The Higher Infinite in Proof Theory

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1 Introduction

The higher infinite usually refers to the lofty reaches of Cantor's paradise, notably to the realm of large cardinals whose existence cannot be proved in the established formalisation of Cantorian set theory, i.e. Zermelo-Fraenkel set theory with the axiom of choice. Proof theory, on the other hand, is commonly associated with the manipulation of syntactic objects, that is finite objects par excellence. However, finitary proof theory became already infinitary in the 1950's when Schütte re-obtained Gentzen's ordinal analysis for number theory in a particular transparent way through the use of an infinitary proof system with the so-called ω-rule (cf. [49]). Nowadays one even finds vestiges of large cardinals in ordinal-theoretic proof theory. Large cardinals have worked their way down through generalized recursion (in the shape of recursively large ordinals) to proof theory wherein they appear in the definition procedures of so-called *collapsing functions* which give rise to ordinal representation systems. The surprising use of ordinal representation systems employing "names" for large cardinals in current proof-theoretic ordinal analyses is the main theme of this paper.

The exposition here diverges from the presentation given at the conference in two regards. Firstly, the talk began with a broad introduction, explaining the current rationale and goals of ordinal-theoretic proof theory, which take the place of the original Hilbert Program. Since this part of the talk is now incorporated in the first two sections of the BSL-paper [43] there is no point in reproducing it here. Secondly, we shall omit those parts of the talk concerned with infinitary proof systems of ramified set theory as they can also be found in [43] and even more detailed in [40]. Thirdly, thanks to the aforementioned omissions, the advantage of present paper over the talk is to allow for a much more detailed account of the actual information furnished by ordinal analyses and the role of large cardinal hypotheses in devising ordinal representation systems.

2 Observations on ordinal analyses

How are ordinals connected with formal systems? Well, this question is way more difficult to answer than "How are vector spaces measured by cardinals?"

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Since the answer is crucial to the branch of logic reported on in this paper, we shall gather together some wellknown and some not so wellknown facts. In doing so, we also aim at averting certain misconceptions about ordinal-theoretic proof theory.[2]

Furthermore, using results of [15], we characterize the provably recursive functions of theories for which an ordinal analysis has been given.

Definition 2.1 For a set X and and a binary relation $\prec$ on X, let $\mathrm{LO}(X, \prec)$ abbreviate that $\prec$ linearly orders the elements of X and that for all u, v, whenever $u \prec v$, then $u, v \in X$.

A *linear ordering* is a pair $\langle X, \prec\rangle$ satisfying $\mathrm{LO}(X, \prec)$.

Let T be a framework for formalizing a certain part of mathematics. T should be a true theory which contains a modicum of arithmetic.

Let A be a subset of $\mathbb{N}$ ordered by $\prec$ such that A and $\prec$ are both definable in the language of T. If the language of T allows for quantification over subsets of $\mathbb{N}$, like that of second order arithmetic or set theory, *well-foundedness* of $\langle A, \prec\rangle$ will be formally expressed by

$$\mathrm{WF}(A, \prec) := \forall X \subseteq \mathbb{N}\, [\forall u{\in}A(\forall v \prec u\, v{\in}X \rightarrow u{\in}X) \rightarrow \forall u{\in}A\, u{\in}X.] \quad (1)$$

If, however, the language of T does not provide for quantification over arbitrary subsets of $\mathbb{N}$, like that of Peano arithmetic, we shall assume that it contains a new unary predicate $\mathbf{U}$. $\mathbf{U}$ acts like a free set variable, in that no special properties of it will ever be assumed. We will then resort to the following formalization of well-foundedness:

$$\mathrm{WF}(A, \prec) := \forall u{\in}A(\forall v \prec u\, \mathbf{U}(v) \rightarrow \mathbf{U}(u)) \rightarrow \forall u{\in}A\, \mathbf{U}(u), \quad (2)$$

where $\forall v \prec u \ldots$ is short for $\forall v(v \prec u \rightarrow \ldots)$.

We also set

$$\mathrm{WO}(A, \prec) := \mathrm{LO}(A, \prec) \wedge \mathrm{WF}(A, \prec). \quad (3)$$

If $\langle A, \prec\rangle$ is well-founded, we use $|\prec|$ to signify its set-theoretic order-type. For $a{\in}A$, the ordering $\prec\restriction a$ is the restriction of $\prec$ to $\{x{\in}A : x \prec a\}$.

The ordering $\langle A, \prec\rangle$ is said to be *provably well-founded in* T if

$$T \vdash \mathrm{WO}(A, \prec). \quad (4)$$

The *proof-theoretic ordinal* $|T|$ of T is often defined as follows:

$$|T| := = \sup\{\alpha : \alpha \text{ provably recursive in } T\} \quad (5)$$

where an ordinal α is said to be provably recursive in T if there is a recursive well–ordering $\langle A, \prec\rangle$ with order–type α such that

$$T \vdash \mathrm{WO}(A, \prec)$$

with A and $\prec$ being provably recursive in T.

The calibration of $|T|$ is then called *ordinal analysis of* T.

[2] The present section is complementary to [43], §2.

The above definition of $|T|$ has the advantage of being mathematically precise.[3] But as to the activity named "ordinal analysis" it is left completely open what constitutes such an analysis. One often encounters this kind of sloppy talk of ordinals in proof theory. Among the uninitiated it might give the impression that the calibration of $|T|$ is akin to computing numerical invariants in other branches of mathematics, i.e. computing dimensions of vector spaces. This likening is not completely mistaken, but what is most problematic about it is that ordinals are not as easily bestowed upon us as natural numbers are. Before one can go about determining the proof-theoretic ordinal of T, one needs to be furnished with representations of ordinals. Not surprisingly, a great deal of ordinally informative proof theory has been concerned with developing and comparing particular ordinal representation systems. Moreover, to obtain the reductions of classical (non-constructive) theories to constructive ones (as related, for instance, in [12], [43],§2) it appears to be pivotal to work with very special and well-structured ordinal representation systems.

But before attempting to delineate the type of ordinal representation systems that are actually used in ordinal analyses, it should be mentioned that, in general, $|T|$ has several equivalent characterizations; though some of these hinge upon the mathematical strength of T.

Proposition 2.2 *(i) Suppose that for every elementary well-ordering $\langle A, \prec\rangle$, whenever $T \vdash \mathrm{WO}(A, \prec)$, then*

$$T \vdash \forall u\,[A(u) \to (\forall v \prec u P(v)) \to P(u)] \to \forall u\,[A(u) \to P(u)]$$

holds for all provably recursive predicates P of T. Then

$$|T| = \sup\{\alpha : \ \alpha \textit{ is provably elementary in } T\} \qquad (6)$$
$$= \sup\{\alpha : \ \alpha \textit{ is provably recursive in } T\}.$$

Moreover, if $T \vdash \mathrm{WO}(A, \prec)$ and *$A, \prec$ are provably recursive in T, then one can find an elementary well-ordering $\langle B, <\rangle$ and a recursive function f such that $T \vdash \mathrm{WO}(B, <)$, f is provably recursive in T, and T proves that f supplies an order isomorphism between $\langle B, <\rangle$ and $\langle A, \prec\rangle$.*

(ii) If T proves comparability of well-orderings, then

$$|T| = \sup\{\alpha : \ \alpha \textit{ is provably arithmetic in } T\}. \qquad (7)$$

(iii) If T proves comprehension for analytic sets of integers, i.e. lightface Σ^1_1 sets of integers, then

$$|T| = \sup\{\alpha : \ \alpha \textit{ is provably analytic in } T\}. \qquad (8)$$

[3] It even rules out some of the pathological candidates of the "dreary list" in [23], p. 334.

Proof: Probably folklore. But the only reference I know is [37]. For (ii) and (iii) see [37], Theorem 1.2 and Corollary 1.3. (i) follows by refining the proof of [37], 1.2.(ii). □

Examples for (i) are the theories $\mathbf{I\Sigma_1}$, $\mathbf{WKL_0}$ and $\mathbf{PA}$. Examples for (ii),(iii) are $\mathbf{ATR_0}$ and $\Pi^1_1 - \mathbf{CA_0}$, respectively.

Definition 2.3 *Elementary recursive arithmetic,* **ERA**, is a weak system of number theory, in a language with $0, 1, +, \times, E$ (exponentiation), $<$, whose axioms are:

1. the usual recursion axioms for $+, \times, E, <$.
2. induction on Δ_0-formulae with free variables.

ERA is referred to as elementary recursive arithmetic since its provably recursive functions are exactly the Kalmar *elementary functions,* i.e. the class of functions which contains the successor, projection, zero, addition, multiplication, and modified subtraction functions and is closed under composition and bounded sums and products (cf. [46]).

The next definition garners some features (similar to [15]) that ordinal representation systems used in proof theory always have, and collectively calls them *"elementary ordinal representation system"*. One reason for singling out this notion is that it leads to an elegant characterization of the provably recursive functions of theories equipped with transfinite induction principles for such ordinal representation systems. Furthermore, though only based on empirical facts about ordinal representation systems surfacing in proof theory, this definition can also be viewed as a first (naive) step towards answering the question: "What is a natural well-ordering?"

Definition 2.4 An *elementary ordinal representation system* (EORS) for a limit ordinal λ is a structure $\langle A, \lhd, n \mapsto \lambda_n, +, \times, x \mapsto \omega^x \rangle$ such that:

(i) A is an elementary subset of $\mathbb{N}$.
(ii) $\lhd$ is an elementary well-ordering of A.
(iii) $|\lhd| = \lambda$.
(iv) Provably in **ERA**, $\lhd \restriction \lambda_n$ is a proper initial segment of $\lhd$ for each n, and $\bigcup_n \lhd \restriction \lambda_n = \lhd$. In particular, $\mathbf{ERA} \vdash \forall y\, \lambda_y \in A \wedge \forall x \in A \exists y\, [x \lhd \lambda_y]$.
(v) $\mathbf{ERA} \vdash \mathrm{LO}(A, \lhd)$
(vi) $+, \times$ are binary and $x \mapsto \omega^x$ is unary. They are elementary functions on elementary initial segments of A. They correspond to ordinal addition, multiplication and exponentiation to base ω, respectively. The initial segments of A on which they are defined are maximal.
$n \mapsto \lambda_n$ is an elementary function.
(vii) $\langle A, \lhd, +, \times, \omega^x \rangle$ satisfies "all the usual algebraic properties" of an initial segment of ordinals. In addition, these properties of $\langle A, \lhd, +, \times, \omega^x \rangle$ can be proved in **ERA**.

(viii) Let $\tilde{n}$ denote the n^{th} element in the ordering of A. Then the correspondence $n \leftrightarrow \tilde{n}$ is elementary.

(ix) Let $\alpha = \omega^{\beta_1} + \cdots \omega^{\beta_k}, \beta_1 \geq \cdots \geq \beta_k$ (Cantor normal form). Then the correspondence $\alpha \leftrightarrow \langle \beta_1, \ldots, \beta_k \rangle$ is elementary.

Elements of A will often be referred to as *ordinals*, and denoted $\alpha, \beta, \ldots$.

In a sense the preceding definition manages to characterize natural well-orderings of order-type ε_0 as any two such well-orderings arising from EORSs are recursively isomorphic (mainly due to 2.4(ix)). Of course, this cannot be expected to hold for larger order-types.

As for the computational complexity of EORSs involved in ordinal analyses, it appears that they are even Δ_0-representable (cf. [52]). Be this as it may, ordinal analysts never expected that the peculiarities of "real" ordinal representation systems, including their naturalness, could be fathomed via complexity theory.[4] Sommer has addressed the issue at great length in [52, 53]. Here are his conclusions:

Observation 2.5 Synopsis of discussion in [52]

- *It is an empirical fact that with regard to complexity measures considered in complexity theory the ordinal representation systems emerging in proof theory are of low computational complexity and their basic properties are provable in weak fragments of arithmetic.*
  The latter includes that computations on ordinals in actual proof-theoretic ordinal analyses can also be handled in such weak theories.
- *The complexity of ordinal representation systems involved in proof-theoretic ordinal analyses cannot be described in terms of the complexity of the representations of these ordinals, but only in terms of the difficulty in recognizing the well-foundedness of these representations.*

We continue to gather information about ordinal analyses.

Definition 2.6 Suppose $\mathrm{LO}(A, \lhd)$ and $F(u)$ is a formula. Then $\mathrm{TI}_{\langle A, \lhd \rangle}(F)$ is the formula

$$\forall n \in A \, [\forall x \lhd n F(x) \rightarrow F(n)] \rightarrow \forall n \in A \, F(n). \tag{9}$$

$\mathrm{TI}(A, \lhd)$ is the schema consisting of $\mathrm{TI}_{\langle A, \lhd \rangle}(F)$ for all F.

Given a linear ordering $\langle A, \lhd \rangle$ and $\alpha \in A$ let $A_\alpha = \{\beta \in A : \beta \lhd \alpha\}$ and $\lhd_\alpha$ be the restriction of $\lhd$ to A_α.

In what follows, quantifiers and variables are supposed to range over the natural numbers. When n denotes a natural number, $\bar{n}$ is the canonical name in the language under consideration which denotes that number.

[4] Though at times they got carried away pointing out the computational complexity of their orderings as if it were their decisive feature.

Observation 2.7 *Every ordinal analysis of a classical (intuitionistic) theory* $\mathbf{T}$ *that has ever appeared in the literature provides an EORS* $\langle A, \lhd, \ldots \rangle$ *such that* $\mathbf{T}$ *and* $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ $(\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}}))$ *prove the same arithmetic sentences.*

Moreover, regardless of the underlying logic, $\mathbf{T}$ *and* $\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ *prove the same* Π^0_2 *statements.*

Proof: $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ can be interpreted in $\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ via the Gödel–Gentzen $\neg\neg$-translation. Since the theorems of $\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ are closed under the Markov-rule for primitive recursive predicates (using, for instance, Friedman's A-translation), it follows that the theories prove the the same Π^0_2 propositions, hence $\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ proves the same Π^0_2 sentences as $\mathbf{T}$. □

The latter result can be considerably improved.

Definition 2.8 For each $\alpha \in A$, $\mathrm{ERWF}(\lhd, \bar{\alpha})$ is the schema

$$\forall \mathbf{x} \exists y [f(\mathbf{x}, y) \underline{\lhd} f(\mathbf{x}, y+1) \vee f(\mathbf{x}, y) \notin A \vee \bar{\alpha} \underline{\lhd} f(\mathbf{x}, y)]$$

for each (definition of an) elementary function f.

$\mathrm{ERWF}(\lhd)$ is the schema

$$\forall \mathbf{x} \exists y [f(\mathbf{x}, y) \underline{\lhd} f(\mathbf{x}, y+1) \vee f(\mathbf{x}, y) \notin A]$$

for each elementary function f.

The schemata $\mathrm{PRWF}(\lhd, \bar{\alpha})$ and $\mathrm{PRWF}(\lhd)$ are defined identically, except that f ranges over the primitive recursive functions.

Definition 2.9 $\mathbf{DRA}_{\langle A, \lhd \rangle}$ (*Descent Recursive Arithmetic*) is the theory whose axioms are $\mathbf{ERA} + \bigcup_{\alpha \in A} \mathrm{ERWF}(\lhd, \bar{\alpha})$.

$\mathbf{DRA}(\lhd^+)$ is the theory whose axioms are $\mathbf{ERA} + \mathrm{ERWF}(\lhd)$.

The difference is that $\mathbf{DRA}(\lhd)$ asserts only the non-existence of elementary infinitely descending sequences below each $\alpha \in A$, where α is given at the meta-level.

Combined with 2.7 the latter result leads to a neat characterization of the provably recursive functions of $\mathbf{T}$ due to the following observation:

Proposition 2.10 *The provably recursive functions of* $\mathbf{DRA}_{\langle A, \lhd \rangle}$ *are all functions* f *of the form*

$$f(\mathbf{m}) = g(\mathbf{m}, \textit{least } n . h(\mathbf{m}, n) \underline{\lhd} h(\mathbf{m}, n+1)) \tag{10}$$

where g *and* h *are elementary functions and* $\mathbf{ERA} \vdash \forall \mathbf{x} y \, h(\mathbf{x}, y) \in A_{\bar{\alpha}}$ *for some* $\alpha \in A$.

The above class of recursive functions will be referred to as the *descent recursive functions over A*.

Proposition 2.11 (Friedman, Sheard [15, 4.4])
$\mathbf{DRA}_{\langle A,\lhd\rangle}$ *and* $\mathbf{PA} + \bigcup_{\alpha\in A}\mathrm{TI}(A_{\bar\alpha},\lhd_{\bar\alpha})$ *prove the same* Π^0_2 *sentences.*

From 2.7 and 2.11 we get:

Observation 2.12 *Suppose an ordinal analysis of the formal system* T *has been attained using an EORS* $\langle A,\lhd,\ldots\rangle$*. Then the provably recursive functions of* T *are the descent recursive functions over* A*.*

We shall list some complimentary results.

Definition 2.13 If T is a theory, the 1-*consistency of* T is the schema

$$\forall u[Pr_T(\ulcorner F(\dot u)\urcorner)\to F(u)]$$

for Σ^0_1 formulae $F(u)$ with one free variable u.

Theorem 2.14 (Friedman and Sheard [15, 4.5]) *The following are equivalent over* **PRA***:*

(i) 1-consistency of $\mathbf{PA} + \bigcup_{\alpha\in A}\mathrm{TI}(A_{\bar\alpha},\lhd_{\bar\alpha})$
(ii) $\mathrm{PRWF}(\lhd^+)$
(ii) $\mathrm{ERWF}(\lhd^+)$.

Observation 2.15 *Again, let* T *be a theory for which an ordinal analysis has been carried out via* $\langle A,\lhd\rangle$*. Then the following are equivalent over* **PRA***:*

(i) 1-consistency of T
(ii) $\mathrm{PRWF}(\lhd^+)$
(ii) $\mathrm{ERWF}(\lhd^+)$.

The ordinal representation systems used in ordinal analyses are distinguished by another property. Suppose **T** successfully underwent an ordinal analysis by employing an EORS $\langle A,\lhd,\ldots\rangle$. Further, assume $T\vdash \mathrm{WO}(B,\prec)$ for some elementary (or recursive) well-ordering $\langle B,\prec\rangle$. Then a question suggesting itself is whether it is possible to determine an initial segment $\lhd_\alpha$ of $\lhd$ and T-provably recursive function f such that

$$\mathbf{T}\vdash \mathbf{f}:\mathbf{B}\xrightarrow{\mathbf{1-1}}\mathbf{A}_{\bar\alpha}\wedge\forall\mathbf{x},\mathbf{y}{\in}\mathbf{B}[\mathbf{x}\prec\mathbf{y}\leftrightarrow\mathbf{f}(\mathbf{x})\lhd_{\bar\alpha}\mathbf{f}(\mathbf{y})]\,? \tag{11}$$

The content of (11) is that $\langle A,\lhd\rangle$ provides a universal measure for the provable well-orderings of T in that each such well-ordering is T-recursively embedded in an initial segment of $\lhd$.

In the case of **PA** a positive answer to (11) can be obtained from Gentzen's proof of $|\mathbf{PA}|\le\varepsilon_0$ (cf. [54, 13.4]). Fortunately, this is not the only example. The proof of the following result requires a refined analysis of infinitary derivations.

Observation 2.16 *In practice, that is to say when a reduction as in 2.7 has been attained, the answer to question (11) is "YES".*[5]

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[5] *Ordinal analyses providing reductions as in 2.7 have also been distinguished in [34], where they were christened "profound".*

A caveat is in order here. Taken in isolation, property (11) does not guarantee a meaningful ordinal notation system as the pathological example (iv) of [23], p. 334 demonstrates.

The preceding pointed out some markings of EORSs found in proof theory. Another feature that we deem more important than the ones mentioned hitherto is their versatileness in establishing equivalences between classical non-constructive theories and intuitionistic constructive theories (cf. [43]) based on radically different ontologies. Thus far we have only given a rather unsatisfying and imprecise answer to the question: "What is so particular about the ordinal representation system used in ordinal analyses?" In connection with this question, it has been suggested (cf. [23], [11]) that it is important to address the broader question of "What is a natural well-ordering?" A criterion for naturalness put forward in [23] is uniqueness up to recursive isomorphism. Furthermore, in [23], Kreisel seems to seek naturalness in algebraic characterizations of ordered structures. Feferman, in [10], discerns the properties of completeness, repleteness, relative categoricity and preservation of these under iteration of the critical process as significant features of systems of natural representation. Girard [17] appears to propose dilators to capture the abstract notion of a notation system for ordinals.

However, in the ensuing sections we shall not be particularly heedful of these suggestions and rather try to reflect on the main question from new angles.

3 Large cardinals and ordinal representation systems I

3.1 A brief history of ordinal representation systems up till the early 1980s

Several natural well-orderings that later came to be used in proof theory had arisen in a purely set-theoretic context. The Cantor normal form of ordinals with exponentiation to the base ω provides an ordinal representation system for ε_0. Veblen's work [55], whose main tools are the operations of derivation and transfinite iteration applied to continuous increasing functions of ordinals, distinguished several ordinals (e.g. Γ_0) which Feferman and Schütte then employed in their investigations on predicativity.

Still from a set-theoretic stance, Bachmann [4] utilized Veblen's methods for building hierarchies of normal functions and added the new procedure of *diagonalization.* A hierarchy of normal functions $\{\varphi_\alpha\}_{\alpha\in B}$ is defined by simultaneously defining the indexing set B such that with each limit $\alpha\in B$ is associated a fundamental sequence $\langle\alpha[\xi] : \xi < \tau_\alpha\rangle$ of ordinals $\alpha[\xi]\in B$ of length τ_α with $\alpha[\xi] < \alpha$. Depending on the type of τ_α the function φ is defined from previously defined functions by one of the procedures. Bachmann's novel idea was the systematic use of uncountable ordinals in the indexing set to keep track of the functions defined by diagonalization.

When in the sixties important proof-theoretic ordinals were located in Bachmann's system, it became the standard source of notations for ordinals required in proof theory. Bachmann's hierarchy was extended by Pfeiffer [32] and Isles [20]. By the end of the 1960s the conceptually straightforward Bachmann method had been pushed as far as it could be. Unfortunately, the dependence of the construction on fundamental sequences for each limit indexing ordinals, with certain additional "dove-tailing" properties, adds enormous complexity to the very definition of the φ_α and severely hampers their applicability in ordinal analyses.

At the end of the 1960s the definitions of ordinal representation systems were so contaminated by details that future progress of ordinal-theoretic proof theory was at stake. Fortunately, around 1970, this impasse was overcome by Feferman who, in unpublished work, made conceptional improvements in the Bachmann approach. In contrast to the definition of Bachmann-style hierarchies, Feferman's definition does not require simultaneous assignment of fundamental sequences to limit ordinals. The definition of the φ_α's is uniform for all α since it does not hinge on a previous assignment of cofinality type τ_α to α.

The new approach was carried out and pushed further by Aczel, Weyhrauch, Bridge and Buchholz (cf. [11]) in the early 1970s. Considerable conceptual improvements and extensions of ordinal representation systems in the late 1970s and early 1980s are due to Buchholz, Jäger, Pohlers and Schütte (cf. [33]).

In this section we shall exhibit three ordinal representation systems which featured in ordinal analyses of extensions of Kripke-Platek set theory from around 1980 on, the first one being an epitome of the finale of the history reported above. Their respective definition procedures make use of weakly inaccessible, weakly Mahlo and weakly compact cardinals. Our objective is to show how large cardinal assumptions are actually employed for devising ordinal representation systems, also with the intention to rectify certain opinions held about ordinal representation systems. Such systems are by no means cooked up or impenetrable. As a rule, they utilize and extend wellknown set-theoretic hierarchies, for instance Mahlo's π- and ρ-number hierarchies [24].

3.2 Ordinal functions based on a weakly inaccessible cardinal

KPi is a set theory which originates from Kripke-Platek set theory and in addition has an axiom which says that any set is contained in an admissible set. Thus the standard models of **KPi** in **L** are the segments $\mathbf{L}_\kappa$ with κ recursively inaccessible. The ordinal analysis for **KPi** (cf. [21]) used an EORS built from ordinal functions which had originally been defined with the help of a weakly inaccessible cardinal. In this subsection we expound on the development of this particular EORS with an eye towards the role of cardinals therein.

Let

$$\mathbf{I} := \text{"first weakly inaccessible cardinal"} \tag{12}$$

and let

$$(\alpha \mapsto \Omega_\alpha)_{\alpha<\mathbf{I}} \tag{13}$$

be a function that enumerates the cardinals below **I**. Further let

$$\Re^{\mathbf{I}} := \{\mathbf{I}\} \cup \{\Omega_{\xi+1} : \xi < \mathbf{I}\}. \tag{14}$$

Variables κ, π will range over $\Re^{\mathbf{I}}$.

Definition 3.1 An ordinal representation system for the analysis of **KPi** can be derived from the following functions and Skolem hulls of ordinals defined by recursion on α:

$$C^{\mathbf{I}}(\alpha,\beta) = \begin{cases} \textit{closure of } \beta \cup \{0,\mathbf{I}\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^{\xi}) \\ (\xi \mapsto \Omega_{\xi})_{\xi<\mathbf{I}} \\ (\xi\pi \longmapsto \psi^{\xi}(\pi))_{\xi<\alpha} \end{cases} \tag{15}$$

$$\psi^{\alpha}(\pi) \simeq \min\{\rho < \pi : C^{\mathbf{I}}(\alpha,\rho) \cap \pi = \rho \ \wedge \ \pi \in C^{\mathbf{I}}(\alpha,\rho)\}. \tag{16}$$

Note that if $\rho = \psi^{\alpha}(\pi)$, then $\psi^{\alpha}(\pi) < \pi$ and $[\rho,\pi) \cap C^{\mathbf{I}}(\alpha,\rho) = \emptyset$, thus the order-type of the ordinals below π which belong to the Skolem hull $C^{\mathbf{I}}(\alpha,\rho)$ is ρ . In more pictorial terms, ρ is the α^{th} *collapse* of π.

Lemma 3.2 *If $\pi \in C^{\mathbf{I}}(\alpha,\pi)$, then $\psi^{\alpha}(\pi)$ is defined; in particular $\psi^{\alpha}(\pi) < \pi$.*

Proof: Note first that for a limit ordinal λ,

$$C^{\mathbf{I}}(\alpha,\lambda) = \bigcup_{\xi<\lambda} C^{\mathbf{I}}(\alpha,\xi)$$

since the right hand side is easily shown to be closed under the clauses that define $C^{\mathbf{I}}(\alpha,\lambda)$. Thus we can pick $\omega \leq \eta < \pi$ such that $\pi \in C^{\mathbf{I}}(\alpha,\eta)$. Now define

$$\begin{aligned} \eta_0 &= \sup C^{\mathbf{I}}(\alpha,\eta) \cap \pi \\ \eta_{n+1} &= \sup C^{\mathbf{I}}(\alpha,\eta_n) \cap \pi \\ \eta^* &= \sup_{n<\omega} \eta_n . \end{aligned} \tag{17}$$

Since the cardinality of $C^{\mathbf{I}}(\alpha,\eta)$ is the same as that of η and therefore less than π, the regularity of π implies that $\eta_0 < \pi$. By repetition of this argument one obtains $\eta_n < \pi$, and consequently $\eta^* < \pi$. The definition of η^* then ensures

$$C^{\mathbf{I}}(\alpha,\eta^*) \cap \pi = \bigcup_n C^{\mathbf{I}}(\alpha,\eta_n) \cap \pi \ = \ \eta^* < \pi.$$

Therefore, $\psi^{\alpha}(\pi) < \pi$. □

Let $\varepsilon_{\mathbf{I}+1}$ be the least ordinal $\alpha > \mathbf{I}$ such that $\omega^{\alpha} = \alpha$. The next definition singles out a subset $\mathcal{T}(\mathbf{I})$ of $C^{\mathbf{I}}(\varepsilon_{\mathbf{I}+1},0)$ which gives rise to an ordinal representation system, i.e., there is an elementary ordinal representation system $\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$, so that

$$\langle \mathcal{T}(\mathbf{I}), <, \Re, \psi, \ldots \rangle \ \cong \ \langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle. \tag{18}$$

"..." is supposed to indicate that more structure carries over to the ordinal representation system.

Definition 3.3 $\mathcal{T}(\mathbf{I})$ is defined inductively as follows:

1. $0, \mathbf{I} \in \mathcal{T}(\mathbf{I})$.
2. If $\alpha_1, \ldots, \alpha_n \in \mathcal{T}(\mathbf{I})$ and $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > \alpha_1 \geq \ldots \geq \alpha_n$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \in \mathcal{T}(\mathbf{I})$.
3. If $\alpha \in \mathcal{T}(\mathbf{I})$, $0 < \alpha < \mathbf{I}$ and $\alpha < \Omega_\alpha$, then $\Omega_\alpha \in \mathcal{T}(\mathbf{I})$.
4. If $\alpha, \pi \in \mathcal{T}(\mathbf{I})$, $\pi \in C^{\mathbf{I}}(\alpha, \pi)$ and $\alpha \in C^{\mathbf{I}}(\alpha, \psi^\alpha(\pi))$, then $\psi^\alpha(\pi) \in \mathcal{T}(\mathbf{I})$.

The side conditions in 3.3.2, 3.3.3 are easily explained by the desire to have unique representations in $\mathcal{T}(\mathbf{I})$. The requirement $\alpha \in C^{\mathbf{I}}(\alpha, \psi^\alpha(\pi))$ in 3.3.4 also serves the purpose of unique representations (and more) but is probably a bit harder to explain. The idea here is that from $\psi^\alpha(\pi)$ one should be able to retrieve the stage (namely α) where it was generated. This is reflected by $\alpha \in C^{\mathbf{I}}(\alpha, \psi^\alpha(\pi))$.

It can be shown that the foregoing definition of $\mathcal{T}(\mathbf{I})$ is deterministic, that is to say every ordinal in $\mathcal{T}(\mathbf{I})$ is generated by the inductive clauses of 3.3 in exactly one way. As a result, every $\gamma \in \mathcal{T}(\mathbf{I})$ has a unique representation in terms of symbols for $0, \mathbf{I}$ and function symbols for $+, (\alpha \mapsto \Omega_\alpha), (\alpha, \pi \mapsto \psi^\alpha(\pi))$. Thus, by taking some primitive recursive (injective) coding function $\ulcorner \cdots \urcorner$ on finite sequences of natural numbers, we can code $\mathcal{T}(\mathbf{I})$ as a set of natural numbers as follows:

$$\ell(\alpha) = \begin{cases} \ulcorner 0, 0 \urcorner & \text{if } \alpha = 0 \\ \ulcorner 1, 0 \urcorner & \text{if } \alpha = \mathbf{I} \\ \ulcorner 2, \ell(\alpha_1), \cdots, \ell(\alpha_n) \urcorner & \text{if } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \\ \ulcorner 3, \ell(\beta) \urcorner & \text{if } \alpha = \Omega_\beta \\ \ulcorner 4, \ell(\beta), \ell(\pi) \urcorner & \text{if } \alpha = \psi^\beta(\pi), \end{cases}$$

where the distinction by cases refers to the unique representation of 3.3. With the aid of ℓ, the ordinal representation system of (18) can be defined by letting $\mathcal{OR}$ be the image of ℓ and setting $\lhd := \{(\ell(\gamma), \ell(\delta)) : \gamma < \delta \wedge \delta, \gamma \in \mathcal{T}(\mathbf{I})\}$ etc. However, for a proof that this definition of $\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$ in point of fact furnishes an elementary ordinal representation system, we have to refer to the literature (cf. [6, 7, 42]).

3.3 Ordinal functions based on a weakly Mahlo cardinal

In a paper from 1911 Mahlo [24] investigated two hierarchies of regular cardinals. In view of its early appearance this work is astounding for its refinement and its audacity in venturing into the higher infinite. Mahlo called the cardinals considered in the first hierarchy π_α*-numbers*. In modern terminology they are spelled out as follows:

κ is 0-*weakly inaccessible* iff κ is regular;

κ is $(\alpha+1)$-*weakly inaccessible* iff κ is a regular limit of α-weakly inaccessibles

κ is λ-*weakly inaccessible* iff κ is α-weakly inaccessible for every $\alpha < \lambda$

for limit ordinals λ. This hierarchy could be extended through diagonalization, by taking next the cardinals κ such that κ is κ-weakly inaccessible and after that choosing regular limits of the previous kind etc.

Mahlo also discerned a second hierarchy which is generated by a principle superior to taking regular fixed-points. Its starting point is the class of ρ_0-numbers which later came to be called *weakly Mahlo cardinals*. Weakly Mahlo cardinals are larger than any of those that can be obtained by the above processes from below. Remarkably, Gaifman [16] showed that in a mathematical precise sense a weakly Mahlo cardinal is the least upper bound of diagonalizing the regular fixed-point operation from below.

Here we shall define an extension of Mahlo's π-hierarchy by using ordinals above a weakly Mahlo to keep track of diagonalization.

The resulting EORS of [35] has been used in [36] to give an ordinal analysis of **KPM**. **KPM** is an extension of **KPi** by a schema stating that for every Σ_1-definable (class) function there exists an admissible set closed under this function. Its canonical models are the sets $\mathbf{L}_\mu$ with μ recursively Mahlo.

Let

$$\mathbf{M} := \text{first weakly Mahlo cardinal} \tag{19}$$

and set

$$\Re^{\mathbf{M}} := \{\pi < \mathbf{M} : \pi \text{ regular}, \pi > \omega\}. \tag{20}$$

Variables κ, π will range over $\Re^{\mathbf{M}}$.

Definition 3.4 An ordinal representation system for the analysis of **KPM** can be derived from the following functions and Skolem hulls of ordinals, defined by recursion on α:

$$C^{\mathbf{M}}(\alpha,\beta) = \begin{cases} \textit{closure of } \beta \cup \{0, \mathbf{M}\} \\ \textit{under:} \\ +, (\xi \longmapsto \omega^\xi) \\ (\xi\delta \mapsto \chi^\xi(\delta))_{\xi<\alpha} \\ (\xi\pi \longmapsto \psi^\xi(\pi))_{\xi<\alpha} \end{cases} \tag{21}$$

$$\chi^\alpha(\delta) \simeq \delta^{th} \text{ regular } \pi < \mathbf{M} \text{ s.t. } C^{\mathbf{M}}(\alpha,\pi) \cap \mathbf{M} = \pi \tag{22}$$

$$\psi^\alpha(\pi) \simeq \min\{\rho < \pi : \ C^{\mathbf{M}}(\alpha,\rho) \cap \pi = \rho \ \wedge \ \pi \in C^{\mathbf{M}}(\alpha,\rho)\}. \tag{23}$$

Lemma 3.5 *For all α,*

$$\chi^\alpha : \mathbf{M} \to \mathbf{M}$$

i.e. χ^α is a total function on $\mathbf{M}$.

Proof: Set

$$X_\alpha := \{\rho < \mathbf{M} : C^{\mathbf{M}}(\alpha,\rho) \cap \mathbf{M} = \rho\}.$$

We want to show that X_α is closed and unbounded in $\mathbf{M}$. As $\mathbf{M}$ is weakly Mahlo the latter will imply that X_α contains $\mathbf{M}$-many regular cardinals, ensuring that χ^α is total on $\mathbf{M}$.

Unboundedness: Given $\eta < \mathbf{M}$, define

$$\begin{aligned} \eta_0 &= \sup(C^{\mathbf{M}}(\alpha, \eta+1) \cap \mathbf{M}) \\ \eta_{n+1} &= \sup(C^{\mathbf{M}}(\alpha, \eta_n) \cap \mathbf{M}) \\ \eta^* &= \sup_n \eta_n. \end{aligned}$$

One easily verifies $C^{\mathbf{M}}(\alpha, \eta^*) \cap \mathbf{M} = \eta^*$. Hence, $\eta < \eta^*$ and $\eta^* \in X_\alpha$.

Closedness: If $X_\alpha \cap \lambda$ is unbounded in a limit $\lambda < \mathbf{M}$, then

$$C^{\mathbf{M}}(\alpha, \lambda) = \bigcup_{\xi \in X_\alpha \cap \lambda} C^{\mathbf{M}}(\alpha, \xi),$$

whence

$$C^{\mathbf{M}}(\alpha, \lambda) \cap \mathbf{M} = \sup\{\xi : \xi \in X_\alpha \cap \lambda\} = \lambda,$$

verifying $\lambda \in X_\alpha$. □

For a comparison with Mahlo's π_α numbers let $\mathbf{I}_\alpha$ be the function that enumerates, monotonically, the α-weakly inaccessibles. Neglecting finitely many exceptions, the function $\mathbf{I}_\alpha$ enumerates Mahlo's π_α numbers.

Proposition 3.6 *Set $\Lambda^* :=$ least $\pi. \mathbf{I}_\pi(0) = \pi$.*

(i) $\forall \alpha < \Lambda^* \forall \xi < \mathbf{M}\ \mathbf{I}_\alpha(\xi) = \chi^\alpha(\xi)$
(ii) $\Lambda^* = \chi^{\mathbf{M}}(0)$.
(iii) For all $\alpha < \mathbf{M}$, a sufficient condition for $\mathbf{I}_\alpha(\xi) = \chi^\alpha(\xi)$ to hold is $\omega^{\alpha+1} \leq \xi < \mathbf{M}$.
(iv) The diagonal set $\{\kappa < \mathbf{M} :\ \kappa$ is κ-weakly inaccessible$\}$ is enumerated by the function $(\alpha \mapsto \chi^{\mathbf{M}}(\alpha))_{\alpha < \mathbf{M}}$.

Ever higher levels of diagonalizations are obtained by the functions χ^{M^M}, $\chi^{M^{M^M}}$, etc.

The preceding gives rise to an EORS $\mathcal{T}(\mathbf{M})$ (similarly as sketched for $\mathcal{T}(\mathbf{I})$) which is essentially order isomorphic to $C^{\mathbf{M}}(\varepsilon_{\mathbf{M}+1}, 0)$. This EORS exactly captures the strength of **KPM**.

3.4 Ordinal functions based on a weakly compact cardinal

Here we shall venture much further, assuming the existence of a weakly compact cardinal. The original impetus was to find an ordinal representation system strong enough for the ordinal analysis of **KP** + Π_3-Reflection (cf. [42]). By *Π_3-Reflection* we mean the schema

$$\phi \rightarrow \exists z\, [\text{“}z \text{ transitive”} \wedge z \neq \emptyset \wedge \phi^z]$$

where ϕ is a set-theoretic Π_3-formula and ϕ^z is the result of restricting all quantifiers to z.

A limit ordinal κ is said to be *Π_3-reflecting* if $\mathbf{L}_\kappa \models \Pi_3$-Reflection.

The connection of weak compactness with Π_3-Reflection was established by Richter and Aczel [45]. The first step to evince this analogy consists in an interesting characterization of the notion of weak compactness (or Π^1_1-Indescribability) in terms of higher type operations.

Let $F : {}^{\kappa}\kappa \to {}^{\kappa}\kappa$. F is *κ-bounded* if for every $f : \kappa \to \kappa$ and $\xi < \kappa$, the value $F(f)(\xi)$ is determined by less than κ values of f, i.e.

$$\forall f \in {}^{\kappa}\kappa\, \exists \gamma < \kappa\, \forall g \in {}^{\kappa}\kappa\, [g \restriction \gamma = f \restriction \gamma \to F(f)(\xi) = F(g)(\xi)].$$

$0 < \alpha < \kappa$ is a *witness for F* if for every $f : \kappa \to \kappa$,

$$f''\alpha \subseteq \alpha \to F(f)''\alpha \subseteq \alpha.$$

Definition 3.7 $\kappa > 0$ is *2-regular* if every κ-bounded $F : {}^{\kappa}\kappa \to {}^{\kappa}\kappa$ has a witness.

Theorem 3.8 ([45], Theorem 1.14) (**ZFC**) *κ is 2-regular iff κ is weakly compact.*

2-regularity has a straightforward analogue in terms of recursion theory on ordinals (cf. [5, 19]). Let κ be an admissible ordinal. A partial function $f \subseteq \kappa \times \kappa$ is said to be *partial κ-recursive* if its graph is *κ-recursively enumerable*, i.e. Σ_1-definable over L_κ (where L_κ denotes the κ^{th} level of Gödel's constructible hierarchy). The partial κ-recursive functions can be parametrized by a κ-recursively enumerable predicate of three arguments, with indices from the ordinals $< \kappa$ (cf. [5], V.4.6 or [47], VII,1.9). In the following definition we write $\{\xi\}_\kappa$ to denote the κ-recursive partial function with index ξ, and write $\{\xi\}_\kappa : \kappa \to \kappa$ to mean that $\{\xi\}_\kappa$ is total on κ.

Definition 3.9 Let κ be an admissible ordinal and $\xi < \kappa$. *$\{\xi\}_\kappa$ maps κ-recursive functions to κ-recursive functions* if

$$\forall \beta < \kappa \left[\{\beta\}_\kappa : \kappa \to \kappa \to \{\{\xi\}_\kappa(\beta)\}_\kappa : \kappa \to \kappa\right].$$

Suppose $\{\xi\}_\kappa$ maps κ-recursive functions to κ-recursive functions. An admissible $\pi < \kappa$ is a *witness for ξ* if $\xi < \pi$ and $\{\xi\}_\pi$ maps π-recursive functions to π-recursive functions.

An admissible κ is *2-admissible* if every $\xi < \kappa$ such that $\{\xi\}_\kappa$ maps κ-recursive functions to κ-recursive functions has a witness.

The next result gives the final link for the analogy.

Theorem 3.10 ([45], Theorem 1.16) *κ is 2-admissible iff κ is Π_3-reflecting.*

Turning back to the main objective of this subsection, we recall Mahlo's second method of generating large cardinals, the ρ-numbers (cf. [24, 25, 26, 16]).

Definition 3.11 Mahlo formulated his ρ numbers by using an operation which is now known as *Mahlo's operation:*

$$M(X) = \{\alpha \in X : X \cap \alpha \text{ is stationary in } \alpha\}.$$

The ρ_α-numbers are obtained by iterating this process:

$$\begin{aligned}
&\kappa \text{ is } 0\text{-}\textit{weakly Mahlo} \text{ iff } \kappa \text{ is regular;}\\
&\kappa \text{ is } (\alpha+1)\text{-}\textit{weakly Mahlo} \text{ iff } \{\tau < \kappa : \tau \text{ is } \alpha\text{-weakly Mahlo}\} \text{ is stationary in } \kappa\\
&\kappa \text{ is } \lambda\text{-}\textit{weakly Mahlo} \text{ iff } \kappa \text{ is } \alpha\text{-weakly Mahlo for every } \alpha < \lambda
\end{aligned}$$

for limit ordinals λ.

Proceeding similarly as with Mahlo's first hierarchy, we shall locate the ρ-number in a hierarchy based on the first weakly compact cardinal. Let

$$\mathbf{K} := \text{first weakly compact cardinal.} \tag{24}$$

Definition 3.12 defined by recursion on α:

$$C^{\mathbf{K}}(\alpha,\beta) = \begin{cases} \textit{closure of } \beta \cup \{0,\mathbf{K}\} \\ \textit{under:} \\ +, (\xi \longmapsto \omega^\xi) \\ (\xi\delta \mapsto \Xi^\xi(\delta))_{\xi<\alpha} \\ (\xi\sigma\pi \longmapsto \Psi^\xi_\sigma(\pi))_{\sigma\leq\xi<\alpha} \end{cases} \tag{25}$$

$$\mathbf{M}^0 = \{\rho < \mathbf{K} : C^{\mathbf{K}}(0,\rho) \cap \mathbf{K} = \rho\} \tag{26}$$

and for $\alpha > 0$:

$$\mathbf{M}^\alpha = \left\{ \pi < \mathbf{K} : \begin{array}{l} C^{\mathbf{K}}(\alpha,\pi) \cap \mathbf{K} = \pi \;\; \wedge \;\; \pi \text{ regular } \wedge \\ (\forall \xi \in C^{\mathbf{K}}(\alpha,\pi) \cap \alpha)(\mathbf{M}^\xi \text{ is stationary in } \pi) \end{array} \right\} \tag{27}$$

$$\Xi^\alpha(\delta) \simeq \delta^{th} \text{ element of } \mathbf{M}^\alpha \tag{28}$$

$$\Psi^\alpha_\beta(\pi) \simeq \min\{\rho \in \mathbf{M}^\beta \cap \pi : \; C^{\mathbf{K}}(\alpha,\rho) \cap \pi = \rho \wedge \pi \in C^{\mathbf{K}}(\alpha,\rho)\} \tag{29}$$

providing $\beta \leq \alpha$ and π is regular and $\omega < \pi < \mathbf{K}$.

The sets $\mathbf{M}^\alpha$ are related to Mahlo's hierarchy as follows:

$$\begin{aligned}
\mathbf{M}^0 &= \varepsilon\text{-numbers below } \mathbf{K} \\
\mathbf{M}^1 &= \text{regular cardinals} > \omega \text{ below } \mathbf{K} \\
\mathbf{M}^2 &= \text{weakly Mahlo cardinals below } \mathbf{K} \\
\mathbf{M}^3 &= 2\text{-weakly Mahlo cardinals below } \mathbf{K} \\
&\;\vdots \\
\mathbf{M}^\alpha &= \alpha\text{-weakly Mahlo cardinals below } \mathbf{K} \\
&\;\vdots \\
\mathbf{M}^{\mathbf{K}} &= \{\kappa < \mathbf{K} : \kappa \text{ is } \kappa\text{-weakly Mahlo}\}
\end{aligned} \tag{30}$$

where $\omega \leq \alpha < \text{least}\rho.\rho \in \mathbf{M}^{\mathbf{K}}$.

Let $V = \bigcup_\alpha V_\alpha$ be the cumulative hierarchy of sets, i.e. $V_0 = \emptyset$, $V_{\alpha+1} = \{X : X \subseteq V_\alpha\}$ and $V_\lambda = \bigcup_{\xi<\lambda} V_\xi$ for limit ordinals λ.

Theorem 3.13 *For all $\alpha < \varepsilon_{\mathbf{K}+1}$, $\mathbf{M}^\alpha$ is stationary in* **K** *and hence $\Xi^\alpha(\delta)$ is defined for all $\delta < \mathbf{K}$.*

Proof: Each ordinal $\mathbf{K} < \beta < \varepsilon_{\mathbf{K}+1}$ has a unique representation of the form $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$ with $\beta > \beta_1 \geq \cdots \geq \beta_n$ and $n > 0$, denoted $\beta =_{NF} \omega^{\beta_1} + \cdots + \omega^{\beta_n}$. Due to uniqueness, we can define an injective mapping

$$f : \varepsilon_{\mathbf{K}+1} \longrightarrow L_{\mathbf{K}}$$

by letting[6]

$$f(\beta) = \begin{cases} \beta & \text{if } \beta < \mathbf{K} \\ \{1\} & \text{if } \beta = \mathbf{K} \\ \langle 2, f(\beta_1), \ldots, f(\beta_n)\rangle & \text{if } \beta =_{NF} \omega^{\beta_1} + \cdots + \omega^{\beta_n} \text{ and } \mathbf{K} < \beta. \end{cases}$$

Putting

$$f(\alpha) \lhd f(\beta) :\Longleftrightarrow \alpha < \beta,$$

$\lhd$ defines a well–ordering on a subset of $L_{\mathbf{K}}$ of order type $\varepsilon_{\mathbf{K}+1}$.

To show the Theorem, we proceed by induction on α, or, equivalently, by induction on $\lhd$.

For any set E that is closed and unbounded in **K**, we have to verify that $\mathbf{M}^\alpha \cap E \neq \emptyset$. Using the induction hypothesis, for all $\beta < \alpha$, $\mathbf{M}^\beta$ is stationary in **K**. Define

$$U_1 := \{f(\alpha)\}, \quad U_2 := \{\langle x, y\rangle : x \lhd y\}, \text{ and } U_3 := \bigcup_{\beta<\alpha} (\mathbf{M}^\beta \times \{f(\beta)\}).$$

In what follows,

- **fun**(G) abbreviates that G is a function;
- **dom**(G), **ran**(G) denote the domain and the range of G, respectively.
- **pow**(a) denotes the powerset of a;
- **club**(X) says that X is a closed and unbounded class.
- $G''x$ is the set $\{G(y) : y \in x\}$.

The following sentences are satisfied in the structure $\langle V_{\mathbf{K}}, \in, U_1, U_2, U_3, E\rangle$:

(1) $\forall G \forall \delta[\mathbf{fun}(G) \wedge \mathbf{dom}(G) = \delta \wedge \mathbf{ran}(G) \subseteq On \rightarrow \exists\gamma(G''\delta \subseteq \gamma)]$
(2) $\forall a \exists b \exists \beta \exists g[b = \mathbf{pow}(a) \wedge \mathbf{fun}(g) \wedge \mathbf{dom}(g) = b \wedge \mathbf{ran}(g) = \beta \wedge g \text{ injective}]$
(3) $U_1 \neq \emptyset \wedge \forall\gamma\exists\delta[\gamma < \delta \wedge \delta \in E]$
(4) $\forall X \forall s \forall t[t \in U_1 \wedge \langle s, t\rangle \in U_2 \wedge \mathbf{club}(X) \rightarrow \{y : \langle y, s\rangle \in U_3\} \cap X \neq \emptyset]$

Employing the Π^1_1–indescribability of **K**, there exists $\pi < \mathbf{K}$ such that the structure

$$\langle V_\pi, \in, U_1 \cap \pi, U_2 \cap \pi, U_3 \cap \pi, E \cap \pi\rangle$$

satisfies:

[6] $\langle x, y\rangle := \{\{x\}, \{x, y\}\}$; $\langle x_1, \ldots, x_{n+1}\rangle := \langle\langle x_1, \ldots, x_n\rangle, x_{n+1}\rangle$ for $n > 2$.

(a) $\forall G \forall \delta[\mathbf{fun}(G) \wedge \mathbf{dom}(G) = \delta \wedge \mathbf{ran}(G) \subseteq On \ \rightarrow \ \exists\gamma(G''\delta \subseteq \gamma)]$

(b) $\forall a \exists b \exists \beta \exists g[b = \mathbf{pow}(a) \wedge \mathbf{fun}(g) \wedge \mathbf{dom}(g) = b \wedge \mathbf{ran}(g) = \beta \wedge g$ injective]

(c) $U_1 \cap \pi \neq \emptyset \ \wedge \ \forall\gamma\exists\delta(\gamma < \delta \ \wedge \ \delta \in E \cap \pi)$

(d) $\forall X \forall s \forall t[t \in U_1 \cap \pi \wedge (\langle s, t\rangle) \in U_2 \cap \pi \wedge \mathbf{club}(X) \ \rightarrow \{y : \langle y, s\rangle \in U_3 \cap \pi\} \cap X \neq \emptyset]$

By virtue of (a), observing that $\forall G$ is second order, and (b), π must be inaccessible. Due to (c), $f(\alpha) \in V_\pi$ and E is unbounded in π; whence $\pi \in E$. (d) ensures that

$$(*) \ \ (\forall\beta < \alpha)[f(\beta) \in V_\pi \rightarrow \mathbf{M}^\beta \text{ stationary in } \pi].$$

Next, we want to verify

$$(+) \ \ (\forall\eta \in C^{\mathbf{K}}(\alpha, \pi))[f(\eta) \in V_\pi].$$

Set $X := \{\eta \in C^{\mathbf{K}}(\alpha, \pi) : f(\eta) \in V_\pi\}$. Clearly, $\pi \cup \{0, \mathbf{K}\} \subseteq X$.

If $\eta =_{NF} \omega^{\eta_1} + \cdots + \omega^{\eta_n}$ and $\eta_1, \ldots, \eta_n \in X$, then $\eta \in X$ since π is closed under $+$ and $\zeta \mapsto \omega^\zeta$ and V_π is closed under $\langle \cdot, \cdot \rangle$.

If $\beta \in X \cap \alpha$, then, according to $(*)$, $\mathbf{M}^\beta$ is stationary in π, yielding

$$\Xi^\beta(\delta) = f(\Xi^\beta(\delta)) < \pi \text{ for all } \delta \in X \cap \mathbf{K}.$$

If $\kappa, \xi, \delta \in X$ und $\xi \leq \delta < \alpha$, then $f(\kappa) = \kappa < \pi$ and therefore $\Psi^\xi_\kappa(\delta) < \pi$. So it turns out that X enjoys all the closure properties defining $C^{\mathbf{K}}(\alpha, \pi)$. This verifies $(+)$. Using $(*)$ and $(+)$, we obtain

$$(\forall\beta \in C^{\mathbf{K}}(\alpha, \pi) \cap \alpha)[\mathbf{M}^\beta \text{ is stationary in } \pi].$$

Whence, $\pi \in \mathbf{M}^\alpha \cap E$. □

The desired EORS, which encapsulates the strength of $\mathbf{KP} + \Pi_3$-Reflection, is essentially isomorphic to $\langle C^{\mathbf{K}}(\varepsilon_{\mathbf{K}+1}, 0), < \rangle$.

4 Recursively large ordinals and ordinal representation systems

The previous section gave ample examples of how large cardinal hypotheses enter the definition procedures of collapsing functions. The latter are then employed in the shape of terms to "name" a countable set of ordinals, and when one succeeds in establishing recursion relations for the ordering between those terms, the set of terms gives rise to an ordinal representation system. It has long been suggested (cf. [11], p. 436) that, instead, one should be able to interpret the collapsing functions as operating directly on the recursively large counterparts of those cardinals. For example, taking such an approach in Definition 3.1 would consist in letting

$$\mathbf{I} := \text{first recursively inaccessible ordinal}$$

and conceiving of $\alpha \mapsto \Omega_\alpha$ as enumerating the admissible ordinals and their limits. The difficulties with this approach arise with the proof of Lemma 3.2. One wants to show that for any admissible π satisfying $\pi \in C^{\mathbf{I}}(\alpha, \pi)$, one has $\psi_\pi(\alpha) < \pi$. In the cardinal setting this comes down to a simple cardinality argument. To get a similar result for an admissible π one would have to work solely with π-recursive operations. How this can be accomplished is far from being clear as the definition of $C^{\mathbf{I}}(\alpha, \rho)$ for $\rho < \pi$ usually refers to higher admissibles than just π. Notwithstanding that, the admissible approach is workable as was shown in [39, 41, 48]. A key idea therein is that the higher admissibles which figure in the definition of $\psi_\pi(\alpha)$ can be mimicked via names within the structure $\mathbf{L}_\pi$ in a π-recursive manner.

The drawback of the admissible approach is that it involves quite horrendous definition procedures and computations, which when taken as the first approach are at the limit of human tolerance.

On the other hand, the admissible approach provides a natural semantics for the terms in the EORSs. Recalling the notion of *good Σ_1-definition* from admissible set theory (see [5], II.5.13), given a set theory T, we say that an ordinal α has a *good Σ_1-definition in T* if there is a Σ_1-formula $\phi(u)$ such that

$$\mathbf{L}_{\mathbf{I}} \models \phi[\alpha] \text{ and } T \vdash \exists! x \phi(x).$$

In case of **KP** it turns out that all the ordinals of the corresponding EORS possess a good Σ_1-definition in **KP** (cf. [38]). As for **KPi**, the admissible approach canonically associates with each ordinal $\alpha \in \mathcal{T}(\mathbf{I}) \cap \mathbf{I}$ a good Σ_1-definition in **KPi**. However, via this interpretation $\mathcal{T}(\mathbf{I}) \cap \mathbf{I}$ only forms a proper subset of the **KPi**-definable ordinals. Therefore, to illuminate the nature of the ordinals in $\mathcal{T}(\mathbf{I})$, it would be desirable to find another property which distinguishes them within the **KPi**-definable ordinals.

In the above **KPi** just served the purpose of an example for a general phenomenon. The same considerations apply to **KPM** etc.

5 Large Cardinals and ordinal representation systems II

This section is devoted to the strongest large cardinal notions that have been used in developing ordinal representation systems. These cardinals exhibit strong indescribability properties which bear some resemblance to supercompact cardinals. The resulting ordinal representation systems have been put to use in ordinal analyses of the subsystems of second order arithmetic based on Π^1_n-Comprehension for $n \geq 2$. When drawing connections to ordinal recursion theory, these cardinals should be viewed as cardinal analogues of stable and n-stable ordinals.(cf. [19])

Proofs for all results in this section are in [44].

To begin with we recall some definitions from ordinal recursion theory.

Definition 5.1 An ordinal κ is said to be stable if $\mathbf{L}_\kappa \prec_1 \mathbf{L}$, i.e. $\mathbf{L}_\kappa$ is a Σ_1-elementary substructure of $\mathbf{L}$.

Let $\rho > \kappa$. κ is ρ-stable if $\mathbf{L}_\kappa \prec_1 \mathbf{L}_\rho$.

Another rendering of stability comes in terms of ordinal recursion theory (cf. [19], VIII.5.1):

κ is stable iff κ is closed under all ∞-partial recursive ordinal functions.

Likewise,

κ is ρ-stable iff κ is closed under all (∞, ρ)-partial recursive functions.

The connection of the system of Π^1_2-Comprehension ($\Pi^1_2 - \mathbf{CA}$ hereafter) with set theory comes through the fact that $\mathbf{KP} + \Sigma_1$-Separation is a conservative extension of $\Pi^1_2 - \mathbf{CA} + \mathbf{BI}$, where $\mathbf{BI}$ is the so-called principle of *Bar Induction.*

Σ_n-separation is the schema of axioms

$$\exists z(z = \{x \in a : \phi(x)\})$$

for all set-theoretic Σ_n-formulae ϕ.

BI is the schema

$$\forall X \left(\mathrm{WO}(<_X) \wedge \forall n \left[\forall m <_X n \Phi(m) \to \Phi(n)\right] \to \forall n \Phi(n)\right)$$

for all formulae Φ of the language of second order arithmetic, where

$$m <_X n := 2^m \cdot 3^n \in X.$$

Assuming *Infinity* to be among the axioms of **KP**, the precise relationship is as follows:

Theorem 5.2 $\mathbf{KP} + \Sigma_1$*-Separation and* $(\Pi^1_2 - \mathbf{CA}) + \mathbf{BI}$ *prove the same sentences of second order arithmetic.*

The ordinals κ such that $\mathbf{L}_\kappa \models \mathbf{KP} + \Sigma_1$*-Separation* are familiar from ordinal recursion theory. They are called *nonprojectible* (cf. [5]) and are exactly those ordinals $\kappa > \omega$ such that κ is a limit of (smaller) κ-stable ordinals.

Stronger comprehension is linked to set theories as follows:

Proposition 5.3 *Let $n > 0$.*

$$\mathbf{KP} + \Sigma_n\text{-}\mathit{Collection} + \Sigma_n\text{-}\mathit{Separation}$$

and

$$(\Pi^1_{n+1} - \mathbf{CA}) + (\mathbf{\Sigma^1_{n+1}} - \mathbf{AC}) + \mathbf{BI}$$

prove the same sentences of second order arithmetic.

To characterize the standard models of $\mathbf{KP} + \Sigma_n$-Collection + Σ_n-Separation, we introduce the notion of *n-stability.*

Definition 5.4 An ordinal κ is said to be *n-stable* if $\mathbf{L}_\kappa \prec_n \mathbf{L}$, i.e. $\mathbf{L}_\kappa$ is a Σ_n-elementary substructure of $\mathbf{L}$.

For $\rho > \kappa$, we say that κ is n-ρ-stable if $\mathbf{L}_\kappa \prec_n \mathbf{L}_\rho$.

n-stability can be reduced to stability in terms of relativized stability.

Let $A \subseteq \mathbf{L}$ be a class. κ is *stable in* A if $\langle \mathbf{L}_\kappa; A_\kappa \rangle \prec_1 \langle \mathbf{L}; A \rangle$, where $A_\kappa = \mathbf{L}_\kappa \cap A$

Let S_1 be the class of stable ordinals, and for $n > 0$, let S_{n+1} be the class of ordinals stable in S_n.

Proposition 5.5 (**ZFC**) *κ is $n+1$-stable iff κ is stable in S_n.*

Similar to the connection between Σ_1-Separation and nonprojectability one has:

Proposition 5.6 *The following are equivalent for limit ordinals κ:*

(i) $\mathbf{L}_\kappa \models \Sigma_n$*-Collection* $+ \Sigma_n$*-Separation.*
(ii) For every $a \in \mathbf{L}_\kappa$ there exists $M \in \mathbf{L}_\kappa$ such that $a \subseteq M$ and $M \prec_n \mathbf{L}_\kappa$.

The next definition introduces what we consider to be the cardinal analogue of stability.

Definition 5.7 Let $\eta > 0$. A cardinal κ is *η-shrewd* if for all $P \subseteq V_\kappa$ and every set-theoretic formula $\phi(v_0, v_1)$, whenever

$$V_{\kappa+\eta} \models \phi[P, \kappa],$$

then there exist $0 < \kappa_0, \eta_0 < \kappa$ such that

$$V_{\kappa_0+\eta_0} \models \phi[P \cap V_{\kappa_0}, \kappa_0].$$

κ is *shrewd* if κ is η-shrewd for every $\eta > 0$.

Corollary 5.8 *If κ is δ-shrewd and $0 < \eta < \delta$, then κ is also η-shrewd.*

Apparently, the notion of shrewdness has not been put into the dictionary of large cardinals. There are some similarities between the notions of η-shrewdness and η-indescribability (see [8], Ch.9, §4). However, the notions are quite different in other aspects. For instance, it is impossible, for any κ, that κ is κ-indescribable. Therefore, if κ is η-indescribable and $\rho < \eta$, it does not necessarily follow that κ is also ρ-indescribable (see [8], 9.4.6). Another difference is that if π is measurable, then for every β, the set $\{\kappa < \pi : \kappa \text{ is } \beta\text{-indescribable}\}$ is stationary in π whereas there need not be any $\pi + 2$-shrewd cardinals below π.

A negative reason for calling the above cardinals *shrewd* is a shortage of names for cardinals. A positive reason is the following: If there is a shrewd cardinal κ in the universe, then, loosely speaking, for any notion of large cardinal N which does not make reference to the totality of all ordinals, whenever there exists an N-cardinal then the least such is below κ. So, for instance, if there are measurable and shrewd cardinals in the universe, then the least measurable is smaller than any of the shrewd cardinals.

A way of evincing the analogy between shrewdness and stability more closely consists in relating shrewdness to power recursion with search over the set-theoretic universe. Power recursion has been studied by Moschovakis [28] and Moss [29]. Central examples of power recursive functions (not requiring search) are $\alpha \mapsto V_\alpha$ and $\alpha \mapsto \aleph_\alpha$. However, limitations of space prevent us from going into details.

As suggested by 5.5, we shall also consider a notion of shrewdness with regard to a given class.

Let $\mathcal{L}_{set}$ denote the language of set theory. Let $\mathbf{U}$ be a fresh unary predicate symbol. Given a language $\mathcal{L}$ let $\mathcal{L}(\mathbf{U})$ denote its extension by $\mathbf{U}$.

If $\mathcal{A}$ is a class, we denote by $\langle V_\alpha; \mathcal{A}\rangle$ the structure $\langle V_\alpha; \in; \mathcal{A} \cap V_\alpha\rangle$. For an $\mathcal{L}_{set}(\mathbf{U})$-sentence ϕ, let the meaning of "$\langle V_\alpha; \mathcal{A}\rangle \models \phi$" be determined by interpreting $\mathbf{U}(t)$ as $t \in \mathcal{A} \cap V_\alpha$.

Definition 5.9 Let $\mathcal{A}$ be a class. Let $\eta > 0$. A cardinal κ is *$\mathcal{A}$-η-shrewd* if for all $P \subseteq V_\kappa$ and every formula $\phi(v_0, v_1)$ of $\mathcal{L}_{set}(\mathbf{U})$, whenever

$$\langle V_{\kappa+\eta}; \mathcal{A}\rangle \models \phi[P, \kappa],$$

then there exist $0 < \kappa_0, \eta_0 < \kappa$ such that

$$\langle V_{\kappa_0+\eta_0}; \mathcal{A}\rangle \models \phi[P \cap V_{\kappa_0}, \kappa_0].$$

κ is *$\mathcal{A}$-shrewd* if κ is $\mathcal{A}$-η-shrewd for every $\eta > 0$.

Corollary 5.10 *If κ is $\mathcal{A}$-δ-shrewd and $0 < \eta < \delta$, then κ is $\mathcal{A}$-η-shrewd.*

To situate the notion of shrewdness with regard to consistency strength in the usual hierarchy of large cardinals, we recall the notion of a subtle cardinal.

Definition 5.11 A cardinal κ is said to be *subtle* if for any sequence $\langle S_\alpha : \alpha < \kappa\rangle$ such that $S_\alpha \subseteq \alpha$ and C closed and unbounded in κ, there are $\beta < \delta$ both in C satisfying

$$S_\delta \cap \beta = S_\beta.$$

Since subtle cardinals are not covered in many of the standard texts dealing with large cardinals, we mention the following facts (see [22], §20):

Remark 5.12 *Let $\kappa(\omega)$ denote the first ω-Erdös cardinal.*

(i) $\{\pi < \kappa(\omega) : \pi$ is subtle$\}$ is stationary in $\kappa(\omega)$.
(ii) "Subtlety" relativises to $\mathbf{L}$, i.e. if π is subtle, then $\mathbf{L} \models$ "π is subtle".

Lemma 5.13 *Assume that π is a subtle cardinal and that $\mathcal{A} \subseteq V_\pi$. Then for every $B \subseteq \pi$ closed and unbounded in π there exists $\kappa \in B$ such that*

$$\langle V_\pi; \mathcal{A}\rangle \models \text{"}\kappa \textit{ is } \mathcal{A}\textit{-shrewd"}.$$

There are similarities between the cardinal notions of shrewdness and supercompactness. To bring out this analogy, we introduce two new cardinal notions. The first of them embodies considerable consistency strength.

Definition 5.14 Let $\mathcal{A}$ be a class. Assume $\eta > 0$. κ is *strongly $\mathcal{A}$-η-reducible* if for every $P \subseteq V_{\kappa+\eta}$ there exist $0 < \kappa_0, \eta_0 < \kappa$ and $Q \subseteq V_{\kappa_0+\eta_0}$ and an elementary embedding i such that $Q \cap V_{\kappa_0} = P \cap V_{\kappa_0}$ and

$$i : \langle V_{\kappa_0+\eta_0}; \in; \mathcal{A}; Q\rangle \longrightarrow \langle V_{\kappa+\eta}; \in; \mathcal{A}; P\rangle$$

with critical point κ_0 and $i(\kappa_0) = \kappa$.

κ is *strongly $\mathcal{A}$-reducible* if κ is strongly $\mathcal{A}$-η-reducible for all $\eta > 0$.

κ is *strongly η-reducible* if κ is strongly V-η-reducible. κ is *strongly reducible* if κ is strongly η-reducible for all $\eta > 0$.

Using elementary equivalence ($\equiv$) of structures instead of elementary embeddability one arrives at the following notion:

Definition 5.15 Let $\mathcal{A}$ be a class. If $\eta > 0$, κ is *$\mathcal{A}$-η-reducible* if for every $P \subseteq V_{\kappa+\eta}$ there exist $0 < \kappa_0, \eta_0 < \kappa$ and $Q \subseteq V_{\kappa_0+\eta_0}$ such that

$$\langle V_{\kappa_0+\eta_0}; \in; \kappa_0; \mathcal{A}; Q; x\rangle_{x\in V_{\kappa_0}} \equiv \langle V_{\kappa+\eta}; \in; \kappa; \mathcal{A}; P; x\rangle_{x\in V_{\kappa_0}}. \tag{31}$$

κ is *$\mathcal{A}$-reducible* if κ is $\mathcal{A}$-η-reducible for every η. κ is η-reducible if κ is V-η-reducible. κ is *reducible* if κ is η-reducible for every η.

Note that $Q \cap V_{\kappa_0} = P \cap V_{\kappa_0}$ springs from (31).

To make the foregoing definition resemble more closely the definition of strong reducibility, notice that in the situation of (31) there exists a *partial* embedding p from $V_{\kappa_0+\eta_0}$ into $V_{\kappa+\eta}$ satisfying $p \restriction V_{\kappa_0+\eta_0} = \mathrm{id} \restriction V_{\kappa_0+\eta_0}$ and $p(\kappa_0) = \kappa$. Moreover, p can be canonically extended so as to being defined on all elements of $V_{\kappa_0+\eta_0}$ which are definable in the structure $\langle V_{\kappa_0+\eta_0}; \in; \mathfrak{V}; \kappa_0; \mathcal{A}; Q; x\rangle_{x\in V_{\kappa_0}}$.

We will use

$$p : \langle V_{\kappa_0+\eta_0}; \in; \mathcal{A}; Q\rangle \underset{\equiv}{\longrightarrow} \langle V_{\kappa+\eta}; \in; \mathcal{A}; P\rangle$$

as a shorthand for conveying the foregoing situation.

The aspired analogy between shrewdness and strong reducibility resides in the fact that (weak) reducibility is closely related to shrewdness.

Proposition 5.16 *If κ is $\mathcal{A}$-ρ-shrewd and $0 < \eta < \rho$, then κ is $\mathcal{A}$-η-reducible.*

The circle of analogies will be completed by the next proposition, which also shows that the notion of a strongly reducible cardinal is equivalent to *supercompactness.*

Definition 5.17 κ is *δ-supercompact* if there is a transitive class M and an elementary embedding

$$j : V \longrightarrow M$$

such that $\mathrm{crit}(j) = \kappa$ and $\delta < j(\kappa)$, and ${}^{\delta}M \subseteq M$.

κ is *supercompact* if κ is δ-supercompact for every $\delta \geq \kappa$.

Proposition 5.18 *κ is strongly reducible iff κ is supercompact.*

A similar equivalence can be shown for $\mathcal{A}$-supercompact cardinals (cf. [51], 6.7).

Proposition 5.19 *κ is strongly $\mathcal{A}$-reducible iff κ is $\mathcal{A}$-supercompact.*

Sufficiently strong ordinal representation systems for the analyses of the systems $(\Pi^1_n - \mathbf{CA})$ utilize the notion of $\mathcal{A}$-reducibility for classes $\mathcal{A}$ which depend on the given n. The pertaining collapsing functions are obtained from inverses of partial elementary embeddings as explained in 5.15. The details will appear in [44].

6 Large sets in constructive set theory

Ideally, one wants to have mathematical results which allow one to state how it is that large cardinals come to be utilized in proof-theoretic ordinal analyses. Something that suggests more than merely an analogue. One idea pursued here is, that one should study the same notion of largeness in different settings. To give an example, we start off with a definition.

Definition 6.1 A non-empty set A is *regular* if A is transitive, and for every $a \in A$ and set $R \subseteq a \times A$ if $\forall x \in a\, \exists y\, (\langle x, y\rangle \in R)$, then there is a set $b \in A$ such that

$$\forall x \in a\, \exists y \in b\, (\langle x, y\rangle \in R) \ \wedge\ \forall y \in b\, \exists x \in a\, (\langle x, y\rangle \in R).$$

In particular, if $R : a \to A$ is a function, then the image of R is an element of A.

In the context of **ZFC** we have that V_κ is regular iff κ is a regular cardinal. The analogy between admissible sets and regular sets is drawn by restricting the class of relations (or functions) to the A-recursive ones. In contradistinction to the latter approach we suggest a study of regularity such that the only changes being made take place in the surrounding environment.[7]

The particular environment will be Aczel's constructive set theory, **CZF**. As for the main question raised above, we have no conclusive answers, but the results presented here might give some new insights. Proofs will be published elsewhere.

This section deals with large cardinal properties in the context of intuitionistic set theories. Since in intuitionistic set theory $\in$ is not a linear ordering on ordinals the notion of a cardinal does not play a central role. Consequently, one talks about *"large set properties"* instead of *"large cardinal properties"*. Friedman and Ščedrov [14] studied large set properties in the context of **IZF**. When stating these properties one has to proceed rather carefully. Classical equivalences of cardinal notion might no longer prevail in the intuitionistic setting , and one therefore wants to choose a rendering which intuitionistically retains the most strength. On the other hand certain notions have to be avoided so as not

[7] Feferman [13] is in a similar vein, but undertakes a different approach.

to imply excluded third. To give an example, cardinal notions like measurability, supercompactness and hugeness have to be expressed in terms of elementary embeddings rather than ultrafilters.

The axioms of **IZF** are Extensionality, Pairing, Union, as usual, and the following:

Infinity $\exists x \forall u[u \in x \leftrightarrow (\emptyset \in x \vee \exists v \in x(u = v \cup \{v\}))]$
Set Induction $\forall x[\forall y \in x \phi(y) \to \phi(x)] \to \forall x \phi(x)$
Separation $\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \phi(x)]$
Collection $\forall a[\forall x \in a \exists y \phi(x,y) \to \exists b \forall x \in a \exists y \in b\, \phi(x,y)]$
Powerset $\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$

for all set-theoretic formulae ϕ.

Regarding proof-theoretic strength, the upshot of [14] is that the equiconsistency of **ZF** and **IZF** propagates to extensions with large set axioms. The proof employs a $\neg\neg$-interpretation.

Theorem 6.2 (Friedman and Ščedrov, [14]) *If* **LSA** *is a large set axiom pertaining to any of the large cardinal axioms asserting the existence of an inaccessible, Mahlo, measurable, supercompact or n-huge cardinal, then:*

$$\mathbf{IZF} + \mathbf{LSA} \text{ and } \mathbf{ZF} + \mathbf{LSA} \text{ are equiconsistent.}$$

To be of interest, the latter systems should not imply excluded third. This follows from the next theorem.

Theorem 6.3 (Friedman and Ščedrov, [14]) *With* **LSA** *as above, the theory* **IZF** + **LSA** *has the disjunction property and the number existence property. Moreover,* **IZF** + **LSA** *is equiconsistent with* **IZF** + **LSA** + *Church's thesis.*

For our purpose the foregoing results appear to be disappointing since large set assumptions retain their consistency strength on the basis of **IZF**. The situation changes radically when we exchange **IZF** for **CZF**. The latter theory is due to Aczel (cf. [1, 2, 3]) and extends Myhill's constructive set theory **CST** (cf. [30]) which grew out of endeavours to discover a (simple) formalism that relates to Bishop's constructive mathematics as **ZFC** relates to classical Cantorian mathematics. The novel ideas were to replace Powerset by the (classically equivalent) Exponentiation Axiom and to discard full Comprehension while retaining full Collection. Aczel extended **CST** to **CZF** and corroborated the constructiveness of the latter theory by interpreting it in Martin-Löf's intuitionistic type theory (cf. [27]).

6.1 The System CZF

In this subsection we will summarize the language and axioms for Aczel's constructive set theory or **CZF**. The language of **CZF** is the first order language of ZF whose only non-logical symbol is $\in$. The logic of **CZF** is intuitionistic first order logic with equality. Its non-logical axioms comprise *Extensionality*, *Pairing*, *Union* in their usual forms, and *Infinity* and *Set Induction* as stated for **IZF**. **CZF** has additionally axiom schemata which we will now proceed to summarize.

Restricted Separation

$$\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \phi(x)]$$

for all *restricted* formulae ϕ. A set-theoretic formula is *restricted* if it is constructed from prime formulae using $\neg, \wedge, \vee, \rightarrow, \forall x{\in}y$, and $\exists x{\in}y$ only.

Strong Collection

$$\begin{aligned}&\forall a\big[\forall x \in a \exists y \phi(x,y) \rightarrow \\ &\quad \exists b\,[\forall x \in a\, \exists y \in b\, \phi(x,y) \wedge \forall y \in b\, \exists x \in a\, \phi(x,y)]\big]\end{aligned}$$

for all formulae ϕ.

Subset Collection

$$\begin{aligned}&\forall a \forall b \exists c \forall u\,\big[\forall x \in a\, \exists y \in b\ \phi(x,y,u) \rightarrow \\ &\quad \exists d \in c\,[\forall x \in a\, \exists y \in d\, \phi(x,y,u) \wedge \forall y \in d\, \exists x \in a\, \phi(x,y,u)]\big]\end{aligned}$$

for all formulae ϕ.

The mathematically important axiom of *Dependent Choices* (**DC**) could be included among the axioms of **CZF** without changing any essential properties of **CZF**, including its interpretation in type theory.

The Subset Collection schema easily qualifies for the most intricate axiom of **CZF**. To explain this axiom in different terms, we introduce the notion of *fullness*.

Definition 6.4 For sets A, B let ${}^{A}B$ be the class of all functions with domain A and with range contained in B.

Let $\mathbf{mv}({}^{A}B)$ be the class of all sets $R \subseteq A \times B$ satisfying $\forall u{\in}A\, \exists v{\in}B\, \langle u, v\rangle{\in}R$.

A set C is said to be *full* in $\mathbf{mv}({}^{A}B)$ if $C \subseteq \mathbf{mv}({}^{A}B)$ and

$$\forall R{\in}\mathbf{mv}({}^{A}B)\, \exists S{\in}C\, S \subseteq R.$$

Additional axioms we shall consider are:

Exponentiation: $\forall x \forall y \exists z\; z = {}^{x}y$.

Fullness: $\forall x \forall y \exists z$ "z full in $\mathbf{mv}({}^{x}y)$".

Proposition 6.5 *Let* $\mathbf{CZF}^{-}$ *be* **CZF** *without* Subset Collection.

(i) $\mathbf{CZF}^{-} \vdash$ Subset Collection $\leftrightarrow$ Fullness.
(ii) $\mathbf{CZF} \vdash$ Exponentiation.

Let **TND** be the principle of excluded third, i.e. the schema consisting of all formulae of the form $A \vee \neg A$.

The first central fact to be noted about **CZF** is:

Proposition 6.6 **CZF + TND = ZF**.

Proof: Note that classically Collection implies Separation. Powerset follows classically from Exponentiation. □

To stay in the world of **CZF** one has to keep away from any principles that imply **TND**. Moreover, it is fair to say that **CZF** is such an interesting theory owing to the non-derivability of Powerset and Separation. Therefore one ought to avoid any principles which imply Powerset or Separation.

In what follows we shall investigate largeness notions corresponding to inaccessibility, Mahloness and weak compactness. Bowing to the demands of brevity, we content ourselves with listing the definitions and results.

6.2 Inaccessibility

Let $\mathbf{Reg}(A)$ be the statement that A is a regular set (cf. (6.1)). The next axiom comprises that the universe is a union of regular sets.

Regular Extension Axiom (REA)

$$\forall x \exists y [x \subseteq y \wedge \mathbf{Reg}(y)]$$

Definition 6.7 A set I is said to be *inaccessible* if $\mathbf{Reg}(I)$ and I is a model of $\mathbf{CZF} + \mathbf{REA}$ in a strong sense, i.e. the structure $\langle I, \in \restriction (I \times I)\rangle$ is a model of Pairing, Union, Infinity, restricted Separation, and **REA** and the following holds:

$$(\mathbf{A}) \quad \forall A, B \in I \, \exists C \in I \text{ “}C \text{ is full in } \mathbf{mv}({}^{A}B)\text{”}$$

Due to $\mathbf{Reg}(I)$ and $(\mathbf{A})$, $\langle I, \in \restriction (I \times I)\rangle$ is also a model of strong collection and subset collection.

Corollary 6.8 *The following theories are the same theories, i.e. they prove the same formulae:*

(i) $\mathbf{CZF} + \exists I\, \mathbf{inac(I)} + \mathbf{TND}$
(ii) $\mathbf{ZF} + \exists \mathbf{I}\, \mathbf{inac(I)}$

They are equiconsistent with $\mathbf{ZFC} + \exists \kappa$ *“κ inaccessible cardinal”*

Theorem 6.9

$$\mathbf{CZF} + \forall x \exists I\, [x \in I \ \wedge \ \text{“}I \textit{ inaccessible”}]$$

can be interpreted in

$$\mathbf{KP} + \forall \alpha \exists \kappa\, [\alpha \in \kappa \ \wedge \ \text{“}\kappa \textit{ recursively inaccessible”}].$$

The interpretation preserves validity of Π_2-sentences. The theories have the same proof-theoretic strength.

6.3 Mahloness

Definition 6.10 A set M is said to be *Mahlo* if it is inaccessible and for each set $R \subseteq M \times M$, whenever

$$\forall x{\in}M\,\exists y{\in}M\,\langle x,y\rangle{\in}R,$$

then for every $u{\in}M$ there exists an inaccessible $I{\in}M$ with $u{\in}I$ and:

$$\forall x{\in}I\,\exists y{\in}I\,\langle x,y\rangle{\in}R.$$

Definition 6.11 Let A, α be sets. A is *α-inaccessible* iff A is inaccessible and for all $\beta{\in}\alpha$:

$$\forall a{\in}A\,\exists B{\in}A[a{\in}B \wedge \text{“}B \text{ is } \beta\text{-inaccessible}].$$

Proposition 6.12 (**CZF**) *If M is Mahlo then M is M-inaccessible.*

Corollary 6.13 **CZF** $+ \exists M$ *“M Mahlo”* $+$ **TND** *and* **ZF** $+ \exists \mathrm{M}$ *“M Mahlo” are the same theories.*

They are equiconsistent with **ZFC** $+ \exists\pi$ *“π Mahlo cardinal”.*

Theorem 6.14

$$\mathbf{CZF} + \forall x \exists M\,[x{\in}M \wedge \text{“}M \textit{ Mahlo”}]$$

can be interpreted in

$$\mathbf{KP} + \forall\alpha\exists\kappa\,[\alpha{\in}\kappa \wedge \text{“}\kappa \textit{ recursively Mahlo ordinal”}].$$

The interpretation preserves validity of Π_2-sentences. The theories have the same proof-theoretic strength.

6.4 Weak compactness

Theorem 3.8 suggests 2-regularity as the natural rendering of weak compactness in **CZF**. However, due to the absence of the axiom of choice in **CZF**, we prefer to introduce a slightly different notion.

Definition 6.15 Recall that $\mathbf{mv}({}^{A}B) = \{R \subseteq A \times B : \forall u{\in}A\,\exists v{\in}B\,\langle u,v\rangle{\in}R\}$.

An inaccessible set K is called *2-strong* if the following holds true for all sets S:

$$\forall r{\in}\mathbf{mv}({}^{K}K)\forall u{\in}K\,\exists x{\in}K\,\exists v{\in}K[x \subseteq R \wedge \langle x,u,v\rangle{\in}S] \rightarrow$$
$$\exists I{\in}K\big(\mathbf{inac(I)} \wedge \forall \mathbf{R}{\in}\mathbf{mv}({}^{\mathbf{I}}\mathbf{I})\forall \mathbf{u}{\in}\mathbf{I}\,\exists \mathbf{x}{\in}\mathbf{I}\,\exists \mathbf{v}{\in}\mathbf{I}[\mathbf{x} \subseteq \mathbf{R} \wedge \langle \mathbf{x},\mathbf{u},\mathbf{v}\rangle{\in}\mathbf{S}]\big).$$

Corollary 6.16 (**CZF**) *If K is 2-strong, then for any formula ϕ,*

$$\forall R{\in}\mathbf{mv}({}^{K}K)\forall u{\in}K\,\exists x{\in}K\,\exists v{\in}K[x \subseteq R \wedge \phi(x,u,v)] \rightarrow$$
$$\exists I{\in}K\big(\mathbf{inac(I)} \wedge \forall \mathbf{r}{\in}\mathbf{mv}({}^{\mathbf{I}}\mathbf{I})\forall \mathbf{u}{\in}\mathbf{I}\,\exists \mathbf{x}{\in}\mathbf{I}\,\exists \mathbf{v}{\in}\mathbf{I}[\mathbf{x} \subseteq \mathbf{R} \wedge \phi(\mathbf{x},\mathbf{u},\mathbf{v})]\big)\big].$$

Lemma 6.17 (**ZFC**) *For all ordinals κ, V_κ is 2-strong iff κ is weakly compact.*

Definition 6.18 Let α, C be sets. C is *α-Mahlo* if C is inaccessible and for all $\beta\in\alpha$:

$$\forall R\in\mathbf{mv}({}^{C}C)\,\exists B\in C\,\big[\text{“}B\text{ is }\beta\text{-Mahlo”} \wedge \forall x\in B\,\exists y\in B\langle x,y\rangle\in R\big].$$

Proposition 6.19 (**CZF**) *If C is 2-strong, then C is C-Mahlo.*

Theorem 6.20

$$\mathbf{CZF} + \forall x\exists K\,[x\in K \wedge \text{“}K\text{ 2-strong”}]$$

can be interpreted in

$$\mathbf{KP} + \forall\alpha\exists\kappa\,[\alpha\in\kappa \wedge \text{“}\kappa\ \Pi_3\text{-reflecting”}].$$

The interpretation preserves validity of Π_2-sentences. The theories have the same proof-theoretic strength.

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