

which is not in accord with our intuitive idea of necessity. It is therefore necessary to restrict Yr to the case of a single constituent on the right in all systems.

This leads to the following statement of rules:

RULES FOR NECESSITY:

Yℓ $$\frac{\mathfrak{X}, A \Vdash \mathfrak{z}}{\mathfrak{X}, \#A \Vdash \mathfrak{z}}$$ Yr $$\frac{\#\mathfrak{X} \Vvdash A}{\#\mathfrak{X} \Vdash \#A}$$

4. It is to be obtained from $\mathfrak{X}$ by prefixing # to all propositions of $\mathfrak{X}$ not already of the form #A.

Here " $\Vdash$ " can refer to either the inner or the outer system; because the inner system can be regarded as a special case of the outer system.

These rules, however, do not admit the methods of proof of II §§ 5-7. So far as Theorem II 2 is concerned, it is clear that we have to admit as primitive the rules Kr and Kℓ. But even this does not save the elimination theorem. Indeed the proof of that theorem in II § 7 depended essentially on the fact that parametric constituents could be added to premises and conclusion of a rule without invalidating the inference. This is not true for the rule Yr. Thus the proof of Theorem II 11 does not carry over to the present case. Whether the elimination theorem can be established by other methods is not known.[5]

On this account we shall not go further with the system LXY. However, the following facts concerning it may be noted incidentally. The elimination theorem can be proved if A is non-modal and LX is such that only prosequences with a single constituent can appear on the right. It can then be shown that

$$\mathfrak{X} \Vdash \mathfrak{Y}$$

is equivalent, for suitable $\mathfrak{M}$, to

$$(4) \qquad \mathfrak{X}, \mathfrak{M} \Vvdash \mathfrak{Y}$$

where every constituent of $\mathfrak{M}$ is either an axiom of the outer system, or is of the form

$$A_1 \supset . A_2 \supset . \ldots \supset . A_m \supset B$$

where

$$(5) \qquad A_1, A_2, \ldots, A_m \vdash B.$$

In case there exists a non-modal proposition M such that 1)

$$(6) \qquad \Vdash M$$

2) if A is an axiom of the outer system, then

$$(7) \qquad M \Vvdash A$$

and 3) whenever (5) holds in the outer system, then

$$(8) \qquad M, A_1, \ldots, A_m \Vvdash B;$$

5. At least it was not at the time these lectures were delivered. Since that time I have proved the elimination theorem for LAY, LMY, LJY. It is probable that the proof can be extended to LDY and LKY, and possibly also to LCY; but the details are not yet clear. Note that a proof for LKY which retained the decidability theorems would give a decision process for the Lewis calculus S4; the relation of this to previously known decision procedures is also not yet clear. The presence of variables seems not to cause any difficulties comparable to those caused by Nx. These matters are left for another publication.

we say that M expresses the axioms and rules of the outer system in the inner. In that case (4) can be replaced by

(9) $$\mathfrak{X}, M \Vdash \mathfrak{Y}.$$

Further, with the addition of the rules Kr and Kℓ there is no trouble about Theorems II 2, II 3, II 4, II 5, II 7, II 8, II 10 in so far as they apply to LX.

3. **The T and H Systems for Necessity.** Since we do not have the elimination theorem for LXY, we cannot derive the rules of a corresponding T system from those of LXY. Instead we shall postulate them directly. It is clear from the semantical discussion in §§1-2 what postulates we should accept. It can be shown - rather more easily than in the previous chapters - that the rules adopted are equivalent to those of LXY if a rule of elimination is postulated for the latter. However this will not be gone into here.

We suppose that a system TX has been defined (where X is A, C, M, J, D, K or one of these with addition of variables). We extend the definition of $\mathfrak{P}(\mathfrak{S})$ as in §2, and we consider two kinds of elementary statements

(10) $$A \ \varepsilon\ \mathfrak{T}\#(\mathfrak{X})$$

(11) $$A \ \varepsilon\ \mathfrak{T}(\mathfrak{X})$$

We postulate that (10) shall obey the rules of TX relative to the inner system, (11) the same rules relative to the outer system. (Thus (10) and (11) differ with regard to the rules t2 and Ei only.) We then adjoin the following:

T-RULES FOR NECESSITY.

Ye $$\frac{\#A}{A}$$ Yi $$\frac{\overset{(\#)}{A}}{\#A}$$

where the "(#)" over the "A" in Yi indicates #A is derived from $\#\mathfrak{X}$ by the rules of the inner system. The rule Yi, written out in full is

$$\frac{A \ \varepsilon\ \mathfrak{T}\#(\#\mathfrak{X})}{\#A \ \varepsilon\ \mathfrak{T}(\#\mathfrak{X})}$$

On the possibility of specializing (11) to (10) we make the same stipulations as in §2.

DEFINITION 1. We define strict implication, A ⥽ B, thus:

$$A \prec B \ .\equiv .\ \#(A \supset B)$$

We then have the following:

THEOREM 1. The following are in $\mathfrak{T}\#(\mathfrak{S})$ whatever the system $\mathfrak{S}$:

(a) $$\# A \supset A,$$

(b) $$\#A \supset \#\#A,$$

(c) $$A \prec B \;.\supset.\; \#A \prec \#B.$$

Further we have the property

(d) If $A \;\varepsilon\; \mathfrak{X}^{\#}(0)$, then $\#A \;\varepsilon\; \mathfrak{X}^{\#}(0)$.

Proof. Property (a) follows at once by Ye and Pi. Property (d) follows at once by Yi. Properties (b) and (c) follow by the following schemes:

(b)
$$\cfrac{\cfrac{\overset{\overset{\checkmark}{1}}{\#A}}{\#\#A}\ \text{Yi}}{\#A \supset \#\#A.}\ \text{Pi - 1}$$

(c)
$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overset{\overset{\checkmark}{1}}{A \prec B}}{A \supset B}\ \text{Ye} \qquad \cfrac{\overset{\overset{\checkmark}{2}}{\#A}}{A}\ \text{Ye}}{B}\ \text{Pe}}{\#B}\ \text{Yi}}{\#A \supset \#B}\ \text{Pi - 2}}{\#A \prec \#B}\ \text{Yi}}{A \prec B \;.\supset.\; \#A \prec \#B}\ \text{Pi - 1}$$

We define the system HXY as the set of propositions A such that

$$A \;\varepsilon\; \mathfrak{X}^{\#}(0)$$

for $\mathfrak{G}^{\#} = \mathfrak{D}$. This system is an algebra $\mathfrak{H}$ closed under the rule Ph (since Ph is a special case of Pe) and the rule Yh:

$$\text{Yh} \qquad \frac{A \;\varepsilon\; \mathfrak{H}}{\#A \;\varepsilon\; \mathfrak{H}}.$$

We consider first the case where only Ph is admitted, so as not to have to revise previous proofs to take care of the new rule.

THEOREM 2. With respect to the rule Ph, a set of prime propositions for HXY consists of all those of the form #A where A is a prime proposition of HX, together with those in the schemes

(a) $$\#A \;.\supset.\; A,$$

(a') $$\#A \;.\prec.\; A,$$

(b') $$\#A \;.\prec.\; \#\#A,$$

(c') $$A \prec B \;.\prec:\; \#A \;.\prec.\; \#B.$$

Proof. Let $\mathfrak{H}(\mathfrak{X})$ be the propositions generated from $\mathfrak{X}$, together with the prime propositions indicated in the theorem, by the rule Ph. Then since, by Theorem 1, every such prime proposition is in $\mathfrak{S}^{\#}(0) \subseteq \mathfrak{S}^{\#}(\mathfrak{X})$,[6] and since $\mathfrak{S}^{\#}(\mathfrak{X})$ is closed with respect to Ph,

$$\mathfrak{H}(\mathfrak{X}) \subseteq \mathfrak{S}^{\#}(\mathfrak{X}).$$

If we show the converse, then the theorem follows by taking $\mathfrak{X}$ void.

To prove the converse we show that $\mathfrak{H}(\mathfrak{X})$ obeys the rules for $\mathfrak{S}^{\#}(\mathfrak{X})$. So far as the non-modal rules are concerned this follows from the relevant cases in the previous chapters. To take care of the modal rules we observe first that the rule

$$\text{Ph}' \qquad \frac{A, A \prec B}{B}$$

is valid by Ph and (a). Then the modal rules are verified as follows:

<u>Ye</u>. This follows at once from (a) and Ph.

<u>Yi</u>. Let $A \,\varepsilon\, \mathfrak{H}(\mathfrak{X})$ and let $B_1, B_2, \ldots, B_n$ be a derivation of A in $\mathfrak{H}(\mathfrak{X})$ - i.e., B_n is A and every B_k is either a prime proposition, a member of $\mathfrak{X}$, or is derived from predecessors B_i, B_j by Ph. If the premises of Rule Yi are fulfilled, every B_k which is a member of $\mathfrak{X}$ is of the form $\# C_k$. If B_k is prime, then either B_k is of the form $\# C_k$ or is an instance of (a). We now show by induction on k that $\# B_k$ is in $\mathfrak{H}(\mathfrak{X})$ for every k. If B_k is of the form $\# C_k$ this follows by (b') and Ph'. If B_k is an instance of (a), then $\# B_k$ is an instance of (a'). If B_k follows from B_i and B_j by Ph, then, without loss of generality, B_j is $B_i \supset B_k$. By the hypothesis of the induction we have $\# B_i$ and $B_i \prec B_k$. From the second of these and (c'), Ph' we have $\# B_i \prec \# B_k$. Hence by Ph' we have $\# B_k$, q.e.d.

THEOREM 3. <u>With respect to the rules Ph and Yh a set of prime propositions for HXY consists of those of HX together with the schemes</u>

(a) $\qquad \# A \supset A,$

(b) $\qquad \# A \,.\supset.\, \#\# A,$

(c) $\qquad A \prec B \,.\supset:\, \# A \,.\prec.\, \# B.$

<u>Proof</u>. Let $\mathfrak{H}_2$, be the system generated in Theorem 2, $\mathfrak{H}_3$ that generated in this theorem. Then, since the prime propositions of $\mathfrak{H}_2$ can be derived in $\mathfrak{H}_3$ by Yh, we have $\mathfrak{H}_2 \subseteq \mathfrak{H}_3$. The converse follows since Yh, which is a special case of Yi, is a derived rule of $\mathfrak{H}_2$, and the prime propositions of $\mathfrak{H}_3$ are valid in $\mathfrak{H}_2$.

The schemes (a) (b) (c) of Theorem 3 are due to Gödel [41]. He stated also the following interesting properties, which it is

6. One uses (d) to derive (a'), (b'), (c').

not difficult to verify: *The system HKY is equivalent to the system S4 of Lewis;*[7] further *the propositions of HJ are all verified in HKY if one makes the following translations:*[8]

HJ	HKY
$\neg A$	$\# \neg A$
$A \supset B$	$\# A \supset \# B$
$A \wedge B$	$A \wedge B$
$A \vee B$	$\# A \vee \# B.$

The above considerations completely determine HKY. But HKY is the algebra we get when $\mathfrak{S}$ is $\mathfrak{D}$. It is possible that in connection with particular systems $\mathfrak{S}$ one would get stronger systems of strict implication. On this point Cf. McKinsey [60]. Carnap [10] arrives at the system S5[9].

4. **Discussion of Possibility.** In conclusion I shall add a few sketchy remarks about possibility.

It is customary to think of possibility and necessity as inter-definable by means of negation. But possibility may also be regarded as an independent notion to be defined according to the preceding methods in its own right. Indeed that is the only course open to us if we have a system without negation. If we were to do that we would proceed somewhat as follows.

Consider a family of formal systems.[10] Let $\mathfrak{S}_1$ be a system of this family. We may regard A as possible in $\mathfrak{S}_1$ if it is true in some stronger $\mathfrak{S}_n$ of the family. Then we can say A is possible on the suppositions $\mathfrak{X}$ if A is derivable from $\mathfrak{X}$ in some stronger $\mathfrak{S}_n$; and that if $\Diamond B$ holds in the suppositions $\mathfrak{X}$, A, then it holds on the suppositions $\mathfrak{X}, \Diamond A$. If we suppose the rules and axioms of $\mathfrak{S}_n$ can be expressed in $\mathfrak{S}_1$ by M_n, and if we follow the same analogy we have been following for multiple right sides, our rules become

$$\frac{\mathfrak{X}, A \Vdash \Diamond B, \mathfrak{Z}}{\mathfrak{X}, \Diamond A \Vdash \Diamond B, \mathfrak{Z}} \qquad\qquad \frac{\mathfrak{X}, M_n \Vdash A, \mathfrak{Z}}{\mathfrak{X} \Vdash \Diamond A, \mathfrak{Z}}$$

With these we can proceed as in the foregoing theory.[11] Evidently there will be no trouble with the elimination theorem. There

7. [57] p. 499.

8. For other similar properties see [62].

9. [A rule-theoretic formulation which leads to S5 has been communicated to me by R. Feys, (Spring, 1949).]

10. On ways of thinking of such a family cf. the remarks in §1.

11. The proofs of the statements in the following remarks have not yet been carried through. Consequently they are only tentative. In particular, no consideration has yet been given to the cases where $\mathfrak{Z}$ is non-void.

would be some revision in the case of the decidability theorems, on account of the elimination of M_n. The rules of the T system would be

$$\frac{\Diamond A \qquad \begin{matrix}[A]\\ \Diamond B\end{matrix}}{\Diamond B} \qquad\qquad \frac{A}{\Diamond A}$$

and the prime propositional schemes for an H-system:

(a) $A \supset \Diamond A$

(b) $\Diamond \Diamond A \supset \Diamond A$

(c) $\Diamond (A \supset B) \;.\supset.\; \Diamond A \supset \Diamond B.$

The following remarks are now relevant:

1) If we have classical negation we arrive again at the Lewis system S4. Further if we have also necessity, (a), (b), (c) are all satisfied if we define

$$\Diamond A \equiv \neg \# \neg A$$

But (b) fails, at least apparently, if we do not have classical negation.

2) If we have an M such that

$$M \Vdash M_n$$

then all properties hold if we define

$$\Diamond A \equiv M \supset A.$$

The existence of such an M may seem problematical in the general case, since any such M might be absurd. But it can be realized in certain cases - e.g., we have three systems - an inner system $\mathfrak{S}_0$, a real system $\mathfrak{S}_1$, and an outer system $\mathfrak{S}_2$, and M be defined as M_2.[12] The situation is parallel to a definition of $\neg A$ as $A \supset F$.

3) If we define

$$\Diamond A \equiv \neg \neg A$$

then (a), (b), (c) are satisfied even in HM.

4) It is clear that the essential element in the Glivenko theorem is that double negation satisfies (a) and (c). (Property (b) is true but incidental.) Property (c) - which both necessity and possibility have in common, is the really essential one. In fact we can argue generally in TA that if

$$A \,\varepsilon\, \mathfrak{X}(\mathfrak{X})$$

12. In such a case we have an interesting distinction between $\Diamond A$ and $\# \Diamond A$. Thus take the system of IV § 2. Let the inner system contain the rules 1, 2, and 3 and the axiom 0 = 0. Let the real system be obtained by adjoining the axiom 0 = 3 and let M be 0 = 2. Then 1 = 3 is necessarily possible, 1 = 2 only contingently so.

then

$$\Diamond A \;\varepsilon\; \mathfrak{X}(\Diamond \mathfrak{x})$$

This argument is the gist of the deduction theorem (Pi in Theorem II 19), the Glivenko theorem, the proof of Πi in Theorem III, and of Yi in Theorem 2.

5) There is an interesting possibility if we have a system with two kinds of negation. Suppose we had refutability and absurdity at the same time. If we use " $-$ " and " $\neg$ " respectively as the connectors in these two cases, then if the formulation is acceptable from the standpoint of interpretation, the $\Diamond A$ defined by

$$\Diamond A = - \neg A$$

will mean that A can be adjoined to the system without making it absurd. This is an example of a definition of possibility which does not have the above properties. But it requires that we have a non-classical negation.