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ALIGNMENT CHARTS

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ALIGNMENT CHARTS

Introduction. From time to time the present author has given a course called Graphical and Numerical Computation. It has usually covered such topics as Alignment Charts, Interpolation, Numerical Differentiation and Integration, and Curve Fitting, as well as various minor topics and applications. It is the object of the present paper to discuss the first of these topics. It is our aim to present the subject with theoretical generality but with all possible simplicity.

The alignment chart, or nomograph, is a tool of considerable value. In a great variety of applied problems where high accuracy is not required, or where it is unattainable, this graphical process has demonstrated its usefulness. It is especially helpful when many numerical problems of a similar sort must be handled. Often the most complicated formula will be solved by reading from a scale. Moreover, the chart can be used by a person without special knowledge or experience and without the mastery of a difficult technique.

There are many books on alignment charts. The subject has appealed to those who are interested in the application of numerical processes to practical problems. One may say of these books that often they have been written by authors whose interests tended in special directions with the result that each presentation has had a particular bias. As a consequence, many of these texts, however excellent in certain of their aspects, have failed to present to the reader certain parts of the underlying theory which seem to the present author to be fundamental. It is because of these facts that this introduction to the subject is being written.

The theory of alignment charts may be reduced to very simple elements. Like all Gaul in the time of Julius Caesar the theory is divided into three parts. They are:

(1) Scales, or curves in parametric form,

(2) Three-rowed determinants,

(3) The transformation theory.

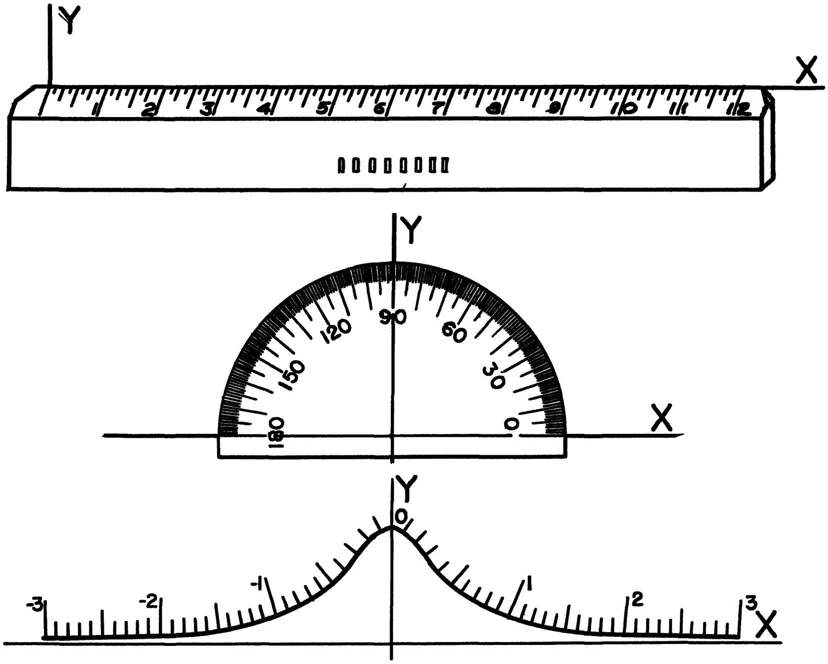
We shall discuss these three parts in order.

Scales. Familiar examples of scales are the common foot rules and measuring tapes with the graduations marked upon them, thermometers, protractors, and the like (see Figure 1). Each exemplifies the fundamental nature of a scale as a curve (including the straight line) with certain marks upon it to which numbers are attached.

Let us think of the curve as lying in a plane in which we have set up a system of rectangular coordinates. The X and Y of each point of the curve depend upon - that is, are functions of - the scale reading at the point. We thus have

$$(1) \qquad X = f(t), \qquad Y = g(t),$$

where t is the scale reading and f and g are suitable functions. We have in (1) the equations of the curve in parametric form.



As an example, the scale furnished by the edge of the foot rule at the top of Figure 1, using the axes indicated, is

$$X = t, \qquad Y = 0,$$

where t is the number of inches as read at the point. The protractor below it carries a scale whose parametric equations are

$$X = r \cos \theta, \qquad Y = r \sin \theta,$$

where θ is the scale reading in degrees and r is the radius of the circle.

Conversely, a pair of relations such as (1) defines a scale. We construct it by plotting the curve which these relations define and marking upon it such values of the parameter t as we wish. For example, at the bottom of Figure 1 we have plotted the scale

$$X = 2t, \qquad Y = \frac{2}{1 + 3t^2} .$$

Consider a uniform scale lying on a line,

$$X = kt, \qquad Y = 0,$$

drawn in a plane in which a coordinate system has been set up. The distance between unit values of t on the scale is k X-units. A large value of k means generous room for the t-graduations, a small k crowds them together. The distance between the unit graduations, measured in terms of the unit used in laying out the coordinate system, is the derivative $dX/dt = k$.

In the general case (1) the ratio of the scale unit to the unit used for the coordinate system is the derivative of arc length

$$\frac{ds}{dt} = \sqrt{f'(t)^2 + g'(t)^2} .$$

Ordinarily, this scale factor will vary from one part of the scale to another.

In the example of the protractor scale of Figure 1, we find

$$\frac{ds}{d\theta} = \sqrt{(-r \sin \theta)^2 + (r \cos \theta)^2} = r.$$

In the example of the protractor scale of Figure 1, we find, since $s = \pi r\theta/180$,

$$\frac{ds}{d\theta} = \frac{\pi r}{180} .$$

The scale factor is thus constant and we have a uniform circular scale.

Slide Rules. The slide rule is so simple an illustration of the use of scales that it seems desirable to digress from our main purpose long enough to set down its general theory. We shall find that we have a mechanical contrivance which can be made to solve a large variety of problems.

Consider two material objects, such as strips of wood or of pasteboard, which can be placed on opposite sides of a line in the plane and can slide freely along this line. We take the line to be the X-axis. Place the objects together in an initial position and mark a common origin O upon the line. Measuring from O we construct lower and upper scales respectively

$$X = f(x), \qquad X = g(y).$$

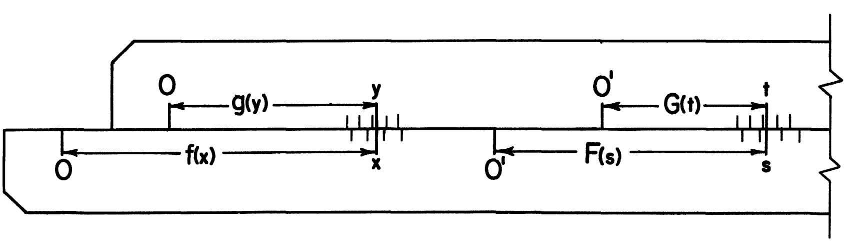
We mark a second origin O' and lay off from it lower and upper scales

$$X' = F(s), \qquad X' = G(t),$$

respectively. We may use any desired functions in constructing the four scales. There results a slide rule whose upper and lower parts bear two scales each.

We proceed to investigate the properties of this mechanism. Let the upper part slide along the line, as illustrated in Figure 2. In the new position suppose that the readings x and y fall together and the readings s and t fall together. Since O and O' on the upper part have been displaced the same distance from their original positions we have immediately that

$$(2) \qquad f(x) - g(y) = F(s) - G(t).$$



This slide rule is a device for solving (2) for any one of the variables when the other three are known. Thus to solve for t when x, y, and s are given we set the rule so that the given values of x and y fall together; coincident with the given value of s we read the required value of t. Any equation in four variables which can be written as a sum of terms each containing only one of the variables can be thrown into the form (2) and hence can be solved by means of a suitable slide rule.

In the preceding construction various simplifications may occur. It may happen that O and O' fall together; that the scales on one of the moving parts may coincide; or the scales above and below the line may be alike. All of these things happen in the case of the simplest form of the slide rule for multiplication. The lower scales $X = \log x$ and $X = \log s$ coincide, as do also the upper scales $X = \log y$ and $X = \log t$. Equation (2) becomes

$$\log x - \log y = \log s - \log t,$$

from which various multiplications and divisions can be performed.

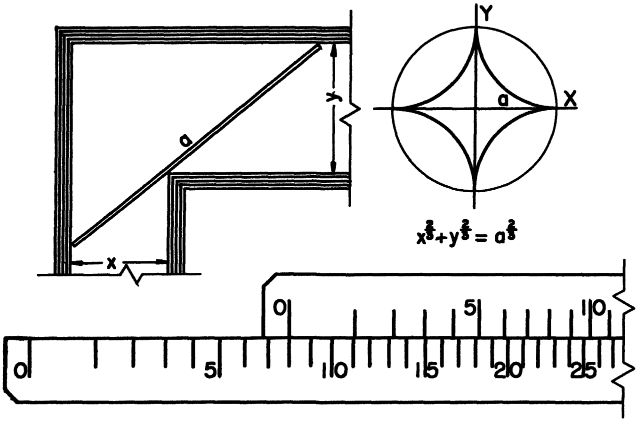
In Figure 3 is shown a slide rule for solving the problem of finding the length a of the longest girder which will go around the corner from a corridor of width x into one of width y at right angles to it. This is a problem frequently met in the calculus. It is allied to the hypocycloid of four cusps, since the equation connecting the variables is

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

We write this in the form (2) as follows:

$$x^{2/3} - 0 = a^{2/3} - y^{2/3},$$

and the scales are of the form $X = x^{2/3}$.



The rule is set for $x = 8$. From it we read that if $y = 10$ then $a = 25.3$; anything shorter than this will pass around the corner. If the girder is 20 feet long and $x = 8$ feet then the second corridor must be about 6.1 feet wide in order that the girder can make the turn.

<u>Three-rowed Determinants</u>. For a comprehension of the essentials of the theory it is important that the student have a knowledge of determinants. They need be only determinants of the third order. A little later he will require the formula for the multiplication of two such determinants, but at present a knowledge of the most elementary properties will suffice.

The equation

$$\begin{vmatrix} X & Y & 1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

where the elements in the second and third rows are constants, represents a straight line; at least, unless the minors by which X and Y are multiplied are both zero. We can make this line pass through a point P_2 (X_2,Y_2) by replacing the elements of either the second row or the third row by $X_2,Y_2,1$. By a similar alteration of the other row we can put the line through another point P_3 (X_3,Y_3). That

$$\begin{vmatrix} X & Y & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = 0$$

passes through P_2 and P_3 is now obvious, since to replace X and Y by the coordinates of either point is to make two rows alike and to cause the equation to be satisfied.

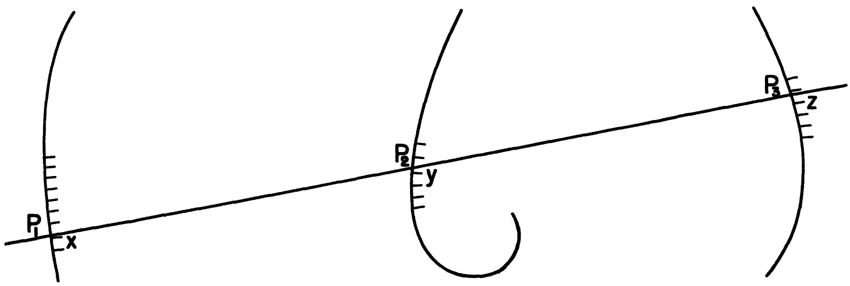
The preceding line will pass through the point $P_1\,(X_1,Y_1)$ if the coordinates of this point satisfy the equation of the line, that is, if

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = 0. \tag{3}$$

This is the necessary and sufficient condition that the three points P_1, P_2, and P_3 lie on a line. In the next section we make use of this condition to lay the foundation of our theory.

<u>Alignment Charts</u>. Given three scales lying in the XY-plane (Figure 4). Let their parametric equations be

$$\begin{aligned} X_1 &= F(x), & Y_1 &= f(x) \\ X_2 &= G(y), & Y_2 &= g(y), \\ X_3 &= H(z), & Y_3 &= h(z). \end{aligned} \tag{4}$$



We lay a straight line across the figure, cutting the scales at the points P_1, P_2, P_3, with scale readings x, y, z. The coordinates of the points, namely, (X_1, Y_1), (X_2, Y_2), (X_3, Y_3) respectively, are given by Equations (4).

From the fact that P_1, P_2, P_3 lie on a line the coordinates of these points satisfy Equation (3). Inserting the values from (4) we have

$$(5) \qquad \begin{vmatrix} F(x) & f(x) & 1 \\ G(y) & g(y) & 1 \\ H(z) & h(z) & 1 \end{vmatrix} = 0$$

Conversely, if this equation is satisfied the points (X_1, Y_1), (X_2, Y_2), (X_3, Y_3), as given by (4), lie on a line.

The three scales constitute an alignment chart for the solution of equation (5). If two of the variables are known the third can be found. We join by a straight line the points on the scales whose parameters are given and where this line meets the third scale we read the value of the remaining parameter.

The determinantal Equation (5) is the central feature of the whole theory. When we set out to construct an alignment chart for the solution of an equation in three variables we endeavor to put the equation in the form (5). If successful, we can then construct the three scales (4) by extracting the functions from the rows of the determinant.

The important fact about the determinant (5) is that each row contains one and only one of the variables and each variable has its row. Given an equation, we endeavor to write it as such a determinant set equal to zero. If we can achieve this in one way we can do it in many ways - by adding columns to columns, which keeps the variables in their appropriate rows, by dividing the elements of a row by a function of the variable appearing in that row, which amounts to dividing the two members of the equation by that function. By this latter device we secure the column of 1's which appears in (5).

The Quadratic and the Reduced Cubic. We illustrate with the equation

$$x^2 + ax + b = 0.$$

Ordinarily, a and b are given and x is to be found. We write this after a few trials as

$$\begin{vmatrix} x^2 & -x & -1 \\ a & 1 & 0 \\ b & 0 & 1 \end{vmatrix} = 0,$$

in which we have the variables suitably distributed into rows. We can complete the determinant in innumerable ways. Thus to remove the 0 in the third column we add the elements of the second column:

$$\begin{vmatrix} x^2 & -x & -x-1 \\ a & 1 & 1 \\ b & 0 & 1 \end{vmatrix} = 0.$$

Now dividing through by $-x-1$ we have the equation in the form (5)

$$\begin{vmatrix} -\frac{x^2}{x+1} & \frac{x}{x+1} & 1 \\ a & 1 & 1 \\ b & 0 & 1 \end{vmatrix} = 0.$$

We would next plot the scales

$$X_1 = -\frac{x^2}{x+1}, \qquad Y_1 = \frac{x}{x+1},$$

$$X_2 = a, \qquad Y_2 = 1,$$

$$X_3 = b, \qquad Y_3 = 0,$$

and our nomograph for solving the quadratic would be completed. The first of these scales is a hyperbola, the second and third are parallel lines. The chart will not be illustrated here.

We can get an alignment chart for solving the reduced cubic

$$x^3 + ax + b = 0$$

by replacing x^2 in the preceding analysis by x^3. The a- and b-scales are the same as before. The x-scale is the cubic curve

$$X_1 = -\frac{x^3}{x+1}, \qquad Y_1 = \frac{x}{x+1}.$$

Parallel Scales. There are many relations in which the variables appear in separate terms, so that the equation is of the type

$$h(z) = f(x) + g(y).$$

This can be written

$$\begin{vmatrix} -1 & f(x) & 1 \\ 1 & g(y) & 1 \\ 0 & \frac{1}{2}h(z) & 1 \end{vmatrix} = 0,$$

from which we can construct an alignment chart.

We have the scales

$$X_1 = -1, \qquad Y_1 = f(x),$$
$$X_2 = 1, \qquad Y_2 = g(y),$$
$$X_3 = 0, \qquad Y_3 = \frac{1}{2}\, h(z).$$

These three scales are parallel, the last being midway between the other two. It is clear from very elementary geometry that any line cuts across the scales so that Y_3 is the average of Y_1 and Y_2,

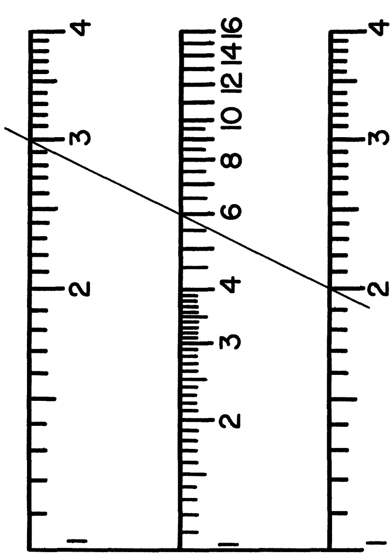
$$Y_3 = \frac{1}{2}\,(Y_1 + Y_2),$$

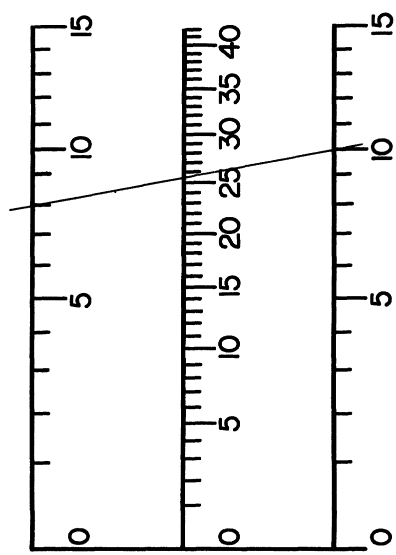
which is identical with the given equation. This type of chart is much used.

Two examples are shown. In Figure 5 we have a chart for multiplication, using

$$\log z = \log x + \log y,$$

the scales thus being the familiar logarithmic scales. In Figure 6 is a chart for the solution of the problem of getting the girder around the corner from one corridor into another,





A chart for multiplication which does not employ logarithmic scales can be devised. The equation

$$z = xy$$

may be written

$$\begin{vmatrix} -x & 1 & 0 \\ 0 & y & 1 \\ z & 0 & 1 \end{vmatrix} = 0.$$

Adding the elements of the second column to those of the third and performing a division in the second row, we get

$$\begin{vmatrix} -x & 1 & 1 \\ 0 & \frac{y}{y+1} & 1 \\ z & 0 & 1 \end{vmatrix} = 0.$$

The scales are again straight lines:

$$\begin{aligned} X_1 &= -x, & Y_1 &= 1, \\ X_2 &= 0, & Y_2 &= \frac{y}{y+1}. \\ X_3 &= z, & Y_3 &= 0 \end{aligned}$$

The first and third are parallel; the second is perpendicular to them.

Mechanical Details. Certain practical hints may be mentioned for the benefit of the novice in the making of charts. The graduations should be fine lines and should be drawn at right angles to the curve on which they are marked. It will simplify the reading if certain important marks, such as every fifth one, are made longer. The ordinary slide rule is well designed in this respect and should be studied by the chartist.

The line across the chart by which a problem is solved should not be actually drawn, since a few lines in pencil would impair the chart's utility. Some sort of mechanical line to be laid across the drawing is required. An opaque straight edge, such as a wooden ruler, is not serviceable, because it covers up part of the scale and makes interpolation difficult. The edge of a transparent ruler is much better. Some persons have had good success with a fine thread, which can be stretched across the nomograph.

The present author, who has studied this matter a good deal, constructed a "line" as follows and found it superior to all other means. On a transparent ruler or celluloid triangle (not too short) let a highly accurate straight line be drawn with the point of a knife. Let graphite from a pencil be worked into the line to give it visibility. The device is used with the side of the ruler on which the line is drawn flat against the paper, so that parallax is prevented.

It need hardly be remarked that a good alignment chart cannot be folded or wrinkled without damage to its accuracy. It is an interesting fact, though, that there are many kinds of stretching and deformation of the paper that will not impair its efficacy. This brings us to the matters treated in the following sections.

The Transformation Theory. If a problem can be solved by an alignment chart it can be solved by an infinitude of different charts. Once we have our basic determinant, it can be altered in innumerable ways by adding columns to columns, and the scales whose equations are extracted from the determinant exhibit great variety. We propose now to treat the process with generality.

Let the equation (5) on which a chart is based be multiplied through by a non-vanishing constant D written in the form of a determinant with constant elements:

(6)
$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$$

The equation which results,

$$\begin{vmatrix} F(x) & f(x) & 1 \\ G(y) & g(y) & 1 \\ H(z) & h(z) & 1 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0,$$

may be rewritten in determinant form by the use of the formula for the multiplication of two determinants. The result is

$$\begin{vmatrix} a_1F(x)+b_1f(x)+c_1 & a_2F(x)+b_2f(x)+c_2 & a_3F(x)+b_3f(x)+c_3 \\ a_1G(y)+b_1g(y)+c_1 & a_2G(y)+b_2g(y)+c_2 & a_3G(y)+b_3g(y)+c_3 \\ a_1H(z)+b_1h(z)+c_1 & a_2H(z)+b_2h(z)+c_2 & a_3H(z)+b_3h(z)+c_3 \end{vmatrix} = 0.$$

We have again a determinant in which each variable is restricted to a single row. We get the desired 1's in the last column by dividing through by the elements that appear there. Obviously, we can divide the first member of this equation by dividing all the elements in a single row. The result is the equation

(7)
$$\begin{vmatrix} X_1' & Y_1' & 1 \\ X_2' & Y_2' & 1 \\ X_3' & Y_3' & 1 \end{vmatrix} = 0,$$

where

$$(8)\quad X_1' = \frac{a_1F(x) + b_1f(x) + c_1}{a_3F(x) + b_3f(x) + c_3}, \quad Y_1' = \frac{a_2F(x) + b_2f(x) + c_2}{a_3F(x) + b_3f(x) + c_3},$$

$$X_2' = \frac{a_1G(y) + b_1g(y) + c_1}{a_3G(y) + b_3g(y) + c_3}, \quad Y_2' = \frac{a_2G(y) + b_2g(y) + c_2}{a_3G(y) + b_3g(y) + c_3},$$

$$X_3' = \frac{a_1H(z) + b_1h(z) + c_1}{a_3H(z) + b_3h(z) + c_3} \quad Y_3' = \frac{a_2H(z) + b_2h(z) + c_2}{a_3H(z) + b_3h(z) + c_3}.$$

Equations (8) are parametric equations for the construction of scales to form an alignment chart for the solution of (7), and (7) is simply an altered form of (5). There are nine constants at our disposal in (8), eight of which are essential. We can select them so as to get a wide variety of different charts.

As a practical matter we observe that we can multiply by D as soon as we have our equation in determinant form with the variables distributed into rows. There is no advantage in getting a column of 1's beforehand. For example, starting with the first determinantal form of the equation of the quadratic in a preceding section, we are led to the following form of Equations (8):

$$X_1' = \frac{a_1x^2 - b_1x - c_1}{a_3x^2 - b_3x - c_3}, \quad Y_1' = \frac{a_2x^2 - b_2x - c_2}{a_3x^2 - b_3x - c_3},$$

$$X_2' = \frac{a_1a + b_1}{a_3a + b_3}, \quad Y_2' = \frac{a_2a + b_2}{a_3a + b_3},$$

$$X_3' = \frac{a_1b + c_1}{a_3b + c_3}, \quad Y_3' = \frac{a_2b + c_2}{a_3b + c_3}.$$

The first of these curves is a conic, the second and third are straight lines. By a suitable choice of the constants, the conic may be any prescribed hyperbola, parabola, or ellipse -

the unit circle with center at the origin, for instance - and the linear scales may have various positions.

We have a similar discussion of the reduced cubic if x^2 in the preceding equations be replaced by x^3. The first scale is then a cubic curve, which may have a variety of forms.

Projective Transformations. Let us consider the scales of Equations (4) in the original chart in the XY-plane and compare them with the new scales of Equations (8). The latter are plotted in rectangular coordinates in an X'Y'-plane. We observe from these two sets of equations that the original scales are carried into the new ones by means of the relations

(9) $$X' = \frac{a_1X + b_1Y + c_1}{a_3X + b_3Y + c_3}, \qquad Y' = \frac{a_2X + b_2Y + c_2}{a_3X + b_3Y + c_3}.$$

These equations map points, lines, scales, etc., lying in the XY-plane upon corresponding objects in the X'Y'-plane. They transform configurations in the XY-plane into configurations in the X'Y'-plane. The particular Equations (9) define what is called a projective transformation.

This important transformation has thus arisen in a simple and natural way in the study of alignment charts. It is a powerful tool to be used in their alteration and improvement. The student should, at this point, devote serious study to the properties of the transformation to the end that he may take full advantage of its possibilities. In the process he will become acquainted with one of the most fascinating parts of geometry.

For many purposes we need the values of X and Y in terms of X' and Y'. Solving (9) we find the so-called inverse transformation:

$$(10) \qquad X = \frac{A_1X' + A_2Y' + A_3}{C_1X' + C_2Y' + C_3}, \qquad Y = \frac{B_1X' + B_2Y' + B_3}{C_1X' + C_2Y' + C_3}.$$

Here the capital letters indicate the co-factors of the corresponding small letters in the determinant D, that is,

$$A_1 = b_2c_3 - b_3c_2, \quad A_2 = b_3c_1 - b_1c_3, \quad A_3 = b_1c_2 - b_2c_1,$$
$$B_1 = a_3c_2 - a_2c_3, \quad B_2 = a_1c_3 - a_3c_1, \quad B_3 = a_2c_1 - a_1c_2,$$
$$C_1 = a_2b_3 - a_3b_2, \quad C_2 = a_3b_1 - a_1b_3, \quad C_3 = a_1b_2 - a_2b_1.$$

Equations (10) would be used to insert the values of X and Y into the equation of a curve in the XY-plane to get the equation of the transformed curve in the X'Y'-plane.

The inverse transformation (10) is itself a projective transformation. Also it is readily found that the succession of two projective transformations is a projective transformation. That is, if the XY-plane is mapped by a projective transformation on the X'Y'-plane and the X'Y'-plane is mapped by a projective transformation on the X"Y"-plane, the resulting transformation of the XY-plane on the X"Y"-plane is projective. We say that all these transformations form a group.

The Straight Line. The general straight line in the XY-plane,

$$pX + qY + r = 0,$$

is carried by the transformation (using Equations (10)) into

$$(pA_1 + qB_1 + rC_1)X' + (pA_2 + qB_2 + rC_2)Y' + pA_3 + qB_3 + rC_3 = 0.$$

We thus have the result that a straight line is carried into a straight line. It is because of this property that an alignment chart is transformed by a projective transformation into an equivalent alignment chart. Three points which lie on a straight line on the scales of one chart are carried into collinear points on the scales of the other chart.

A line of special import is

$$a_3X + b_3Y + c_3 = 0, \tag{11}$$

which we get by setting the denominators in (9) equal to zero. We dispose of this case by postulating a "line at infinity" in the X'Y'-plane into which this line goes. Similarly, from (10), there is a line

$$C_1X' + C_2Y' + C_3 = 0,$$

which corresponds to a line at infinity in the XY-plane. If $a_3 = b_3 = 0$, so that the denominator of (9) is constant, the line at infinity goes into the line at infinity and the transformation is called <u>affine</u>.

The use of infinite elements is one of the most intriguing aspects of projective geometry for the beginner. Parallel lines meet at infinity and two distinct lines thus always have one common point. We transform concurrent linear scales into parallel scales by making the line (11), pass through the point of intersection of the original scales. We can indeed transform a whole linear scale to the line at infinity, lines through a point on the scale would then merely have a specified direction.

<u>Conics</u>. The degree of a curve is unchanged by a projective transformation. That the degree is not increased is seen by substituting from (10) and simplifying, that it is not decreased follows from the fact that the inverse transformations cannot raise it again. It follows that conics are transformed into conics.

Conics are classified according to their behavior with reference to the line at infinity. An ellipse does not meet that line, a hyperbola meets it in two points, a parabola meets it in one point. What a conic is carried into by the projective transformation (9) is determined by the number of its intersections with the line (11).

The beginner will find it instructive to apply some simple transformation, such as

$$X' = \frac{1}{X}, \qquad Y' = \frac{Y}{X}$$

to familiar conics to see what results. Here the Y-axis, $X = 0$, is carried to infinity. Conics which meet the line at two points, such as

$$X^2 + Y^2 = 1, \qquad Y^2 = X + 1,$$

go into hyperbolas

$$X'^2 - Y'^2 = 1, \qquad Y'^2 - X'^2 = X'.$$

Conics which do not meet it go into ellipses, for example, the parabola

$$Y^2 = X - 1$$

is transformed into the circle

$$X'^2 + Y'^2 = X'.$$

Conics which touch the line, as the following circle, parabola, and hyperbola,

$$X^2 + Y^2 + aX = 0, \quad Y^2 = X, \quad XY = 1,$$

go into parabolas:

$$Y'^2 + aX' + 1 = 0, \quad Y'^2 = X', \quad Y' = X'^2.$$

A scale which lies on a conic - a circular scale, for instance - may be projected into one lying on any other conic whatsoever, and this may be done in infinitely many ways. Let us, for instance, transform a given conic into $Y' = X'^2$ so that three points P, Q, R, on the conic go into the end of the Y-axis, the origin, and the point (1,1) respectively. Let

$$a_3X + b_3Y + c_3 = 0$$

be the tangent at P, let

$$a_2X + b_2Y + c_2 = 0$$

be the tangent at Q, and let

$$a_1X + b_1Y + c_1 = 0$$

be the line PQ. Now the transformation

$$X' = h\,\frac{a_1X + b_1Y + c_1}{a_3X + b_3Y + c_3}, \qquad Y' = k\,\frac{a_2X + b_2Y + c_2}{a_3X + b_3Y + c_3}$$

carries the given conic into a parabola touching both the X'-axis and the line at infinity on the Y'-axis. It has then the equation $Y' = KX'^2$. If now we choose h and k so that at R we have $X' = 1$, $Y' = 1$, then the transform is $Y' = X'^2$. Since two conics can thus be transformed into the same parabola then one can be projected into the other so that three prescribed points of the first go into three prescribed points of the second. This can be done, in fact, in only one way. In particular, a conic may be thus mapped on itself. For example, a circle, in addition to being carried into itself by rotations about its center, can be projected into itself in infinitely many other ways.

The Problem of the Rectangular Page. We use the projective transformation to shift the positions of scales and to put a chart in more manageable form. We bring distant portions of a chart back on the page, we change the positions of scales so as to make best use of the space on which our chart is to be drawn. As an indication of what we can achieve we quote two theorems:

There is one and only one projective transformation which carries four given points, no three collinear, of the XY-plane into four given points, no three collinear, of the X'Y'-plane.

There is one and only one projective transformation which carries four given lines, no three concurrent, of the XY-plane into four given lines, no three concurrent, of the X'Y'-plane.

Let the portion of the chart that we wish to use be enclosed in a convex quadrilateral PQRS, as shown in Figure 7. We wish to map this on the rectangular page P'Q'R'S', as in the figure. We call this the problem of the rectangular page. It can be solved, according to the preceding theorems, in exactly one way. We give here its analytic solution.

Let PS and PQ have equations

$$a_1X + b_1Y + c_1 = 0, \qquad a_2X + b_2Y + c_2 = 0,$$

respectively. Let M,N be the points of intersection of opposite sides of the quadrilateral, and let the equation of the line MN be

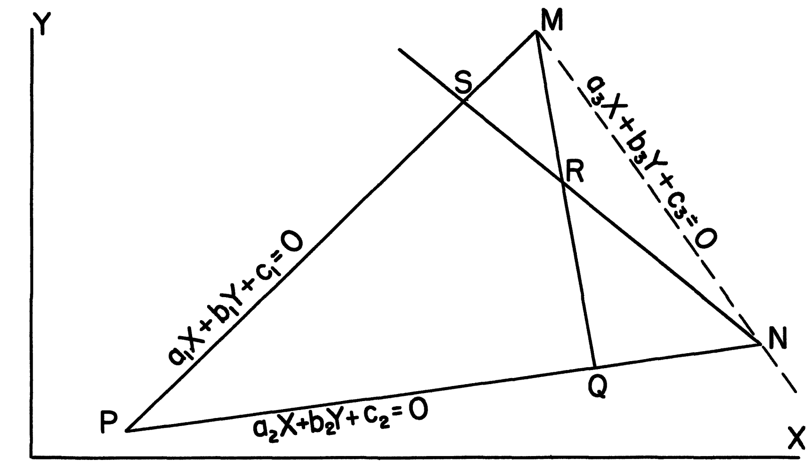
$$a_3X + b_3Y + c_3 = 0.$$

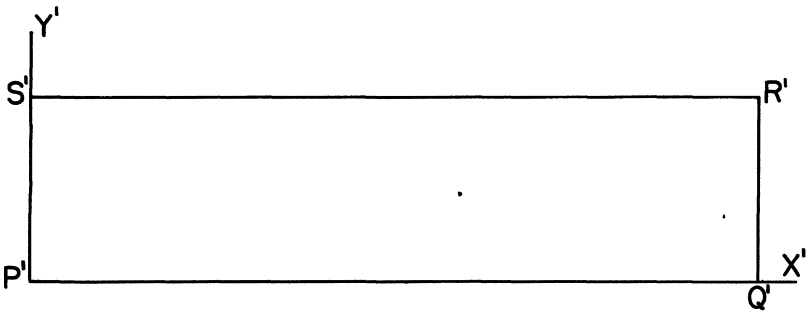
We set up the transformation

$$X' = h\,\frac{a_1X + b_1Y + c_1}{a_3X + b_3Y + c_3}, \qquad Y' = k\,\frac{a_2X + b_2Y + c_2}{a_3X + b_3Y + c_3}.$$

It is clear that the line PS goes into the line $X' = 0$, as required, and that the line PQ goes into $Y' = 0$. The line MN goes to infinity.

Since M is carried to infinity, the line QR is carried into a parallel to the Y'-axis. Similarly, since N goes to





infinity, the line SR is carried into a parallel to the X'-axis. The original quadrilateral is thus mapped on a rectangle two of whose sides lie along the axes. We have now but to choose h and k so that the remaining sides pass through the points Q' and S', and the problem is solved.

In Figure 7 the points and lines all appear in the finite plane, but this need not be the case. One or more of them may lie at infinity. This leads to no difficulty. For example, if PQ and RS are parallel but PS and QR are not, then N is at infinity and MN is a line through M parallel to PQ. If PQRS is a parallelogram then MN is the line at infinity; in this case we set $a_3 = b_3 = 0$. We handle similarly the cases in which a vertex or a side of PQRS is at infinity.

It is essential that the quadrilateral be such that the line MN, which is to be carried to infinity, should not cut across the interior of the region to be mapped on the page.

High Accuracy Charts. It was remarked in the beginning that alignment charts are valuable if great accuracy is not required. Let us see what can be done to increase the accuracy that one ordinarily expects. We could, of course, make larger and larger charts, but there are obvious objections to excessive size.

One thing that occurs to us is that there is a good deal of blank space on an alignment chart. There are ordinarily three scales surrounded by unused areas. Is it not possible to break scales into pieces and fit them into these areas with the result that we get the advantages of long scales without the disadvantages of long charts? That this is sometimes possible will be shown by an example.

We consider the case of parallel scales and the equation

$$h(z) = f(x) + g(y).$$

Let us write this in the form of a determinant as follows:

$$\begin{vmatrix} -1+2ma & f(x)-mb & 1 \\ 1+2na & g(y)-nb & 1 \\ (m+n)a & \frac{1}{2}[h(z)-(m+n)b] & 1 \end{vmatrix} = 0.$$

For $m = n = 0$, this gives the chart with three parallel scales explained earlier in this paper. Let b be the height of the page and let the original chart be so designed that the scales extend upward from the X-axis to a height p times the height of the page, where p is an integer. Taking the X-axis at the bottom of the page, we plot the scales

$$X_1 = -1 + 2ma, \qquad Y_1 = f(x) - mb,$$

for $m = 0,1,\ldots,p-1$. The long x-scale thus appears in p parts along vertical lines spaced a distance 2a apart.

Similarly, we plot

$$X_2 = 1 + 2na, \qquad Y_2 = g(y) - nb,$$

for $n = 0,1,\ldots,p-1$, getting p parallel y-scales 2a apart. The z-scales,

$$X_3 = (m+n)a, \qquad Y_3 = \frac{1}{2}[h(z) - (m+n)b],$$

$m,n = 1,2,\ldots,p-1$ are $2p-1$ in number and are spaced a units apart. In designing the chart, a is so chosen that the scales do not interfere with one another.

For a given m and n we note that

$$X_3 = \frac{1}{2}(X_1 + X_2).$$

In using the chart we need remember only one thing, namely, that the z-scale which is used with given x- and y-scales is the one lying midway between them.

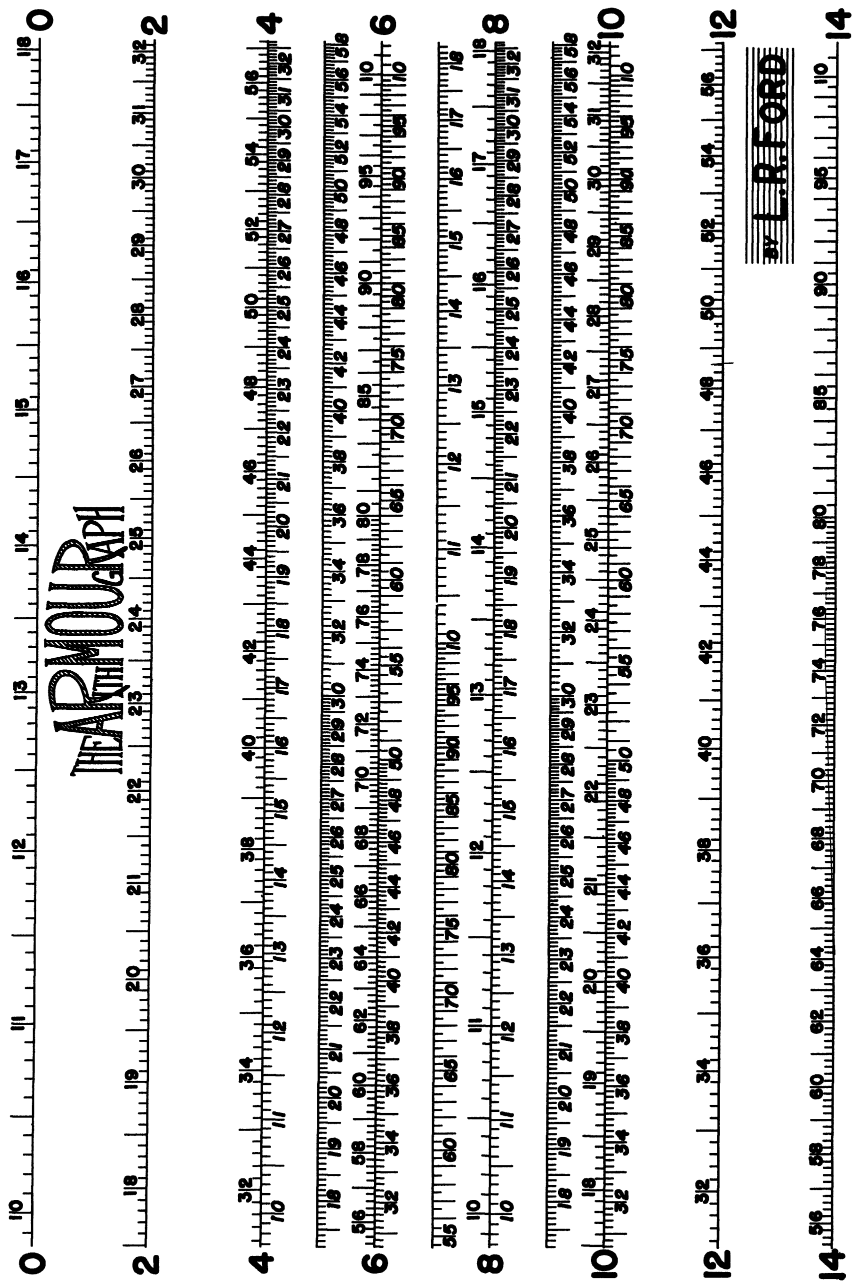
The accompanying "Arithmougraph" is a chart which is constructed on these principles. It is based on the equation

$$\log z = \log x + \log y.$$

By its use, considerable accuracy in multiplication and division is possible. The values of x appear at the tops of scales 0,2,4,6; the values of y are read on the tops of scales 8,10,12,14; the values of z are on the bottoms of scales 4 to 10.

For example, to multiply 12.13 by 11.15, we lay a line across 12.13 on scale 0 and 11.15 on the top of scale 8. This line cuts the lower scale of 4 (midway between 0 and 8) at the product 135.2.

Other types of charts will require quite different handling. These matters have been little studied, but in this author's judgment they would well repay investigation. By suitable design a device for rough calculation may be converted into an instrument of precision.


THE ARITHMOGRAPH
BY L.R. FORD