ON THE ARITHMETICAL PROPERTIES OF CERTAIN ENTIRE CHARACTERISTIC FUNCTIONS

EUGENE LUKACS
THE CATHOLIC UNIVERSITY OF AMERICA

1. Introduction

Let $f(t)$ be a characteristic function and suppose that the corresponding distribution function has a finite second moment. It is then known that

$$T_1 f(t) = \frac{f'(t) - f'(0)}{tf''(0)} \tag{1.1a}$$

and

$$T_2 f(t) = \frac{f''(t)}{f''(0)} \tag{1.1b}$$

are also characteristic functions. These operators may be applied to analytic characteristic functions repeatedly and yield again analytic characteristic functions. The present study is motivated by the wish to investigate the arithmetical properties of the two families of characteristic functions which are obtained if the operator T_2^k or $T_1 T_2^{k-1}$ is applied to the characteristic function $f(t) = \exp(-t^2/2)$.

It is, however, convenient to investigate a more general class of characteristic functions, namely the family of entire characteristic functions of order two which have only a finite number of zeros. (In view of a theorem of Marcinkiewicz (see [4], p. 156), this family is identical with the entire characteristic functions of finite order which have only a finite number of zeros.) In the following, we denote this class by $\mathfrak{G}_2$. In section 2, we give some formulae concerning Hermite polynomials and discuss the construction of characteristic functions which belong to $\mathfrak{G}_2$. Section 3 deals with factorization problems of characteristic functions of $\mathfrak{G}_2$.

2. The class $\mathfrak{G}_2$

It is well known (see [4], p. 134) that the zeros of analytic characteristic functions cannot have an arbitrary location but are subject to the following restrictions:

This paper was prepared with support from the National Science Foundation under grants GP-96 and GP-3725.

(i) the zeros of an analytic characteristic function are located symmetrically with respect to the imaginary axis.

(ii) An analytic characteristic function has no zeros on the segment of the imaginary axis inside its strip of regularity. (This implies that an entire characteristic function has no purely imaginary zeros.)

We show in this section that it is possible to construct entire characteristic functions of order two which have preassigned zeros, subject only to the restrictions (i) and (ii).

An entire function of order 2 with a finite number of zeros has, according to Hadamard's factorization theorem, the form $f(t) = P(t)e^{A(t)}$ where $P(t)$ and $A(t)$ are polynomials and where $A(t)$ is of second degree. If $f(t)$ is also a characteristic function (that is, $f(t) \in \mathfrak{G}_2$), then it has the Hermitian property [namely, $f(-t) = \overline{f(t)}$] and $f(0) = 1$; moreover, $f(t)$ must also satisfy conditions (i) and (ii). Therefore, $P(t)$ is of even degree, say $2m$, and has the form

$$P(t) = \prod_{\nu=1}^{m} \left(1 - \frac{t}{\zeta_\nu}\right)\left(1 + \frac{t}{\bar{\zeta}_\nu}\right) = \prod_{\nu=1}^{m} \left(1 + 2i\frac{b_\nu}{c_\nu^2}t - \frac{t^2}{c_\nu^2}\right) \tag{2.1}$$

where $\zeta_\nu = a_\nu + ib_\nu$, $-\bar{\zeta}_\nu = -a_\nu + ib_\nu$ (a_ν, b_ν real, $c_\nu^2 = a_\nu^2 + b_\nu^2$; $\nu = 1, 2, \cdots, m$) are the zeros of $P(t)$. In view of the properties of $f(t)$, one can write

$$A(t) = -\sigma^2 t^2/2 + i\beta t \tag{2.2}$$

where β is real and $\sigma^2 > 0$. If we study the decomposition of a characteristic function, we can disregard factors of the form $e^{i\beta t}$ (β real) and write, therefore,

$$f(t) = e^{-\sigma^2 t^2/2}P(t) \tag{2.3}$$

where $P(t)$ is given by (2.1). It is sometimes convenient to write $P(t)$ in an alternative form by expanding it; then

$$P(t) = \sum_{\nu=0}^{2m} \lambda_\nu (it)^\nu. \tag{2.1a}$$

Here the λ_ν are real and $\lambda_0 = 1$.

We introduce next the Hermite polynomials

$$H_k(x) = e^{x^2/2}\frac{d^k}{dx^k}(e^{-x^2/2}). \tag{2.4}$$

It is not difficult to compute the coefficients of these polynomials (see for instance, [5], pp. 104–105); one obtains

$$H_k(x) = (-1)^k 2^{-k/2} k! \sum_{\nu=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^\nu}{\nu!}\frac{(x\sqrt{2})^{k-2\nu}}{(k-2\nu)!}. \tag{2.5}$$

We see from (2.4) that

$$\frac{d^k}{dt^k}(e^{-t^2/2}) = H_k(t)e^{-t^2/2}, \tag{2.4a}$$

and can therefore derive easily the families mentioned in the introduction. We get

$$h_{2k}(t) = T_2^k e^{-t^2/2} = \frac{(-1)^k}{\alpha_{2k}} H_{2k}(t)e^{-t^2/2}, \qquad (k = 1, 2, \cdots) \tag{2.6}$$

and

$$h_{2k-1}(t) = T_1T_2^{k-1}e^{-t^2/2} = \frac{(-1)^k}{\alpha_{2k}}\frac{H_{2k-1}(t)}{t}e^{-t^2/2}, \qquad (k = 1, 2, \cdots) \tag{2.7}$$

where $\alpha_{2k} = (-1)^kH_{2k}(0) = (2k)!/2^kk!$ is the moment of order $2k$ of the normal distribution with zero mean and variance one. Since

$$e^{-u^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iuy}e^{-y^2/2}\,dy, \tag{2.8}$$

we conclude from (2.4a) that

$$H_k(u)e^{-u^2/2} = \frac{(-i)^k}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iuy}y^ke^{-y^2/2}\,dy. \tag{2.9}$$

It follows easily that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}t^ke^{-\sigma^2t^2/2}\,dt = \frac{1}{\sigma^{k+1}(-i)^k\sqrt{2\pi}}H_k\left(\frac{x}{\sigma}\right)e^{-x^2/(2\sigma^2)}. \tag{2.9a}$$

If the function $f(t)$, given by (2.3), is a characteristic function, then

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}f(t)\,dt \tag{2.10}$$

is the frequency function of the corresponding distribution. This follows immediately from the fact that $f(t)$ is absolutely integrable. Using (2.10), (2.1a), and (2.3), we get

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}Q(x)e^{-x^2/2\sigma^2} \tag{2.11}$$

where

$$Q(x) = \sum_{k=0}^{2m}(-1)^k\lambda_k\sigma^{-k}H_k\left(\frac{x}{\sigma}\right). \tag{2.11a}$$

An entire function $f(t)$ of order two which has a finite number of zeros satisfying (i) and (ii) is not necessarily a characteristic function. Such a function determines, by (2.11a), a polynomial $Q(x)$ which we call the polynomial associated with $f(t)$. We see from (2.11) that $f(t)$ is a characteristic function if and only if its associated polynomial $Q(x)$ is nonnegative for all real x.

We wish to construct entire characteristic functions of order two having a finite number of preassigned zeros. These must of course satisfy the conditions (i) and (ii). For this construction, we need a lemma due to D. Dugué [2].

LEMMA 2.1. *Let $g(z)$ be an analytic characteristic function which has the strip of regularity $-\alpha < \mathrm{Im}(z) < \beta$ and choose a real η such that $-\alpha < \eta < \beta$. Then $h(z) = g(z + i\eta)/g(i\eta)$ is an analytic characteristic function whose strip of regularity is given by $-\alpha - \eta < \mathrm{Im}(z) < \beta - \eta$.*

For the proof of the lemma we refer to [2] or to ([4], p. 194).

We put $k = 1$ in (2.6) and see that $h_2(t) = (1 - t^2)\exp(-t^2/2)$, and therefore also

$$h_2\left(\frac{t}{a}\right) = \left(1 - \frac{t^2}{a^2}\right)e^{-t^2/2a^2}, \tag{2.12}$$

are (entire) characteristic functions. We apply lemma 2.1 with $\eta = -b$ to $h_2(z/a)$ and see that

$$\frac{h_2\left(\frac{z - ib}{a}\right)}{h_2(-ib/a)} = \left(1 - \frac{z^2 - 2ibz}{a^2 + b^2}\right) \exp\left[-\frac{1}{2a^2}(z^2 - 2ibz)\right] \tag{2.13}$$

is an entire characteristic function which has two complex zeros $\zeta = a + ib$ and $-\bar{\zeta} = -a + ib$. It follows immediately that

$$\left(1 - \frac{t^2 - 2ibt}{a^2 + b^2}\right) \exp\left(-\frac{t^2}{2a^2}\right) \tag{2.14}$$

and

$$\left(1 - \frac{t^2 - 2ibt}{a^2 + b^2}\right) e^{-\sigma^2 t^2/2}, \qquad \left(\sigma^2 \geq \frac{1}{a^2}\right) \tag{2.15}$$

are characteristic functions. We see, therefore, that the function $f(t) = P(t) \exp(-\sigma^2 t^2/2)$, determined by (2.1) and (2.3) belongs to $\mathfrak{G}_2$, provided that

$$\sigma^2 \geq \sum_{\nu=1}^{m} a_\nu^{-2} \tag{2.16}$$

holds.

We note that condition (2.16) is only sufficient but not necessary. This is, however, irrelevant in the present context since we are here only interested in the construction of a function of $\mathfrak{G}_2$.

3. Factorization of functions of $\mathfrak{G}_2$

It is well known that every factor of an entire characteristic function of finite order ρ is also an entire function of finite order not exceeding ρ. We conclude from this result that a function $f(t) \in \mathfrak{G}_2$ can have only factors which belong to $\mathfrak{G}_2$. It follows that the distribution functions of all factors of $f(t)$ are absolutely continuous.

Suppose that $f(t) \in \mathfrak{G}_2$, then it has necessarily the form

$$f(t) = P(t) e^{-\sigma^2 t^2/2}. \tag{3.1}$$

There are two possibilities: either $f(t)$ is indecomposable or $f(t)$ admits a decomposition

$$f(t) = f_1(t) f_2(t). \tag{3.2}$$

In the last mentioned case there are again two possibilities which are not mutually exclusive. The function $f(t)$ may have a normal factor say $f_1(t) = \exp(-\sigma^2 t^2/2)$, whereas the second factor has the form $f_2(t) = \exp(-\sigma_2^2 t^2/2) P(t)$ with $\sigma_1^2 + \sigma_2^2 = \sigma^2$, or the functions $f_j(t)$ may both have the form (3.1); that is, $f_j(t) = P_j(t) \exp(-\sigma_j^2 t^2/2) \in \mathfrak{G}_2$, $(j = 1, 2)$ where $\sigma_1^2 + \sigma_2^2 = \sigma^2$, $P_1(t) P_2(t) = P(t)$. Here and in the following, we assume that the polynomials $P_1(t)$, $P_2(t)$, and $P(t)$ satisfy the conditions (i) and (ii).

We consider first the case where $f(t)$ is decomposable so that at least one of the factors, say $f_2(t)$ of (3.2) has the form (3.1), and derive the following result.

THEOREM 1. *Suppose that the characteristic function $f(t) \in \mathfrak{G}_2$ admits a non-trivial decomposition; then its associated polynomial $Q(x)$ has no real zeros.*

Let $p(x)$ and $p_2(x)$ be the frequency function of $f(t)$ and $f_2(t)$ respectively; it follows from (2.10) that

$$\begin{cases} p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} Q(x) \exp\left(-\dfrac{x^2}{2\sigma^2}\right), \\ p_2(x) = \dfrac{1}{\sigma_2\sqrt{2\pi}} Q_2(x) \exp\left(-\dfrac{x^2}{2\sigma_2^2}\right). \end{cases} \tag{3.3}$$

Here $Q(x)$ and $Q_2(x)$ are nonnegative polynomials whose coefficients are determined by the zeros of $f(t)$ and $f_2(t)$ respectively. Let $F_1(x)$ be the distribution function which corresponds to $f_1(t)$. We see from (3.2) and (3.3) that

(3.4)

$$p(x) = \int_{-\infty}^{\infty} p_2(x-y)\,dF_1(y) = \frac{1}{\sigma_2\sqrt{2\pi}} \int_{-\infty}^{\infty} Q_2(x-y) \exp\left[-\frac{(x-y)^2}{2\sigma_2^2}\right] dF_1(y).$$

We give an indirect proof for the theorem and assume, therefore, tentatively that $Q(x_0) = 0$ for some real x_0. In view of (3.3) and (3.4), one has

$$\int_{-\infty}^{\infty} Q_2(x_0-y) \exp\left[-\frac{(x_0-y)^2}{2\sigma_2^2}\right] dF_1(y) = 0. \tag{3.5}$$

Since $Q_2(x)$ is a nonnegative polynomial, we see that $Q_2(x_0 - y)$ is a nonnegative polynomial for all real y. The integral in (3.5) can therefore vanish only if $F_1(y)$ is a purely discrete distribution whose discontinuity points are zeros of $Q_2(x_0 - y)$. However, all factors of $f(t)$ are absolutely continuous; this contradiction proves theorem 1.

COROLLARY. *The characteristic functions*

$$h_{2k}(t) = \frac{(-1)^k}{(\alpha_{2k})} H_{2k}(t)e^{-t^2/2} \tag{3.6}$$

are indecomposable.

We differentiate (2.8) $2k$-times and see that

$$H_{2k}(t)e^{-t^2/2} = \frac{(-1)^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ity}y^{2k}e^{-y^2/2}\,dy, \tag{3.7}$$

and conclude from (2.6) that $p_{2k}(x) = (1/\alpha_{2k})x^{2k} \exp(-x^2/2)$ is the frequency function of $h_{2k}(t)$ so that $p_{2k}(0) = 0$. In view of theorem 1, $h_{2k}(t)$ is indecomposable.

In order to derive the converse to theorem 1, it is necessary to obtain an explicit expression for the polynomial $Q(x)$. We combine formulae (2.5) and (2.11a) and obtain

$$(3.8)\qquad Q(x) = \sum_{k=0}^{2m} \lambda_k \sigma^{-k} k! \sum_{\nu=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^\nu 2^{-\nu}}{\nu!(k-2\nu)!} \frac{x^{k-2\nu}}{\sigma^{k-2\nu}}.$$

It is then easy to verify that

$$(3.9)\qquad \sigma^{2m}Q(x) = \sum_{\mu=0}^{m}\left[\sum_{\nu=0}^{m-\mu} \frac{(-1)^\nu 2^{-\nu}}{\nu!} \lambda_{2\mu+2\nu}(2\mu+2\nu)!\sigma^{2m-2\mu-2\nu}\right]\left(\frac{x}{\sigma}\right)^{2\mu}\frac{1}{(2\mu)!}$$
$$+ \sum_{\mu=0}^{m-1}\left[\sum_{\nu=0}^{m-\mu-1} \frac{(-1)^\nu 2^{\nu}}{\nu!} \lambda_{2\mu+2\nu+1}(2\mu+2\nu+1)!\sigma^{2m-2\mu-2\nu-1}\right]\left(\frac{x}{\sigma}\right)^{2\mu+1}\frac{1}{(2\mu+1)!}.$$

We introduce the polynomials

$$(3.10)\qquad c_{m,k}(z) = \frac{1}{k!} \sum_{\nu=0}^{m-\left[\frac{k+1}{2}\right]} \frac{(-1)^\nu}{2^\nu \nu!} \lambda_{k+2\nu}(k+2\nu)! z^{2m-k-2\nu}.$$

The $c_{m,k}(z)$ are polynomials of degree $2m - k$ and contain for k even (resp. odd) only even (resp. odd) powers of z. We write $c_{m,k} = c_{m,k}(\sigma)$ and $S(y) = \sum_{k=0}^{2m} c_{m,k} y^k$. Using this notation in (3.9) we get

$$(3.11)\qquad \sigma^{2m}Q(x) = \sum_{k=0}^{2m} c_{m,k}\left(\frac{x}{\sigma}\right)^k = S\left(\frac{x}{\sigma}\right).$$

Now let $f(t)$ be a function of $\mathfrak{G}_2$ which is given by

$$(3.12)\qquad f(t) = e^{-\sigma^2 t^2/2}P(t) = e^{-\sigma^2 t^2/2} \sum_{\nu=0}^{2m} \lambda_\nu (it)^\nu.$$

According to (2.11) and (3.11), the corresponding frequency function $p(x)$ is given by

$$(3.13)\qquad p(x) = \sigma^{-2m-1}(2\pi)^{-1/2} e^{-x^2/(2\sigma^2)} S\left(\frac{x}{\sigma}\right).$$

We wish to prove the following statement which is somewhat stronger than the converse of theorem 1.

THEOREM 2. *Suppose that $f(t) \in \mathfrak{G}_2$ and that the polynomial associated to $f(t)$ has no real zero. Then $f(t)$ has a normal factor.*

We note that the function $f(t)$, given by (2.3), has a normal factor if and only if it can be written in the form $f(t) = n_\theta(t) f_\theta(t)$ where $n_\theta(t) = \exp[-\sigma^2(1-\theta^2)t^2/2]$ with $0 < \theta < 1$, and where

$$(3.14)\qquad f_\theta(t) = \exp(-\sigma^2\theta^2 t^2/2) \sum_{\nu=0}^{2m} \lambda_\nu (it)^\nu$$

is a characteristic function. We can therefore establish the existence of a normal factor of $f(t)$ by showing that $f_\theta(t)$ is a characteristic function for some $\theta \in (0, 1)$. This is the case if the function

$$(3.15)\qquad p_\theta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} f_\theta(t)\, dt$$

is real and nonnegative for all real x. It is easily seen (by replacing σ by $\theta\sigma$ in (3.13)) that

$$p_\theta(x) = (\sigma\theta)^{-2m-1}(2\pi)^{-1/2} \exp\left[-x^2/2\sigma^2\theta^2\right] \sum_{k=0}^{2m} c_{m,k}(\theta\sigma)\left(\frac{x}{\theta\sigma}\right)^k. \tag{3.16}$$

The function $p_\theta(x)$ is a frequency function if and only if

$$S_\theta(y) = \sum_{k=0}^{2m} c_{m,k}(\theta\sigma)y^k \tag{3.17}$$

is nonnegative for all real y.

We note that $p_1(x) = p(x)$ and $S_1(y) = S(y)$. Since $p(x)$ is the frequency function of $f(t)$, it follows from the assumption of theorem 2 that

$$S_1(y) = S(y) > 0 \tag{3.18}$$

for all real y. Since $S_1(y) = \sum_{k=0}^{2m} c_{m,k}y^k$ is a polynomial of degree $2m$, we conclude from (3.18) that

$$c_{m,2m} > 0. \tag{3.19}$$

Let $\eta_\nu(\nu = 1, 2, \cdots, r; r \leq 2m)$ be the roots of $S_1(y)$ and assume that the root η_ν has the multiplicity n_ν so that $\sum_{\nu=1}^{r} n_\nu = 2m$. We have then

$$S(y) = c_{m,2m} \prod_{\nu=1}^{r} (y - \eta_\nu)^{n_\nu}. \tag{3.20}$$

We select $\rho > 0$ so that $2\rho < \min_\nu |\mathrm{Im}(\eta_\nu)|$, and let C_j be the circle $|z - \eta_j| < \rho$ and $G = \bigcup_{j=1}^{r} C_j$ the union of these circles. Since the roots of a polynomial are continuous functions of the coefficients, it is possible to find a $\delta > 0$ such that the roots of the polynomial $T(y) = \sum_{k=0}^{2m} d_k y^k$ will also be located in the region G, provided that the relations

$$|c_{m,k} - d_k| < \delta, \qquad (k = 1, 2, \cdots, 2m) \tag{3.21}$$

are satisfied. We put $d_k = c_{m,k}(\theta\sigma)$; since the $c_{m,k}(z)$ are continuous functions of z, it is possible to find an $\epsilon > 0$ which is so small that (3.21) is satisfied as soon as $\theta > 1 - \epsilon$. Now let θ_1 be a real number such that $1 > \theta_1 > 1 - \epsilon$. Then the polynomial $\sum_{k=0}^{2m} c_{m,k}(\theta_1\sigma)y^k$ has all its roots in the region G; therefore, it has no real roots and does not change its sign. In view of (3.19), it is always positive; hence, the polynomial

$$\sum_{k=0}^{2m} c_{m,k}(\theta_1\sigma)\left(\frac{x}{\theta_1\sigma}\right)^k > 0 \tag{3.22}$$

for all real x. This means that $p_{\theta_1}(x)$ is a frequency function and that $f(t)$ has the normal factor $n_{\theta_1}(t)$ so that the statement of theorem 2 is proven.

COROLLARY. *The characteristic functions*

$$h_{2k-1}(t) = \frac{(-1)^k}{\alpha_{2k}} \frac{H_{2k-1}(t)}{t} e^{-t^2/2} \tag{3.23}$$

are decomposable and have normal factors.

To prove the corollary, we must compute the polynomial associated to $h_{2k-1}(t)$ and show that it is always positive. We see from (2.5) and (2.7) that

$$h_{2k-1}(t) = \left[(k-1)! \sum_{j=0}^{k-1} \frac{2^j}{(k-1-j)!(2j+1)!} (it)^{2j}\right] e^{-t^2/2}, \tag{3.24}$$

so that $\sigma^2 = 1$, $m = k - 1$, $\lambda_{2j-1} = 0$, whereas

$$\lambda_{2j} = \frac{2^j m!}{(m-j)!(2j+1)!}, \qquad (j = 0, 1, \cdots, (k-1)). \tag{3.25}$$

It follows immediately then from (3.10) that

$$\begin{cases} c_{m,2k-1}(z) = 0, & (k = 1, 2, \cdots, m), \\ c_{m,2k}(z) = \dfrac{m!2^k}{(2k)!(m-k)!} \displaystyle\sum_{\nu=0}^{m-k} (-1)^\nu \binom{m-k}{\nu} \frac{1}{2k+2\nu+1} z^{2m-2k-2\nu}, & \\ & (k = 0, 1, \cdots, m). \end{cases} \tag{3.26}$$

We put

$$a_{m,l}(z) = \sum_{\nu=0}^{l} (-1)^\nu \binom{l}{\nu} \frac{1}{2m+2\nu-2l+1} z^{2l-2\nu} \tag{3.27}$$

and see from (3.26) that

$$\begin{cases} c_{m,2k-1}(z) = 0, & (k = 1, 2, \cdots, m), \\ c_{m,2k}(z) = \dfrac{m!2^k}{(2k)!(m-k)!} a_{m,m-k}(z), & (k = 0, 1, \cdots, m). \end{cases} \tag{3.28}$$

It is easily seen that the $a_{m,l}(z)$ satisfy the recurrence relation

$$a_{m,l}(z) = z^2 a_{m-1,l-1}(z) - a_{m,l-1}(z), \qquad (l = 1, 2, \cdots, m), \tag{3.29}$$

while

$$a_{m,0}(z) = \frac{1}{2m+1}. \tag{3.30}$$

Relation (3.29) and the initial condition (3.30) determine the functions $a_{m,l}(z)$, and therefore also the $c_{m,2k}(z)$ uniquely.

We write $a_{m,l} = a_{m,l}(1)$ and note that $\sigma = 1$; it follows then from (3.11) that the polynomial associated to $h_{2k-1}(t)$ is given by

$$Q(y) = S(y) = m! \sum_{k=0}^{m} \frac{2^k}{(2k)!(m-k)!} a_{m,m-k} y^{2k}. \tag{3.31}$$

We put $z = 1$ in (3.29) and obtain a recurrence relation for $a_{m,l}$; using (3.30), we see easily that

$$a_{m,l} = 2^l l! \left[\prod_{j=0}^{l} (2m+1-2j)\right]^{-1}, \qquad (l = 0, 1, \cdots, m). \tag{3.32}$$

Therefore, $S(y)$ is an even polynomial with positive coefficients and is therefore positive for all y, so that the corollary is proven.

We combine theorem 1 and theorem 2 and obtain theorem 3.

THEOREM 3. *A characteristic function $f(t) \in \mathfrak{G}_2$ is indecomposable if and only if its associated polynomial has at least one real zero.*

We consider next a characteristic function $f(t)$ (which does not necessarily belong to $\mathfrak{G}_2$), and suppose that it has a factor $f_2(t) \in \mathfrak{G}_2$ so that $f(t) = f_1(t)f_2(t)$.

The characteristic function $f(t)$ has then an absolutely continuous distribution. Let $p(x)$ be the corresponding frequency function and denote the distribution function of $f_1(t)$ by $F_1(x)$. According to (2.11), $p(x)$ is then given by

$$p(x) = \frac{1}{\sigma_2\sqrt{2\pi}} \int_{-\infty}^{\infty} Q_2(x - y) \exp [-(x - y)^2/(2\sigma_2^2)]\, dF_1(y) \tag{3.33}$$

where Q_2 is the polynomial associated with f_2. Therefore,

$$p(0) = \frac{1}{\sigma_2\sqrt{2\pi}} \int_{-\infty}^{\infty} Q_2(-y)e^{-y^2/(2\sigma_2^2)}\, dF_1(y). \tag{3.34}$$

It follows that $p(0) = 0$ implies that $F_1(x)$ is a discrete distribution which has discontinuity points only at the zeros of $Q_2(x)$. If we assume that $f_2(t)$ is not indecomposable, then, according to theorem 3, $p(0) > 0$, and we have obtained the following result.

COROLLARY. *Suppose that a characteristic function $f(t)$ has a factor which is not indecomposable and which belongs to $\mathfrak{G}_2$. Then the frequency function $p(x)$, corresponding to $f(t)$, satisfies the relation $p(0) > 0$.*

This is a generalization of the well-known fact that a frequency function which has a normal component cannot vanish for $x = 0$.

Let $f(t)$ be a characteristic function of $\mathfrak{G}_2$ and suppose that its associated polynomial has no real zeros. According to theorem 2, $f(t)$ has a normal factor. The next theorem deals with the determination of the normal factor which has maximal variance.

We consider a characteristic function $f(t) = \exp(-\sigma^2t^2/2)P(t) \in \mathfrak{G}_2$ and assume that its associated polynomial has no real zeros. Let

$$S_\theta(y) = \sum_{k=0}^{2m} c_{m,k}(\theta\sigma)y^k \tag{3.35}$$

be the polynomial defined by (3.17). The polynomial $S_1(y)$ has then no real zeros and is positive for all real y. Denote by θ_0 the greatest lower bound of all θ for which $S_\theta(y)$ is nonnegative for all real y, and all θ such that $\theta_0 < \theta \leq 1$. For every $\epsilon > 0$ there exists then a τ such that $\theta_0 - \epsilon \leq \tau < \theta_0$ for which $S_\tau(y)$ assumes a negative value for at least one real y. Since $S_\theta(y)$ is a continuous function of y and θ, we see that $S_{\theta_0}(y)$ is also a nonnegative polynomial. We show next that $S_{\theta_0}(y)$ has at least one real zero. We give an indirect proof and assume, therefore, that this is not the case. Then we can again construct circles G_j such that the union G of these circles contains all zeros of $S_{\theta_0}(y)$ in its interior and has no points in common with the real axis. We conclude from the fact that the roots of a polynomial are continuous functions of its coefficients that there exists an $\epsilon > 0$ such that all roots of $S_\theta(y)$ are inside G, provided $|\theta - \theta_0| < \epsilon$. But then $S_\theta(y)$ does not change its sign if $\theta_0 - \epsilon < \theta < \theta_0$; moreover, it is seen from (3.10), (2.1), and (2.1a) that $c_{m,2m}(z) = \lambda_{2m} > 0$ is independent of z so that $S_\theta(y)$ is a nonnegative polynomial if $\theta \epsilon (\theta_0 - \epsilon, \theta_0)$. This is in contradiction to the definition of θ_0 so that we have shown that $S_{\theta_0}(y)$ has at least one real zero.

It follows then from (3.13), (3.16), and theorem 3 that $p_{\theta_0}(x)$ is the frequency function of an indecomposable characteristic function $f_{\theta_0}(t)$, and that $f(t) = n_{\theta_0}(t)f_{\theta_0}(t)$ where

$$n_{\theta_0}(t) = \exp[-\sigma^2(1 - \theta_0^2)t^2/2] \tag{3.36a}$$

while

$$f_{\theta_0}(t) = P(t)\exp[-\sigma^2\theta_0^2t^2/2]. \tag{3.36b}$$

We still have to show that $f(t)$ can have no normal factor whose variance exceeds $\sigma^2(1 - \theta_0^2)$. We give again an indirect proof and assume that there exists a positive α, less than θ_0, such that $n_\alpha(t) = \exp[-\sigma^2(1 - \alpha^2)t^2/2]$ is a factor of $f(t)$. Then $f(t) = n_\alpha(t)f_\alpha(t)$ where $f_\alpha(t) = P(t)\exp[-\sigma^2\alpha^2t^2/2]$ is a characteristic function. We choose an $\epsilon > 0$ such that $\alpha < \theta_0 - \epsilon < \theta_0$; according to the definition of θ_0, there exists a τ, satisfying the inequality $\theta_0 - \epsilon < \tau < \theta_0$ for which $S_\tau(y)$ assumes also negative values. Since $f_\alpha(t)$ is a characteristic function. we see that

$$f_0(t) = f_\alpha(t)\exp[-\sigma^2(\tau^2 - \alpha^2)t^2/2] = P(t)\exp[-\sigma^2\tau^2t^2/2] \tag{3.37}$$

is also a characteristic function. But this is impossible since the associated polynomial $S_\tau(y)$ is not nonnegative. This contradiction shows that $f(t)$ can have no normal factor $n_\alpha(t)$ with $\alpha < \theta_0$. We formulate this result in the following manner.

THEOREM 4. *Let $f(t) = P(t)\exp[-\sigma^2t^2/2]$ be a characteristic function of the class $\mathfrak{G}_2$, and suppose that its associated polynomial has no real zeros. Then $f(t)$ is decomposable,*

$$f(t) = n_{\theta_0}(t)f_{\theta_0}(t), \tag{3.38}$$

where

$$f_{\theta_0}(t) = P(t)\exp[-\sigma^2\theta_0^2t^2/2] \tag{3.39}$$

is an indecomposable factor of $f(t)$, while

$$n_{\theta_0}(t) = \exp[-\sigma^2(1 - \theta_0^2)t^2/2] \tag{3.40}$$

is the normal component with maximal variance. Here θ_0 is defined in the preceding discussion.

REMARK. We note that the decomposition of $f(t)$ into a normal and an indecomposable factor is unique. Suppose that this is not the case. Then there exists an $\alpha \neq \theta_0$ such that

$$f(t) = f_{\theta_0}(t)n_{\theta_0}(t) = f_\alpha(t)n_\alpha(t) \tag{3.41}$$

where $n_\alpha(t) = \exp[-\sigma^2(1 - \alpha^2)t^2/2]$, and where $f_\alpha(t)$ and $f_{\theta_0}(t)$ are indecomposable. Without loss of generality we may assume that $\alpha^2 > \theta_0^2$. Then

$$f_\alpha(t) = f_{\theta_0}(t)\exp[-\sigma^2t^2(\alpha^2 - \theta_0^2)/2] \tag{3.42}$$

is the product of two characteristic functions. This contradicts the assumption that $f_\alpha(t)$ is indecomposable so that the uniqueness of the decomposition (3.38) is proven.

Let $f_1(t)$ and $f_2(t)$ be two characteristic functions belonging to $\mathfrak{G}_2$, and suppose that both are indecomposable. The function $f(t) = f_1(t)f_2(t)$ belongs also to $\mathfrak{G}_2$, and its associated polynomial has, according to theorem 1, no real zero. We conclude then from theorem 2 that $f(t)$ has necessarily a normal factor. This shows that a product of two indecomposable characteristic functions can have a normal factor, even if none of its components admits a normal factor. A different example of this phenomenon, not involving functions of $\mathfrak{G}_2$, was given by R. A. Fisher and D. Dugué [3].

In conclusion, we indicate a method which permits the numerical determination of the maximal normal component of a characteristic function $f(t) \in \mathfrak{G}_2$; this amounts to the computation of the constant θ_0. We have seen that the function $S_{\theta_0}(y)$ is a nonnegative polynomial which assumes the value zero for some real value y^* of the variable y. Since $S_{\theta_0}(y)$ is nonnegative, it must have a minimum at $y = y^*$ so that $S_{\theta_0}(y^*)$ as well as $S'_{\theta_0}(y^*)$ are zero. To find θ_0 we consider the polynomial (in y) $S_\theta(y)$ and determine a value of the parameter θ for which $S_\theta(y)$ and $S'_\theta(y)$ have a common zero. One forms, therefore, the resultant $R(\theta)$ of $S_\theta(y)$, and $S'_\theta(y)$, which is a polynomial in θ. In view of theorem 4, the polynomial $R(\theta)$ has exactly one root in the interval (0, 1) which leads to a minimum of $s_\theta(y)$, and this is the desired value θ_0.

We illustrate this procedure by two examples.

EXAMPLE 1. The function $h_2(t) = (1 - t^2) \exp(-t^2/2)$ is, according to the corollary to theorem 1, an indecomposable characteristic function. We consider the function

$$f(t) = \left[h_2\left(\frac{t}{\sqrt{2}}\right)\right]^2 = \left(1 - t^2 + \frac{t^4}{4}\right)e^{-t^2/2} \in \mathfrak{G}_2. \tag{3.43}$$

We then have $m = 2$, $\sigma^2 = 1$, $\lambda_0 = \lambda_2 = 1$, $\lambda_4 = 1/4$, $\lambda_1 = \lambda_3 = 0$, and see easily that $c_{2,0} = \theta^4 - \theta^2 + 3/4$, $c_{2,2} = \theta^2 - 3/2$, $c_{2,4} = 1/4$, whereas $c_{2,1} = c_{2,3} = 0$ so that

$$S_\theta(y) = (\theta^4 - \theta^2 + \tfrac{3}{4}) + (\theta^2 - \tfrac{3}{2})\, y^2 + \tfrac{1}{4}y^4, \tag{3.44}$$

and

$$S'_\theta(y) = (2\theta^2 - 3)y + y^3. \tag{3.45}$$

An elementary computation shows that

$$R(\theta) = 4^{-2}(4\theta^4 - 4\theta^2 + 3)(4\theta^2 - 3)^2. \tag{3.46}$$

The only positive real root of $R(\theta)$ is $\theta_0^2 = 3/4$. We see, therefore, that $f(t)$ admits, in addition to the representation (3.43) as the product of two indecomposable factors, the decomposition

$$f(t) = \left[\left(1 - t^2 + \frac{t^2}{4}\right)e^{-3t^2/8}\right][e^{-t^2/8}]. \tag{3.47}$$

The factor $(1 - t^2 + t^2/4) \exp(-3t^2/8)$ is indecomposable while $\exp(-t^2/8)$ is the normal component of $f(t)$ which has maximal variance. This example was given by D. Dugué [1] in connection with the problem of "deflation" (see also [4]).

EXAMPLE 2. The function $h_{2k-1}(t)$, given by formula (2.7), has according to the corollary to theorem 2, a normal factor. We wish to determine the maximal normal factor of $h_5(t) = [t^4 - 10t^2 + 15/15] \exp(-t^2/2)$. As in the previous example, we have $m = 2$, $\sigma^2 = 1$, and see from (3.26) and (3.28) that

$$c_{2,0}(\theta) = \theta^4 - \tfrac{2}{3}\theta^2 + \tfrac{1}{5}, \qquad c_{2,2} = \tfrac{2}{3}\theta^2 - \tfrac{2}{5}, \qquad c_{2,4} = \tfrac{1}{15}. \tag{3.48}$$

Therefore,

$$S_\theta(y) = \tfrac{1}{15}y^4 + (\tfrac{2}{3}\theta^2 - \tfrac{2}{5})\, y^2 + (\theta^4 - \tfrac{2}{3}\theta^2 + \tfrac{1}{5}), \tag{3.49}$$

and

$$S'_\theta(y) = \tfrac{4}{15}y^3 + (\tfrac{4}{3}\theta^2 - \tfrac{4}{5})\, y. \tag{3.50}$$

A simple computation shows that the resultant $R(\theta)$ of these two polynomials is

$$R(\theta) = C(15\theta^4 - 10\theta^2 + 3)(5\theta^4 - 10\theta^2 + 3)^2 \tag{3.51}$$

where C is a real constant. The first factor of $R(\theta)$ has no real roots; the second factor has the roots $\pm\sqrt{1 \pm 2/\sqrt{10}}$ so that $\theta_0^2 = 1 - \sqrt{10}/5$ is the only root of $R(\theta)$ which is in the interval (0, 1). The maximal normal factor of $h_5(t)$ is therefore $\exp[t^2/\sqrt{10}]$.

Theorem 4 permits the factorization of every decomposable characteristic function of $\mathcal{G}_2$ into a normal component with maximal variance and an indecomposable factor belonging to $\mathcal{G}_2$. We still have to indicate a method which permits us to ascertain whether a characteristic function $f(t) = P(t) \exp(-\sigma^2t^2/2) \in \mathcal{G}_2$ admits a decomposition of the form

$$f(t) = f_1(t)f_2(t) \tag{3.52}$$

where $f_j(t) = P_j(t) \exp(-\sigma t_j^{2 2}/2) \in \mathcal{G}_2$, $(j = 1, 2)$ and where $P_1(t)$ and $P_2(t)$ are polynomials such that $P_1(t)P_2(t) = P(t)$, $P_j(t) \not\equiv 1$, $P_1(0) = P_2(0) = 1$ while $\sigma_1^2 + \sigma_2^2 = \sigma^2$. The polynomial $P_1(t)$ is determined by certain (symmetrically located) pairs of zeros of $P(t)$ while $P_2(t)$ depends on the remaining zeros of $P(t)$. If $P(t)$ has multiple zeros, then $P_1(t)$ and $P_2(t)$ may have some zeros in common. Let $\zeta_\nu = a_\nu + ib_\nu$ and $-\bar{\zeta}_\nu = -a_\nu + ib_\nu$, $(\nu = 1, 2, \cdots, m)$ be the zeros of $P(t)$ where the ζ_ν need not be different. We consider now a partition of the zeros of $P(t)$ into two groups and investigate whether this partition can yield a decomposition of $f(t)$ of the form (3.52). Let $\zeta_\nu t_1$, $(\nu = 1, \cdots, k)$ and $\zeta_{\mu,2}$, $(\mu = 1, \cdots, m - k)$ be this partition and suppose that the polynomials $P_1(t)$ and $P_2(t)$ are determined by the zeros of the first and second group respectively so that

$$\begin{aligned} P_1(t) &= \prod_{\nu=1}^{k} \left(1 - \frac{t}{\zeta_{\nu,1}}\right)\left(1 + \frac{t}{\bar{\zeta}_{\nu,1}}\right), \\ P_2(t) &= \prod_{\mu=1}^{m-k} \left(1 - \frac{t}{\zeta_{\mu,2}}\right)\left(1 + \frac{t}{\bar{\zeta}_{\mu,2}}\right) \end{aligned} \tag{3.53}$$

are polynomials of degree $2k$ and $2m - 2k$ respectively. Let $a_{\nu,1} = \operatorname{Re} \zeta_{\nu,1}$, $(\nu = 1, \cdots, k)$ and $a_{\mu,2} = \operatorname{Re} \zeta_{\mu,2}$, $(\mu = 1, \cdots, m - k)$ and form

$$\tilde{\sigma}_1^2 = \sum_{\nu=1}^{k} a_{\nu,1}^{-2} \tag{3.54}$$

and

$$\tilde{\sigma}_2^2 = \sum_{\mu=1}^{k} a_{\mu,2}^{-2}. \tag{3.55}$$

The functions $\tilde{f}_j(t) = P_j(t) \exp(-\tilde{\sigma}_j^2 t^2/2)$, $(j = 1, 2)$ satisfy the condition (2.16) and are therefore characteristic functions of $\mathcal{G}_2$. If $\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 \leq \sigma^2$, then we choose $\sigma_1^2 \geq \tilde{\sigma}_1^2$, $\sigma_2^2 \geq \tilde{\sigma}_2^2$ so that $\sigma_1^2 + \sigma_2^2 = \sigma^2$ and obtain a decomposition of the form (3.52). If, however, $\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 > \sigma^2$, then one has to continue the investigation in order to arrive at a decision. According to theorem 4, it is possible to decompose each of the functions $f_1(t)$ and $f_2(t)$ into an indecomposable factor and a normal component with maximal variance. Let

$$g_j(t) = P_j(t) \exp(-\tau_j^2 t^2/2), \qquad (j = 1, 2) \tag{3.56}$$

be the indecomposable component in this factorization of $f_j(t)$. Clearly $\tau_j^2 \leq \tilde{\sigma}_j^2$, $(j = 1, 2)$. If $\tau_1^2 + \tau_2^2 \leq \sigma^2$, then one can obtain a decomposition of the form (3.52); if, however, $\tau_1^2 + \tau_2^2 > \sigma^2$, then a decomposition of the form (3.52) involving the partition of the zeros of $P(t)$ into the two groups $\zeta_{1,1}, \cdots, \zeta_{k,1}$ and $\zeta_{1,2}, \cdots, \zeta_{m-k,2}$ is impossible.

As an illustration we give two examples.

Example 3. We see from example 1 that $f_1(t) = (1(t^2/2))^2 \exp(-3t^2/8)$ and $f_2(t) = (1 - t^2)^2 \exp(-6t^2/8)$ are indecomposable functions of $\mathcal{G}_2$. We consider $f(t) = f_1(t)f_2(t) = (1 - t^2/2)^2(1 - t^2)^2 \exp(-\tau^2 t^2/2)$ where $\tau^2 = \frac{9}{4}$. It is easily seen that for $\sigma^2 = \frac{3}{2}$ the function

$$g(t) = (1 - t^2/2)(1 - t^2) \exp(-\sigma^2 t^2/2) \in \mathcal{G}_2; \tag{3.57}$$

however, $f(t) \not\equiv [g(t)]^2$. We use the procedure of theorem 4 to determine the decomposition of $g(t)$ into an indecomposable factor and a normal factor with maximal variance. After a somewhat tedious elementary computation we get

$$R(\theta) = \tfrac{1}{16}(9\theta^4 - 9\theta^2 + 6)(\tfrac{9}{4}\theta^4 - 36\theta^2 + 24). \tag{3.58}$$

The only real root of the equation $R(\theta) = 0$ which satisfies the conditions of the theorem is $\theta_0^2 = (24 - 4\sqrt{30})/3$; therefore $\theta_0^2\sigma^2 = 12 - 2\sqrt{30}$. We see then that $\tau^2/(2\theta_0^2\sigma^2) = 9/(8\theta_0^2\sigma^2) = 3(6 + \sqrt{30})/32 > 1$ so that a decomposition of the form $f(t) = h_1(t)h_2(t)$ with $h_j(t) = (1 - (t^2/2)\ (1 - t^2) \exp(\rho_j^2 t^2/2))$ is possible.

The method used in this example can also be used to show that the decomposition of the functions of $\mathcal{G}_2$ into indecomposable factors is in general not unique.

Example 4. Let $f_a(t) = (1 - t^2/2)^2 \exp(-3t^2/8)$ and

$$f_b(t) = [1 - 2t^2/(2b^2)]^2 \exp[-3t^2/(8b^2)] \tag{3.59}$$

with $b^2 = 7 + 4\sqrt{3}$ be two indecomposable functions of $\mathcal{G}_2$ and put $f(t) = f_a(t)f_b(t)$. It can then be shown that $f(t) = [g(t)]^2$ where

$$g(t) = \left(1 - \frac{t^2}{2}\right)\left(1 - \frac{t^2}{2b^2}\right)\exp\left[-\tfrac{3}{4}(2 - \sqrt{3})t^2\right] \in \mathcal{G}_2 \tag{3.60}$$

and where $g(t)$ is indecomposable.

REFERENCES

[1] D. Dugué, "Sur certains examples de décomposition en arithmétique des lois de probabilité," *Ann. Inst. H. Poincaré*, Vol. 12 (1951), pp. 159–169.

[2] ———, "Sur le théorème de Lévy-Cramér," *Publ. Inst. Statist. Univ. Paris*, Vol. 6 (1957), pp. 213–225.

[3] R. A. Fisher and D. Dugué, "Un résultat assez inattendu d'arithmétique des lois de probabilité," *C. R. Acad. Sci. Paris*, Vol. 227 (1948), pp. 1205–1206.

[4] E. Lukacs, *Characteristic Functions*, London, Griffin, 1960.

[5] G. Szegö, *Orthogonal Polynomials*, New York, American Mathematical Society, 1959 (rev. ed.).