THE MOTION OF CHARGED PARTICLES IN A RANDOM MAGNETIC FIELD

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1. Introduction

There are in nature several instances of charged particles with very high energies, distributed over a wide range: cosmic rays, the electrons of the outer Van Allen belt, and protons emitted by the sun in association with flares. The origin of these particles has long presented a puzzle. In 1949, Enrico Fermi pointed out that the motion of charged particles in a randomly changing magnetic field ought to lead, through Faraday's law of induction, to a gradual but unlimited increase in their mean energy [1]. Fermi proposed this mechanism as the source of cosmic rays, the random magnetic field being identified with the field expected to exist in interstellar space, according to Alfvén's ideas on cosmical electrodynamics [2].

If one wishes to study quantitatively the problem of the motion of charged particles in a random magnetic field, one is faced with the obvious difficulty that the equations of motion are complicated and cannot be integrated. It is therefore necessary to introduce certain simplifying assumptions. We shall begin right away by listing the principal assumptions made in this paper, assumptions which are nearly the same as those laid down by Fermi in his treatment of the cosmic ray problem.

2. Outline of the problem

The motion of a particle of charge q and momentum $\vec{p}$ in a magnetic field $\vec{B}$ is associated with a characteristic length $l = 2\pi pc/|q|B$, which is the distance traveled during one cyclotron period (c is the speed of light).

We shall adopt three postulates concerning the nature of the random magnetic field and the charged particles moving in it.

POSTULATE 1. *If L is a characteristic length, suitably defined, associated with the fluctuations of the magnetic field, then $l/L \ll 1$.*

This assumption clearly breaks down for particles of sufficiently large momentum, a restriction which should be kept in mind.

POSTULATE 2. *The magnetic field is embedded in a plasma of infinite electrical conductivity, whose hydrodynamical motion is described by a velocity field $\vec{U}$, small compared with the particle velocity v: $U/v \ll 1$.*

It may be pointed out here that $\vec{U}$ is related to the induced electric field $\vec{E}$ by the magnetohydrodynamic relation

$$\vec{E} = -\frac{1}{c}\vec{U} \times \vec{B}, \tag{1}$$

which will be derived in section 3. Thus the particle motion, which is of course influenced by $\vec{E}$ as well as $\vec{B}$, is inevitably coupled to the plasma velocity. All other interactions between plasma and particles will be neglected.

POSTULATE 3. *The component $\vec{v}_\perp$ of the particle velocity $\vec{v}$ normal to the magnetic field remains always small compared with c, that is $v_\perp/c \ll 1$.*

We shall see that this assumption appears to be essentially consistent with the equations of motion.

Our procedure, suggested by the last postulate, will be to linearize these equations with respect to $\vec{v}_\perp/c$. We shall then be able to derive a simple linearized equation of motion,

$$\frac{d\theta}{ds} = K, \tag{2}$$

in which θ is a kinematic variable related to v by the relation $\theta = \tanh^{-1}(v/c)$. Here K is an auxiliary random field determined locally by the electromagnetic field and its gradient, and s is the coordinate of the particle measured along its trajectory.

It turns out that equation (2) is integrable, so that the increment $\Delta\theta$ can be calculated and a Fokker-Planck equation derived for the probability distribution $P(\theta, s)$ of θ.

When the fields $\vec{B}$ and $\vec{U}$ are stationary random functions in both space and time, we shall be able to prove, by appealing to the fundamental equation of magnetohydrodynamics,

$$\frac{\partial \vec{B}}{\partial t} = \operatorname{curl}(\vec{U} \times \vec{B}), \tag{3}$$

that the Fokker-Planck equation reduces to the simple form

$$\frac{\partial P}{\partial s} = \varphi \frac{\partial^2 P}{\partial \theta^2}, \qquad \varphi > 0, \quad \text{constant}. \tag{4}$$

Diffusion equations of this type, where the distance traveled occurs in place of the time, play an important role in the theory of neutron diffusion in piles where they are called "Fermi age equations," the distance traveled by the neutron being known as its age.

3. The linearized equation of motion

In a frame of reference moving with the plasma velocity $\vec{U}$, the electric current density $\vec{j}$ may be written, according to Ohm's law, $\vec{j} = \sigma\vec{E}$, where σ is the electrical conductivity. According to postulate 2, σ is infinite and $\vec{E}$ must therefore vanish. A Lorentz transformation to a frame of reference at rest then yields exactly equation (1) for the electric field.

The force $d\vec{p}/dt$ experienced by the particle is

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right), \tag{5}$$

the momentum and velocity being related relativistically according to the expression

$$\vec{p} = \frac{m\vec{v}}{\left[1 - \left(\frac{v}{c}\right)^2\right]^{1/2}}, \tag{6}$$

where m is the rest mass of the particle. The other equation of motion is of course $d\vec{r}/dt = \vec{v}$, where $\vec{r}$ is the particle position vector.

It follows from (1) that $\vec{E}$, $\vec{B}$ are mutually orthogonal,

$$\vec{E} \cdot \vec{B} = 0. \tag{7}$$

From (5) and (7) we have therefore

$$\vec{B} \cdot \frac{d\vec{p}}{dt} = 0. \tag{8}$$

The energy W of the particle increases at the rate $dW/dt = d\vec{p}/dt \cdot \vec{v}$. From (8) it follows that $\vec{v}$ may here be replaced by $\vec{v}_\perp$, yielding

$$\frac{dW}{dt} = \frac{d\vec{p}}{dt} \cdot \vec{v}_\perp. \tag{9}$$

With the help of the relation $\vec{p} = W\vec{v}/c^2$, equation (9) may be written

$$\frac{dW}{dt} = \frac{d}{dt}\left(W\frac{\vec{v}}{c}\right) \cdot \frac{\vec{v}_\perp}{c}. \tag{10}$$

We shall now, as suggested by postulate 3, linearize this equation with respect to $\vec{v}_\perp/c$. This may be done by inserting $\vec{v} = \pm\vec{b}v$, $\vec{b} = \vec{B}/B$, into (10), which becomes

$$c^2\frac{dW}{dt} = \pm Wv\frac{d\vec{b}}{dt} \cdot \vec{v}_\perp. \tag{11}$$

We have

$$\frac{d\vec{b}}{dt} = \frac{\partial\vec{b}}{\partial t} + \vec{v} \cdot \operatorname{grad}\vec{b}. \tag{12}$$

Again inserting $\vec{v} = \pm\vec{b}v$, we obtain finally

$$c^2 \frac{dW}{dt} = \pm W v \left(\frac{\partial \vec{b}}{\partial t} \pm v \vec{b} \cdot \operatorname{grad} \vec{b} \right) \cdot \vec{v}_\perp. \tag{13}$$

The justification of postulate 3, which underlies the linearization procedure, results from the following considerations.

In a frame of reference moving with the velocity $\vec{U}$, the quantity $p_\perp^2/B$ is known [3], with certain restrictions, to be an adiabatic invariant of the particle motion to all orders in the small quantity l/L. Even this restricted invariance is not exact, however, as the corresponding power series in l/L does not converge to a limit. Nevertheless, one has the impression that $p_\perp^2/B$ is effectively a "good" constant and we shall assume that this is the case. Thus, if $p_\perp/mc \ll 1$ is satisfied at one instant, it will continue to be, provided the fluctuations of B are not extraordinarily severe.

The inequality $v_\perp/c < p_\perp/mc$ then guarantees that $v_\perp/c \ll 1$ continues to hold. This is the case however in a frame of reference moving with the local plasma velocity $\vec{U}$. To obtain $\vec{v}_\perp$ in a stationary frame of reference we merely have to add to $\vec{v}_\perp$ the normal plasma velocity component $\vec{U}_\perp$. Since $U/c \ll 1$, the inequality $v_\perp/c \ll 1$ holds also in the frame of reference at rest.

Returning to equation (13), we notice that the ratio of the first to the second term in the parentheses is of the order of magnitude U/v. According to postulate 2, we may therefore neglect the first term and rewrite (13) in the simpler form

$$\frac{c^2}{Wv^2} \frac{dW}{dt} = (\vec{b} \cdot \operatorname{grad} \vec{b}) \cdot \vec{v}_\perp. \tag{14}$$

With the help of the usual relativistic relations between energy, momentum, and velocity, this may be rewritten

$$\frac{d}{dt} \log p = (\vec{b} \cdot \operatorname{grad} \vec{b}) \cdot \vec{v}_\perp. \tag{15}$$

To facilitate the integration of (15) we may now utilize postulate 1. The time taken for the particle to traverse the distance l is one cyclotron period $T = l/v$. During this time, the increment of $\log p$, according to (15), is of the order of magnitude $(v_\perp/v)(l/L)$, which is negligible; furthermore $\vec{b} \cdot \operatorname{grad} \vec{b}$ is essentially constant over the distance l. If we now proceed to average (15) over one cyclotron period, we may therefore neglect not only the change in $\log p$ but also the change in $\vec{b} \cdot \operatorname{grad} \vec{b}$. The averaged equation (15) may therefore be written

$$\frac{d}{dt} \log p = (\vec{b} \cdot \operatorname{grad} \vec{b}) \cdot \overline{\vec{v}}_\perp, \tag{16}$$

where

$$\overline{\vec{v}}_\perp(t) = \frac{1}{T} \int_{-T/2}^{T/2} d\tau \, \vec{v}_\perp(t + \tau). \tag{17}$$

Now it may be shown [4], with the help of postulate 1, that $\overline{\vec{v}}_\perp$ is actually independent of the particle velocity and is equal to the so-called electric drift velocity $c\vec{E} \times \vec{B}/B^2$, which, according to (1), is simply

$$\vec{U}_\perp = c\frac{\vec{E} \times \vec{B}}{B^2}. \tag{18}$$

Thus (16) becomes

$$\frac{d}{dt}\log p = (\vec{b} \cdot \operatorname{grad} \vec{b}) \cdot \vec{U}_\perp. \tag{19}$$

However, since $\vec{b} \cdot \operatorname{grad} \vec{b}$ is orthogonal to $\vec{B}$, we may write this simply

$$\frac{d}{dt}\log p = (\vec{b} \cdot \operatorname{grad} \vec{b}) \cdot \vec{U}. \tag{20}$$

Thus the right side of (20) depends, according to (18), only on the electromagnetic field and its gradient at the position of the particle.

By using the relativistic relation (6), and writing $v = ds/dt$, we may rewrite (20) in the form

$$\frac{d\theta}{ds} = (\vec{b} \cdot \operatorname{grad} \vec{b}) \cdot \frac{\vec{U}}{c} \equiv K(\vec{r}, t), \tag{21}$$

where

$$\theta = \tanh^{-1}\frac{v}{c} \tag{22}$$

and s is the distance traveled by the particle.

4. The Fokker-Planck equation

To construct the Fokker-Planck equation for equation (21) we must compute the mean value limits [5], [6]

$$\begin{aligned} f_1 &= \lim_{\Delta s \to 0} \frac{\langle \Delta\theta \rangle}{\Delta s}, \qquad & \Delta\theta &= \theta - \theta_0 \\ f_2 &= \lim_{\Delta s \to 0} \frac{\langle \Delta\theta^2 \rangle}{\Delta s}, \qquad & \Delta s &= s - s_0. \end{aligned} \tag{23}$$

Here, the limit $\Delta s \to 0$ is to be understood with the qualification $\Delta s/L \gg 1$. We shall assume that U/v is sufficiently small that

$$(U/v)(\Delta s/L) \ll 1. \tag{24}$$

This condition is somewhat stronger than is needed to satisfy the requirement that $\langle \Delta\theta^2 \rangle^{1/2}$ be small. However, it insures also that the time elapsed, $\sim\Delta s/v$, is small compared with the characteristic time L/U with which the fields $\vec{B}$ and $\vec{U}$ fluctuate. Thus the time dependence of $K(\vec{r}, t)$ may be neglected. Furthermore, the particle trajectory which, in the linearized theory follows the equation

$$d\vec{r} = \pm\vec{b}\, ds, \tag{25}$$

is determined entirely by the particular realization of the magnetic field and the value of $\vec{r}$ at the instant t_0. In fact the trajectory lies on the magnetic "line of force" going through $\vec{r}_0$. Thus, when we integrate (21), the increment

$$\Delta\theta = \int_{s_0}^{s_0+\Delta s} ds\, K(\vec{r}, t), \qquad t \sim t_0, \tag{26}$$

depends only upon $\vec{r}_0$, t_0, and Δs [and on the sign in (25)].

If we now assume that $\vec{B}$ and $\vec{U}$ are stationary random functions [7] in both $\vec{r}$ and t, and average over $\vec{r}_0$, t_0, and the sign, it follows that f_1, f_2 are constants.

Neglecting small quantities of order Δs^2, $\langle\Delta\theta^3\rangle$, the probability distribution $P(\theta, s)$ of the kinematic variable θ then satisfies the Fokker-Planck equation

$$\frac{\partial P}{\partial s} = \frac{1}{2} f_2 \frac{\partial^2 P}{\partial \theta^2} - f_1 \frac{\partial P}{\partial \theta}. \tag{27}$$

We shall now prove that f_1 vanishes.

5. Calculation of f_1

According to (26) we have

$$\langle\Delta\theta\rangle = \left\langle \int_{s_0}^{s_0+\Delta s} ds\, K(\vec{r}, t_0)\right\rangle. \tag{28}$$

From ergodic theory we may assume that $\langle\Delta s\rangle$ is equal to the limit of the space average of the random function $\Delta\theta(\vec{r}_0, t_0)$ taken over a volume V, as V becomes infinite. Using the coordinate s_0 of $\vec{r}_0$ along the magnetic lines, and a surface element dA_0 orthogonal to these lines, we have

$$\begin{aligned} \langle\Delta\theta\rangle &= \lim_{V\to\infty} \frac{1}{V} \int dA_0 \int ds_0 \int_{s_0}^{s_0+\Delta s} ds\, K(\vec{r}, t_0) \\ &= \Delta s \lim_{V\to\infty} \frac{1}{V} \int dA_0 \int ds_0\, K(\vec{r}_0, t_0) \\ &= \Delta s \langle K\rangle. \end{aligned} \tag{29}$$

We need therefore only calculate $\langle K\rangle$.

Because of the presence of irreversible processes, such as viscous dissipation, it is not obvious that $\langle K\rangle$ must vanish in a time-stationary situation, and to prove that it does we must appeal to the fundamental equation of magnetohydrodynamics,

$$\frac{\partial \vec{B}}{\partial t} = \text{curl}\, \vec{U} \times \vec{B} = \text{curl}\, \vec{U}_\perp \times \vec{B}. \tag{30}$$

This equation is a direct consequence of (1) and of the Maxwell-Faraday equation

$$-\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \operatorname{curl} \vec{E}. \tag{31}$$

With the help of the Maxwell equation div $\vec{B} = 0$, (30) may be written

$$\frac{\partial \vec{B}}{\partial t} = \vec{B} \cdot \operatorname{grad} \vec{U}_\perp - \vec{B} \operatorname{div} \vec{U}_\perp - \vec{U}_\perp \cdot \operatorname{grad} \vec{B}. \tag{32}$$

Substituting

$$\operatorname{grad} \vec{B} = (\operatorname{grad} \vec{b})B + (\operatorname{grad} B)\vec{b}, \tag{33}$$

this becomes

$$\frac{\partial \vec{B}}{\partial t} = \vec{B} \cdot \operatorname{grad} \vec{U}_\perp - \vec{B} \operatorname{div} \vec{U}_\perp - \vec{U}_\perp \cdot (\operatorname{grad} \vec{b})B - \vec{U}_\perp \cdot (\operatorname{grad} B)\vec{b}. \tag{34}$$

Taking the scalar product of this equation with $\vec{B}$ and dividing by B^2, we get

$$\begin{aligned} &\frac{\partial}{\partial t}\log B \\ &\quad = \vec{b} \cdot (\operatorname{grad} \vec{U}_\perp) \cdot \vec{b} - \operatorname{div} \vec{U}_\perp - \vec{U}_\perp \cdot (\operatorname{grad} \vec{b}) \cdot \vec{b} - \vec{U}_\perp \cdot \operatorname{grad} \log B. \end{aligned} \tag{35}$$

Noting that $(\operatorname{grad} \vec{b}) \cdot \vec{b}$ vanishes and that

$$(\operatorname{grad} \vec{U}_\perp) \cdot \vec{b} = -(\operatorname{grad} \vec{b}) \cdot \vec{U}_\perp, \tag{36}$$

we obtain

$$\frac{\partial}{\partial t}\log B + \vec{U}_\perp \cdot \operatorname{grad} \log B + \operatorname{div} \vec{U}_\perp = -cK. \tag{37}$$

Noting that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{U}_\perp \cdot \operatorname{grad} \tag{38}$$

represents time differentiation following a point moving with the velocity $\vec{U}_\perp$, we obtain

$$\frac{d}{dt}\log B + \operatorname{div} \vec{U}_\perp = -cK. \tag{39}$$

It is of interest to note [8] that $-cK$ is the local rate at which a magnetic line of force stretches, each point on the line being assumed to move with the velocity $\vec{U}_\perp$.

The expectation value of div $\vec{U}_\perp$ vanishes, since $\vec{U}_\perp$ is a stationary random function of $\vec{r}$. Therefore

$$\left\langle \frac{d}{dt} \log B \right\rangle = \langle -cK \rangle. \tag{40}$$

Here $\langle -cK \rangle$ must be constant, since K is a stationary random function in space and time. If $|\langle \log B \rangle|$ is not ultimately to become infinite, this constant must be zero. The average $\langle \log B \rangle$ is taken over a random sample of points moving with the velocity $\vec{U}_\perp$. If, for instance on physical grounds, we impose upon $\log B$ an upper bound $\log B_0$, then unless $\langle -cK \rangle$ vanishes, $\langle \log B \rangle$ must at some time exceed $\log B_0$, a contradiction. Thus we must have

$$\langle K \rangle = 0 \tag{41}$$

and therefore $\langle \Delta\theta \rangle$ and f_1 must vanish.

Writing $f_2 = 2\varphi$, the Fokker-Planck equation becomes

$$\frac{\partial P}{\partial s} = \varphi \frac{\partial^2 P}{\partial \theta^2}, \tag{42}$$

equation (4).

6. The Fermi acceleration

In the absence of boundary conditions, the principal solutions of (42) are too well known to deserve special comment [5], [6]. We must however make some mention of the consequences of the fact that the variable θ is nonnegative so that a special boundary condition must prevail at the value $\theta = 0$. We shall write down the principal solutions for the two cases: (A) particles reflected at $\theta = 0$, and (B) particles absorbed at $\theta = 0$. A simple discussion leads to the following results.

Case (A).

$$P(\theta, s) = \frac{1}{(4\pi\varphi s)^{1/2}} \left\{ \exp\left[-\frac{(\theta - \theta_0)^2}{4\varphi s} \right] + \exp\left[-\frac{(\theta + \theta_0)^2}{4\varphi s} \right] \right\}. \tag{43}$$

Case (B).

$$P(\theta, s) = \frac{1}{(4\pi\varphi s)^{1/2}} \left\{ \exp\left[-\frac{(\theta - \theta_0)^2}{4\varphi s} \right] - \exp\left[-\frac{(\theta + \theta_0)^2}{4\varphi s} \right] \right\}. \tag{44}$$

In case (B), probability is not conserved, and

$$\int_0^\infty d\theta P(\theta, s) \tag{45}$$

decreases with increasing s.

In case (A), the principal solutions satisfy the boundary condition

$$\left. \frac{\partial P}{\partial \theta} \right|_{\theta=0} = 0. \tag{46}$$

In case (B), the boundary condition is

$$P(0, s) = 0. \tag{47}$$

We shall now derive two mean value theorems of physical interest. The relativistic relations between momentum, energy, and velocity lead, with the help of (22), to the expressions

$$\frac{W}{mc^2} = \cosh\theta, \qquad \frac{P}{mc} = \sinh\theta. \tag{48}$$

The mean energy is therefore

$$\overline{W} = mc^2 \int_0^\infty d\theta P \cosh\theta. \tag{49}$$

From (42) we have

$$\begin{aligned} \frac{d}{ds}\frac{\overline{W}}{mc^2} &= \int_0^\infty d\theta \frac{\partial P}{\partial s}\cosh\theta = \varphi \int_0^\infty d\theta \frac{\partial^2 P}{\partial\theta^2}\cosh\theta \\ &= \varphi \int_0^\infty d\theta P \frac{\partial^2 \cosh\theta}{\partial\theta^2} - \varphi \frac{\partial P}{\partial\theta}\bigg|_{\theta=0} \\ &= \varphi \frac{\overline{W}}{mc^2} - \varphi \frac{\partial P}{\partial\theta}\bigg|_{\theta=0}. \end{aligned} \tag{50}$$

Thus, in case (A) we obtain

$$\overline{W}(s) = \overline{W}(0)e^{\varphi s}, \tag{51}$$

a remarkable expression first derived by Fermi [1] by intuitive arguments.

A similar result follows at once for the momentum in case (B),

$$\bar{p}(s) = \bar{p}(0)e^{\varphi s}. \tag{52}$$

The exponential increase of the mean particle energy with distance, as given by (51), is known as the Fermi acceleration.

7. Discussion

It is worthwhile to examine the extent to which the theory developed in the foregoing sections would have to be modified if we were to change slightly the assumptions made. On the whole we would expect a slight departure from the validity of postulates 1 to 3 to affect the theory and equation (42) only slightly. It is important to note however that this is not the case if the time stationarity assumption is dropped.

The vanishing of f_1 depended upon the assumption that the fields $\vec{U}$ and $\vec{B}$ are stationary in time as well as in space. If we drop the requirement of time stationarity, we may estimate the error incurred in adopting equation (42) by estimating the ratio $|f_1|/f_2$. The order of magnitude of f_2 is given by $(U/c)^2/L$. If time stationarity fails to hold even approximately, we may estimate the order of magnitude $|f_1|$ as U/cL. Thus

$$\frac{|f_1|}{f_2} \sim \frac{c}{U}. \tag{53}$$

Recalling our assumption (24), we see that this ratio is truly enormous. We may therefore expect that even a small departure from time stationarity affects the theory in a radical manner and may altogether mask the effect of Fermi acceleration.

In astrophysical problems, we frequently encounter situations in which time stationarity may not be assumed to prevail to a high level of accuracy. It is therefore of the utmost importance in applying the theory to such situations to verify that the condition of time stationarity is fulfilled. This point was mentioned but not sufficiently stressed by Fermi, and seems to have been generally overlooked in the literature [9], [10], [11]. For example, the application of the concept of Fermi acceleration to the acceleration of cosmic rays by such an essentially nonstationary object as the Crab Nebula should be done with great caution. It should be stressed that nonstationarity may produce either a deceleration or an acceleration of charged particles, often much larger than the Fermi acceleration. On the other hand a nonstationary phenomenon may often not last for a very long time. In such cases, the energy of a typical charged particle may not change by more than an order of magnitude.

I am indebted to the Office of Scientific Research of the U.S. Air Force for supporting this work, and to Professor L. G. Henyey and Professor K. M. Watson for valuable discussions.

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