

SECOND ORDER SYMMETRIES OF THE CONFORMAL LAPLACIAN

JEAN-PHILIPPE MICHEL[†], FABIAN RADOUX[†] and JOSEF SILHAN[‡]

[†]*Department of Mathematics, University of Liège, Grande Traverse 12
4000 Liège, Belgium*

[‡]*Department of Algebra and Geometry of the Masaryk University in Brno
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic*

Abstract. Let (M, g) be an arbitrary pseudo-Riemannian manifold of dimension at least three. We determine the form of all the conformal symmetries of the conformal Laplacian on (M, g) , which are given by differential operators of second order. They are constructed from conformal Killing two-tensors satisfying a natural and conformally invariant condition. As a consequence, we get also the classification of the second order symmetries of the conformal Laplacian. We illustrate our results on two families of examples in dimension three. Besides, we explain how the (conformal) symmetries can be used to characterize the R -separation of some PDEs.

MSC: 58J10, 53A30, 70S10, 53D20, 53D55

Keywords: Laplacian, Quantization, Conformal geometry, separation of variables

1. Introduction

We work over a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$, with Levi-Civita connection ∇ and scalar curvature Sc . Our main result is the classification of all the differential operators D_1 of second order such that the relation

$$\Delta_Y D_1 = D_2 \Delta_Y \tag{1}$$

holds for some differential operator D_2 , where $\Delta_Y := \nabla_a g^{ab} \nabla_b - \frac{n-2}{4(n-1)} Sc$ is the conformal Laplacian. Such operators D_1 are called conformal symmetries of order two of Δ_Y . They preserve the kernel of Δ_Y , i.e., the solution space of the equation $\Delta_Y \psi = 0$, $\psi \in C^\infty(M)$.

Over flat pseudo-Euclidean space, the classification of conformal symmetries up to second order is due to Boyer, Kalnins and Miller [6], who use it to study the R -separation of variables of the Laplace equation $\Delta\Psi = 0$, where Δ denotes the Laplace-Beltrami operator. More generally, Kalnins and Miller provide an intrinsic characterization for R -separation of this equation on (M, g) in terms of second order conformal symmetries [17]. Thus, classifying those symmetries happens to be a basic problem in the theory of separation of variables.

A new input into the quest of conformal symmetries has been given by the work of Eastwood [13]. He classified indeed the conformal symmetries of any order over the conformally flat space and exhibited their interesting algebraic structure. Actually, using the principal symbol map, we obtain from the equation (1) that the principal symbol of D_1 has to be a conformal Killing two-tensor, i.e., a constant of motion of the geodesic flow, restricted to the null cone. In the conformally flat case, there is never any obstruction to the existence of conformal symmetries. Namely, for all conformal Killing two-tensor K , there exists always a conformal symmetry that has K as principal symbol.

If the pseudo-Riemannian manifold (M, g) is Einstein, i.e., if $\text{Ric} = \frac{1}{n}\text{Sc} g$ (where Ric denotes the Ricci tensor), Carter proved in [8] that there is never any obstruction to the existence of second order symmetries of Δ_Y . In other words, on these pseudo-Riemannian manifolds, for all Killing two-tensor K , there exists always a symmetry whose the principal symbol is given by K .

The description of conformal symmetries on an arbitrary pseudo-Riemannian manifold (M, g) is more involved, even at order two. Indeed, if K is a (conformal) Killing two-tensor on M , the tensor K has to verify an additional property to be the principal symbol of a (conformal) symmetry. The main goal of this paper is to give this additional condition and to give the structure of the space of all the (conformal) symmetries that have K as principal symbol if this condition is satisfied.

We illustrate our result on two examples in dimension three. In the first one, there is no obstruction to the existence of symmetries of Δ_Y . In the second one, we exhibit a family of pseudo-Riemannian metrics on \mathbb{R}^3 that admit conformal Killing two-tensors. In a generic situation, if K is the conformal Killing tensor associated with a metric inside this family, there is no conformal symmetry of Δ_Y that has K as principal symbol.

At the end of the paper, we explain how the second-order (respectively conformal) symmetries of $\Delta_Y + V$ can be used to characterize the existence of R -separating coordinates systems for the Schrödinger equation $(\Delta_Y + V)\Psi = E\Psi$ (respectively the Schrödinger equation at zero energy $(\Delta_Y + V)\Psi = 0$), extending in this way the results in [17] and [19].

We detail now the content of the paper.

In Section 2, we introduce the basic notions that are necessary for a good understanding of the paper. We define first the space of tensor densities $\mathcal{F}_\lambda(M)$ of weight $\lambda \in \mathbb{R}$, the one of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ acting between λ - and μ -densities, the one of symbols $\mathcal{S}_\delta(M)$ with $\delta = \mu - \lambda$. Then, we define the conformal Laplacian Δ_Y as an element of $\mathcal{D}_{\lambda_0,\mu_0}(M)$, with $\lambda_0 = \frac{n-2}{2n}$ and $\mu_0 = \frac{n+2}{2n}$, so that it is a conformally invariant operator. At the end of this section, we define the spaces of conformal symmetries of Δ_Y and the spaces of conformal Killing tensors, which can be viewed as the symmetries of the null geodesic flow.

In Section 3 lies our main result. We introduce first the natural and conformally invariant quantization

$$\mathcal{Q}_{\lambda,\mu}(g): \mathcal{S}_\delta(M) \rightarrow \mathcal{D}_{\lambda,\mu}(M)$$

which will be used to analyze the existence and the structure of the conformal symmetries. Next, we give the condition under which a conformal Killing tensor K is the principal symbol of a conformal symmetry of Δ_Y in terms of some curvature tensors arising in conformal geometry. Finally, we give the structure of the space of conformal symmetries in the case where this condition is satisfied using $\mathcal{Q}_{\lambda,\mu}(g)$.

In Section 4, we provide two examples illustrating our main result. In the first one, there is no obstruction to the existence of symmetries of Δ_Y . In this example, we consider Di Pirro metrics. These metrics are diagonal metrics on \mathbb{R}^3 admitting diagonal Killing tensors. In the second one, there is in general obstruction to the existence of conformal symmetries. The metrics occurring in this example are some conformal Stäckel metrics on \mathbb{R}^3 (a conformal Stäckel metric is a metric for which the Hamilton-Jacobi equation admits an additive separation in some orthogonal coordinate system).

In Section 5, we explain how the (conformal) symmetries of $\Delta_Y + V$ can be used to characterize the R -separation of the equations $(\Delta_Y + V)\Psi = E\Psi$ and $(\Delta_Y + V)\Psi = 0$, where $V \in C^\infty(M)$ is an arbitrary potential and where E is an arbitrary real parameter. In particular, we can thus obtain a simple condition by means of curvature tensors for the R -separation of the equations $\Delta_Y\Psi = E\Psi$ and $\Delta_Y\Psi = 0$ because the R -separation of these equations can be characterized by means of the existence of (conformal) symmetries of Δ_Y .

2. Problem Setting

Throughout this paper, we employ the abstract index notation from [25]. That is, on a smooth manifold M , v^a denotes a section of the tangent bundle TM , v_a a section of the cotangent bundle T^*M and e.g. $v^{ab}{}_c$ a section of $TM \otimes TM \otimes T^*M$. The letters a, b, c, d and r, s, t are reserved for abstract indices. Repetition of an abstract index in the covariant and contravariant position means contraction, e.g. v_b^{ab} is a section of TM . In few places we use concrete indices attached to a coordinate

system. This is always explicitly stated and we denote such indices by letters i, j, k, l to avoid confusion with abstract indices. We always use the Einstein's summation convention for indices, except if stated otherwise.

2.1. Differential Operators and Symbols

Let M be a n -dimensional smooth manifold. If $\lambda \in \mathbb{R}$, the vector bundle of λ -densities, $F_\lambda(M) \rightarrow M$, is a line bundle associated with P^1M , the linear frame bundle over M

$$F_\lambda(M) = P^1M \times_\rho \mathbb{R}$$

where the representation ρ of the group $\mathrm{GL}(n, \mathbb{R})$ on \mathbb{R} is given by

$$\rho(A)e = |\det A|^{-\lambda}e, \quad A \in \mathrm{GL}(n, \mathbb{R}), \quad e \in \mathbb{R}.$$

We denote by $\mathcal{F}_\lambda(M)$ the space of smooth sections of this bundle. Since $F_\lambda(M)$ is associated with P^1M , the space $\mathcal{F}_\lambda(M)$ is endowed with canonical actions of $\mathrm{Diff}(M)$ and of $\mathrm{Vect}(M)$. If (x^1, \dots, x^n) is a coordinate system on M , we denote by $|Dx|^\lambda$ the local λ -density equal to $[(\mathrm{Id}, 1)]$, where Id is the identity frame in the coordinates system (x^1, \dots, x^n) .

Actually, a λ -density φ at a point $x \in M$ can be viewed as a map on $\wedge^n T_x M$ with values in \mathbb{R} such that

$$\varphi(cX_1 \wedge \dots \wedge X_n) = |c|^\lambda \varphi(X_1 \wedge \dots \wedge X_n)$$

for all $X_1, \dots, X_n \in T_x M$ and $c \in \mathbb{R}$. The λ -density $|Dx|^\lambda$ is then the λ -density equal to one on $\partial_1 \wedge \dots \wedge \partial_n$, where $\partial_1, \dots, \partial_n$ denotes the canonical basis of $T_x M$ corresponding to the coordinate system (x^1, \dots, x^n) .

If a λ -density φ reads locally $f|Dx|^\lambda$, where f is a local function, then the Lie derivative of φ in the direction of a vector field X reads locally

$$L_X^\lambda \varphi = (X.f + \lambda(\partial_i X^i)f)|Dx|^\lambda. \quad (2)$$

It is possible to define the multiplication of two densities. If φ_1 reads locally $f|Dx|^\lambda$ and if φ_2 reads locally $g|Dx|^\delta$, then $\varphi_1\varphi_2$ reads locally $fg|Dx|^{\lambda+\delta}$.

On a pseudo-Riemannian manifold (M, g) , it is possible to define in a natural way a λ -density. In a coordinate system, this λ -density reads

$$|\mathrm{vol}_g|^\lambda = |\det g|^{\frac{\lambda}{2}} |Dx|^\lambda$$

where $|\det g|$ denotes the absolute value of the determinant of the matrix representation of g in the coordinate system.

We shall denote by $\mathcal{D}_{\lambda, \mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. It is the space of linear maps between $\mathcal{F}_\lambda(M)$ and $\mathcal{F}_\mu(M)$ that read in trivialization charts as differential operators. The actions of $\mathrm{Vect}(M)$ and $\mathrm{Diff}(M)$ on $\mathcal{D}_{\lambda, \mu}(M)$ are induced by the actions on tensor densities: if $\mathcal{L}_X D$ denotes the

Lie derivative of the differential operator D in the direction of the vector field X , we have

$$\begin{aligned} \mathcal{L}_X D &= L_X^\mu \circ D - D \circ L_X^\lambda, & D \in \mathcal{D}_{\lambda,\mu}(M) & \text{ and } X \in \text{Vect}(M). \\ \phi \cdot D &= \phi \circ D \circ \phi^{-1}, & D \in \mathcal{D}_{\lambda,\mu}(M) & \text{ and } \phi \in \text{Diff}(M). \end{aligned}$$

The space $\mathcal{D}_{\lambda,\mu}(M)$ is filtered by the order of differential operators. We denote by $\mathcal{D}_{\lambda,\mu}^k(M)$ the space of differential operators of order k . It is well-known that this filtration is preserved by the action of local diffeomorphisms.

On a pseudo-Riemannian manifold (M, g) , it is easy to build an isomorphism between $\mathcal{D}_{\lambda,\mu}(M)$ and $\mathcal{D}(M)$, the space of differential operators acting between functions. Indeed, thanks to the canonical densities built from $|\text{vol}_g|$, all operators $D \in \mathcal{D}_{\lambda,\mu}(M)$ can be pulled-back on functions as follows

$$\begin{array}{ccc} \mathcal{F}_\lambda(M) & \xrightarrow{D} & \mathcal{F}_\mu(M) \\ \uparrow |\text{vol}_g|^\lambda & & \uparrow |\text{vol}_g|^\mu \\ \mathcal{C}^\infty(M) & \xrightarrow{|\text{Vol}_g|^{-\mu} \circ D \circ |\text{vol}_g|^\lambda} & \mathcal{C}^\infty(M) \end{array} \tag{3}$$

The space of *symbols* is the graded space associated with $\mathcal{D}_{\lambda,\mu}(M)$: it is then equal to

$$\text{gr}\mathcal{D}_{\lambda,\mu}(M) := \bigoplus_{k=0}^{\infty} \mathcal{D}_{\lambda,\mu}^k(M) / \mathcal{D}_{\lambda,\mu}^{k-1}(M).$$

The canonical projection $\sigma_k : \mathcal{D}_{\lambda,\mu}^k(M) \rightarrow \mathcal{D}_{\lambda,\mu}^k(M) / \mathcal{D}_{\lambda,\mu}^{k-1}(M)$ is called the principal symbol map. As the actions of $\text{Diff}(M)$ and of $\text{Vect}(M)$ preserve the filtration of $\mathcal{D}_{\lambda,\mu}(M)$, they induce actions of $\text{Diff}(M)$ and $\text{Vect}(M)$ on the space of symbols.

Let $\delta = \mu - \lambda$ be the shift of weights. If the sum of the k -order terms of $D \in \mathcal{D}_{\lambda,\mu}^k$ in a coordinate system (x^1, \dots, x^n) reads

$$D^{i_1 \dots i_k} \partial_{i_1} \dots \partial_{i_k}$$

and if (x^i, p_i) is the coordinate system on T^*M canonically associated with (x^1, \dots, x^n) , then we get the following identification

$$\sigma_k(D) \longleftrightarrow D^{i_1 \dots i_k} p_{i_1} \dots p_{i_k}.$$

Thus, the space of symbols of degree k can be viewed as the space $\mathcal{S}_\delta^k(M) := \text{Pol}^k(T^*M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{F}_\delta(M)$, where $\text{Pol}^k(T^*M)$ denotes the space of real functions on T^*M which are polynomial functions of degree k in the fibered coordinates of T^*M . The algebra $\mathcal{S}(M) := \text{Pol}(T^*M)$ is clearly isomorphic to the algebra $\Gamma(STM)$ of symmetric tensors and depending on the context we will refer to its elements as symbols, functions on T^*M or symmetric tensors on M .

Let us recall that, if $S_1, S_2 \in \mathcal{S}(M)$, then the Poisson bracket of S_1 and S_2 , denoted by $\{S_1, S_2\}$, is defined in a canonical coordinate system (x^i, p_i) of T^*M in the following way:

$$\{S_1, S_2\} = (\partial_{p_i} S_1)(\partial_{x^i} S_2) - (\partial_{p_i} S_2)(\partial_{x^i} S_1). \quad (4)$$

We conclude this subsection by two properties of the principal symbol map linked to the composition and to the commutator of differential operators. For all $k, l \in \mathbb{N}$, we have:

$$\sigma_{k+l}(A \circ B) = \sigma_k(A)\sigma_l(B) \quad (5)$$

$$\sigma_{k+l-1}([A, B]) = \{\sigma_k(A), \sigma_l(B)\} \quad (6)$$

where A and B are elements of $\mathcal{D}(M)$ of order k and l respectively.

2.2. Pseudo-Riemannian and Conformal Geometry

Let (M, g) be a pseudo-Riemannian manifold. The isometries Φ of (M, g) are the diffeomorphisms of M that preserve the metric g , i.e., $\Phi^*g = g$. Their infinitesimal counterparts $X \in \text{Vect}(M)$ are called Killing vector fields, they satisfy $L_X g = 0$, with $L_X g$ the Lie derivative of g along X .

Given the Levi-Civita connection ∇ corresponding to the metric g , the Riemannian curvature tensor, which reads as $R_{ab}{}^c{}_d$ in abstract index notation, is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d$ for a tangent vector field v^c . Then, one gets the Ricci tensor by taking a trace of the Riemann tensor, which is indicated by repeated indices: $\text{Ric}_{bd} = R_{ab}{}^a{}_d$. By contraction with the metric, the Ricci tensor leads to the scalar curvature $\text{Sc} = g^{ab}\text{Ric}_{ab}$.

The Riemann tensor admits the following decomposition

$$R_{ab}{}^c{}_d = C_{ab}{}^c{}_d + 2\delta_{[a}^c P_{b]d} + 2g_{d[b} P_{a]}{}^c \quad (7)$$

where $C_{ab}{}^c{}_d$ is the totally trace-free Weyl curvature

$$P_{ab} = \frac{1}{n-2} \left(\text{Ric}_{ab} - \frac{1}{2(n-1)} \text{Sc} g_{ab} \right)$$

is the Schouten tensor, δ_a^b is the Kronecker delta and square brackets denote anti-symmetrization of enclosed indices. The Weyl tensor C_{abcd} is zero for the dimension $n = 3$. Note also that C_{abcd} obeys the same symmetries of indices as R_{abcd} does.

In the sequel, we will also need the Cotton-York tensor A_{abc} . It is defined in the following way

$$A_{abc} = 2\nabla_{[b} P_{c]a}.$$

Recall simply that the Weyl tensor $C_{ab}{}^c{}_d$ is always conformally invariant, whereas A_{abc} is conformally invariant in dimension three.

A conformal structure on a smooth manifold M is given by the conformal class $[g]$ of a pseudo-Riemannian metric g , where two metrics g and \hat{g} are conformally related if $\hat{g} = e^{2\Upsilon}g$, for some function $\Upsilon \in C^\infty(M)$. The conformal diffeomorphisms Φ of $(M, [g])$ are those which preserve the conformal structure $[g]$, i.e., there exists $\Upsilon \in C^\infty(M)$ such that $\Phi^*g = e^{2\Upsilon}g$. Their infinitesimal counterparts $X \in \text{Vect}(M)$ are called conformal Killing vector fields, they satisfy $L_Xg = f_Xg$, for some function $f_X \in C^\infty(M)$.

According to the transformation rule $|\text{vol}_{\hat{g}}|^\lambda = e^{n\Upsilon}|\text{vol}_g|^\lambda$ if $\hat{g} = e^{2\Upsilon}g$, we have the conformally invariant object $\mathbf{g}_{ab} = g_{ab} \otimes |\text{vol}_g|^{-\frac{2}{n}}$, termed *conformal metric*, with the inverse \mathbf{g}^{ab} in $\Gamma(S^2TM) \otimes \mathcal{F}_{2/n}$. The conformal metric gives a conformally invariant identification $TM \cong T^*M \otimes F_{-2/n}$. In other words, we can raise and lower indices, with expense of the additional density, in a conformally invariant way. For example, we get $C_{abcd} \in \Gamma(S^2(\Lambda^2T^*M)) \otimes \mathcal{F}_{-2/n}$. Note also that \mathbf{g}_{ab} and \mathbf{g}^{ab} are parallel for any choice of a Levi-Civita connection from the conformal class.

2.3. The Conformal Laplacian

Starting from a pseudo-Riemannian manifold (M, g) of dimension n , one can define the Yamabe Laplacian, acting on functions, in the following way

$$\Delta_Y := \nabla_a g^{ab} \nabla_b - \frac{n-2}{4(n-1)} \text{Sc}$$

where ∇ denotes the Levi-Civita connection of g and Sc the scalar curvature. For the conformally related metric $\hat{g} = e^{2\Upsilon}g$, the associated Yamabe Laplacian is given by

$$\widehat{\Delta}_Y = e^{-\frac{n+2}{2}\Upsilon} \circ \Delta_Y \circ e^{\frac{n-2}{2}\Upsilon}.$$

According to the transformation law $|\text{vol}_{\hat{g}}| = e^{n\Upsilon}|\text{vol}_g|$ and to the diagram (3), this translates into the conformal invariance of Δ_Y viewed as an element of $\mathcal{D}_{\lambda_0, \mu_0}(M)$, for the specific weights

$$\lambda_0 = \frac{n-2}{2n}, \quad \mu_0 = \frac{n+2}{2n} \quad \text{and} \quad \delta_0 = \mu_0 - \lambda_0 = \frac{2}{n}. \quad (8)$$

Thus, the data of a conformal manifold $(M, [g])$ is enough to define the operator $\Delta_Y \in \mathcal{D}_{\lambda_0, \mu_0}(M)$. We write it below as $\Delta_Y^M(g)$ and we refer to it as the Yamabe or conformal Laplacian. One easily gets

Proposition 1. *The conformal Laplacian is a natural conformally invariant operator; i.e.,*

- *it satisfies the naturality condition*

$$\Delta_Y^N(\Phi^*g) = \Phi^* (\Delta_Y^M(g)) \quad (9)$$

for all diffeomorphisms $\Phi : N \rightarrow M$ and for all pseudo-Riemannian metric g on M

- it is conformally invariant, $\Delta_Y^M(e^{2\Upsilon}g) = \Delta_Y^M(g)$ for all $\Upsilon \in C^\infty(M)$.

More generally, a natural operator over pseudo-Riemannian manifolds is an operator that acts between natural bundles, is defined over any pseudo-Riemannian manifold (M, g) and satisfies an analogue of the naturality condition (9). It is said to be conformally invariant if it depends only on the conformal class of g . For a general study of natural operators in the pseudo-Riemannian setting, see the book [20].

From Proposition 1, we deduce that the conformal Laplacian Δ_Y is invariant under the action of conformal diffeomorphisms, which reads infinitesimally as

$$L_X^{\mu_0} \circ \Delta_Y = \Delta_Y \circ L_X^{\lambda_0} \quad (10)$$

for all conformal Killing vector fields X . Here, as introduced in (2), L^{λ_0} and L^{μ_0} denote the Lie derivatives of λ_0 - and μ_0 -densities.

2.4. The Algebra of Symmetries of the Conformal Laplacian

Let $(M, [g])$ be a conformal manifold of dimension n . Fixing a metric $g \in [g]$, we can regard the conformal Laplacian, $\Delta_Y = \nabla_a g^{ab} \nabla_b - \frac{n-2}{4(n-1)} \text{Sc}$, as acting on functions. The symmetries of Δ_Y are defined as differential operators which commute with Δ_Y . Hence, they preserve the eigenspaces of Δ_Y .

More generally, conformal symmetries D_1 are defined by the weaker algebraic condition

$$\Delta_Y \circ D_1 = D_2 \circ \Delta_Y \quad (11)$$

for some differential operator D_2 , so that they only preserve the kernel of Δ_Y . The operator Δ_Y can be considered in equation (11) as acting between different line bundles and in particular as an element of $\mathcal{D}_{\lambda_0, \mu_0}$, where $\lambda_0 = \frac{n-2}{2n}$, $\mu_0 = \frac{n+2}{2n}$. With this choice, Δ_Y is conformally invariant and the space of conformal symmetries depends only on the conformal class of the metric g . It is stable under linear combinations and compositions.

The operators of the form $P\Delta_Y$, i.e., in the left ideal generated by Δ_Y , are obviously conformal symmetries. Since they act trivially on the kernel of Δ_Y , they are considered as trivial. Following [13, 14, 23], this leads to

Definition 2. Let $(M, [g])$ be a conformal manifold with conformal Laplacian $\Delta_Y \in \mathcal{D}_{\lambda_0, \mu_0}$. The algebra of conformal symmetries of Δ_Y is defined as

$$\mathcal{A} = \{D_1 \in \mathcal{D}_{\lambda_0, \lambda_0}; D_2 \in \mathcal{D}_{\mu_0, \mu_0}, D_2 \circ \Delta_Y = \Delta_Y \circ D_1\}$$

and the subspace of trivial symmetries as

$$(\Delta_Y) = \{A\Delta_Y; A \in \mathcal{D}_{\mu_0, \lambda_0}\}.$$

Thus, \mathcal{A} is a subalgebra of $\mathcal{D}_{\lambda_0, \lambda_0}$ and (Δ_Y) is the left ideal generated by Δ_Y in $\mathcal{D}_{\lambda_0, \lambda_0}$. The filtration by the order on $\mathcal{D}_{\lambda_0, \lambda_0}$ induces a filtration on \mathcal{A} and we denote by

$$\mathcal{A}^k := \mathcal{A} \cap \mathcal{D}_{\lambda_0, \lambda_0}^k$$

the algebra of conformal symmetries of order k . Obviously, $\mathcal{A}^0 \simeq \mathbb{R}$ is the space of constant functions, identified with zero order operators on λ_0 -densities. Moreover, the invariance of Δ_Y under the action of conformal Killing vector fields, see (10), shows that \mathcal{A}^1 is the direct sum of \mathcal{A}^0 with the space of Lie derivatives $L_X^{\lambda_0} \in \mathcal{D}_{\lambda_0, \lambda_0}^1$ along conformal Killing vector fields X . Since \mathcal{A} is an algebra, \mathcal{A}^2 contains in particular $L_X^{\lambda_0} \circ L_Y^{\lambda_0}$ for X, Y conformal Killing vector fields.

2.5. The Algebra of Symmetries of the Null Geodesic Flow

Let (M, g) be a pseudo-Riemannian manifold and (x^i, p_i) denote a canonical coordinate system on T^*M . The inverse metric g^{-1} pertains to $\Gamma(S^2TM)$ and identifies with $H := g^{ij}p_i p_j \in \mathcal{S}_0$, where $\mathcal{S}_0 = \text{Pol}(T^*M) \cong \Gamma(STM)$ (see Section 2.1). Along the isomorphism $T^*M \cong TM$ provided by the metric, the Hamiltonian flow of H corresponds to the geodesic flow of g .

The symmetries of the geodesic flow are given by functions $K \in \mathcal{S}_0$ which Poisson commute with H . They coincide with the symmetric Killing tensors. The null geodesic flow, i.e., the geodesic flow restricted to the level set $H = 0$, depends only on the conformal class of g . It admits additional symmetries, namely all the functions $K \in \mathcal{S}_0$ such that

$$\{H, K\} \in (H)$$

where $\{\cdot, \cdot\}$ stands for the canonical Poisson bracket on T^*M , defined in (4), and (H) for the ideal spanned by H in \mathcal{S}_0 . The linearity and Leibniz property of the Poisson bracket ensure that the space of symmetries of the null geodesic flow is a subalgebra of \mathcal{S}_0 . Besides, remark that all the functions in (H) are symmetries which act trivially on the null geodesic flow.

Definition 3. Let (M, g) be a pseudo-Riemannian manifold and $H \in \mathcal{S}_0$ the function associated to g . The algebra of symmetries of the null geodesic flow of g is given by the following subalgebra of \mathcal{S}_0 ,

$$\mathcal{K} = \{K \in \mathcal{S}_0 ; \{H, K\} \in (H)\}.$$

In particular, the algebra \mathcal{K} contains the ideal (H) of trivial symmetries. It inherits the gradation of \mathcal{S}_0 by the degree

$$\mathcal{K}^k := \mathcal{K} \cap \mathcal{S}_0^k.$$

The space \mathcal{K}^0 is the space of constant functions on T^*M . The Hamiltonian flows of functions in \mathcal{K}^1 coincide with the Hamiltonian lift to T^*M of the conformal Killing vectors on $(M, [g])$. For higher degrees, the elements in \mathcal{K} are symmetric conformal Killing tensors whose Hamiltonian flows do not preserve the configuration manifold M . They are symmetries of the whole phase space but not of the configuration manifold and often named hidden symmetries by physicists.

The next proposition is essential to determine the algebra \mathcal{A} of conformal symmetries.

Proposition 4. *If $D_1 \in \mathcal{A}^k$ then $\sigma_k(D_1) \in \mathcal{K}^k$. Under the identification $\text{gr}\mathcal{D}_{\lambda_0, \lambda_0} \cong \mathcal{S}_0$, the associated graded algebra $\text{gr}\mathcal{A}$ becomes a subalgebra of \mathcal{K} and $\text{gr}(\Delta_Y)$ identifies with (H) .*

Proof: Suppose that D_1 is a conformal symmetry of order k , i.e., satisfies $\Delta_Y \circ D_1 = D_2 \circ \Delta_Y$ for some D_2 . Working in the algebra $\mathcal{D}_{\lambda_0, \lambda_0}$ we deduce that $[\Delta_Y, D_1] \in (\Delta_Y)$ and the property (6) leads then to $\{H, \sigma_k(D_1)\} \in (H)$, i.e., $\sigma_k(D_1) \in \mathcal{K}^k$. The inclusion $\text{gr}\mathcal{A} \leq \mathcal{K}$ follows. As $\sigma_2(\Delta_Y) = H$, the property (5) of the principal symbol maps implies that $\text{gr}(\Delta_Y) \cong (H)$. ■

3. Structure of the Second Order Symmetries of Δ_Y

We give first the fundamental tool which will allow us to analyze the existence of conformal symmetries and to give the structure of the space \mathcal{A}^2 .

3.1. Natural and Conformally Invariant Quantization

Recall first the definition of a quantization on a smooth manifold M .

Definition 5. *Let $\lambda, \mu \in \mathbb{R}$ and $\delta = \mu - \lambda$. A quantization on M is a linear bijection $\mathcal{Q}_{\lambda, \mu}^M$ from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda, \mu}(M)$ such that*

$$\sigma_k(\mathcal{Q}_{\lambda, \mu}^M(S)) = S, \quad S \in \mathcal{S}_\delta^k(M), \quad k \in \mathbb{N}. \quad (12)$$

On locally conformally flat manifolds $(M, [g])$, for generic weights λ, μ , there exists a unique conformally equivariant quantization [10], i.e., a unique quantization which intertwines the actions of the conformal Killing vector fields on \mathcal{S}_δ and on $\mathcal{D}_{\lambda, \mu}$. In the following, we need an extension of the conformally equivariant quantization to arbitrary conformal manifolds. This is provided by the notion of *natural and conformally invariant quantization*. The definition and the conjecture of the existence of such a quantization were given for the first time in [21].

Definition 6. *A natural and conformally invariant quantization is the data for every pseudo-Riemannian manifold (M, g) of a quantization $\mathcal{Q}_{\lambda, \mu}^M(g)$, which satisfies*

- *the naturality condition*

$$\mathcal{Q}_{\lambda,\mu}^N(\Phi^*g)(\Phi^*S) = \Phi^*(\mathcal{Q}_{\lambda,\mu}^M(g)(S)), \quad S \in \mathcal{S}_\delta(M) \quad (13)$$

for all diffeomorphisms $\Phi : N \rightarrow M$ and for all pseudo-Riemannian metric g on M .

- *the conformal invariance: $\mathcal{Q}_{\lambda,\mu}^M(e^{2\Upsilon}g) = \mathcal{Q}_{\lambda,\mu}^M(g)$ for all $\Upsilon \in \mathcal{C}^\infty(M)$.*

In the following we refer to a quantization map by $\mathcal{Q}_{\lambda,\mu}$, the dependence in the chosen pseudo-Riemannian manifold (M, g) being understood. Accordingly, we drop the reference to M in the spaces of densities \mathcal{F}_λ , symbols \mathcal{S}_δ and differential operators $\mathcal{D}_{\lambda,\mu}$.

The concept of natural and conformally invariant quantization is an extension to quantizations of the more usual one of natural conformally invariant operator, introduced in the previous section. Restricting to conformally flat manifolds $(M, [g])$ and to $\Phi \in \text{Diff}(M)$ preserving $[g]$, the naturality condition (13) reads as conformal equivariance of the quantization map $\mathcal{Q}_{\lambda,\mu}$. Thus, the problem of the natural and conformally invariant quantization on an arbitrary manifold generalizes the problem of the conformally equivariant quantization on conformally flat manifolds.

The proof of the existence of such a quantization, at an arbitrary order and for generic values of λ, μ , was given in [22], [7] and [27] in different ways.

The proof of the existence of a natural and conformally invariant quantization at the second order was first given in [11], together with an explicit formula. We provide this formula in the theorem below, because we will need it later on.

Theorem 7 ([11]). *Let $\delta \notin \{\frac{2}{n}, \frac{n+2}{2n}, 1, \frac{n+1}{n}, \frac{n+2}{n}\}$. A natural and conformally invariant quantization $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}_\delta^{\leq 2} \rightarrow \mathcal{D}_{\lambda,\mu}^2$ is provided, on a pseudo-Riemannian manifold (M, g) of dimension n , by the formulas*

$$\begin{aligned} \mathcal{Q}_{\lambda,\mu}(f) &= f, & \mathcal{Q}_{\lambda,\mu}(X) &= X^a \nabla_a + \frac{\lambda}{1-\delta} (\nabla_a X^a) \\ \mathcal{Q}_{\lambda,\mu}(S) &= S^{ab} \nabla_a \nabla_b + \beta_1 (\nabla_a S^{ab}) \nabla_b + \beta_2 g^{ab} (\nabla_a \text{Tr} S) \nabla_b \\ &\quad + \beta_3 (\nabla_a \nabla_b S^{ab}) + \beta_4 g^{ab} \nabla_a \nabla_b (\text{Tr} S) + \beta_5 \text{Ric}_{ab} S^{ab} + \beta_6 \text{Sc}(\text{Tr} S) \end{aligned} \quad (14)$$

where f, X, S are symbols of degrees 0, 1, 2 respectively and $\text{Tr} S = g_{ab} S^{ab}$. Moreover the coefficients β_i entering the last formula are given by

$$\begin{aligned} \beta_1 &= \frac{2(n\lambda + 1)}{2 + n(1 - \delta)} \\ \beta_2 &= \frac{n(\lambda + \mu - 1)}{(2 + n(1 - \delta))(2 - nd)} \end{aligned}$$

$$\begin{aligned}\beta_3 &= \frac{n\lambda(n\lambda + 1)}{(1 + n(1 - \delta))(2 + n(1 - \delta))} \\ \beta_4 &= \frac{n\lambda(n^2\mu(2 - \lambda - \mu) + 2(n\lambda + 1)^2 - n(n + 1))}{(1 + n(1 - \delta))(2 + n(1 - \delta))(2 + n(1 - 2\delta))(2 - n\delta)} \\ \beta_5 &= \frac{n^2\lambda(\mu - 1)}{(n - 2)(1 + n(1 - \delta))} \\ \beta_6 &= \frac{n^2\lambda(\mu - 1)(n\delta - 2)}{(n - 1)(n - 2)(1 + n(1 - \delta))(2 + n(1 - 2\delta))}.\end{aligned}$$

3.2. Second Order (Conformal) Symmetries

We saw in Section 2 that the principal symbol of a (conformal) symmetry of Δ_Y has to be a (conformal) Killing tensor. On an arbitrary pseudo-Riemannian manifold, a (conformal) Killing tensor has to satisfy some condition to be the principal symbol of a (conformal) symmetry. This condition can be expressed by means of a natural and conformally invariant operator which is denoted here by **Obs** and which is defined below.

Definition 8. *The operator **Obs** is defined as follows*

$$\mathbf{Obs} : \mathcal{S}_0^2 \rightarrow \mathcal{S}_{2/n}^1 : S \mapsto \frac{2(n-2)}{3(n+1)} \mathbf{F}(S)$$

where $(\mathbf{F}(S))^a = C^r_{st}{}^a \nabla_r S^{st} - 3A_{rs}{}^a S^{rs}$.

We are now in position to prove our main theorem, which provides a full description of the conformal symmetries of Δ_Y given by second order differential operators. The proof of the following result is quite long and technical, it is the reason for which it is omitted here. The reader who would to know more details about it may consult the reference [24]. In the following statement, $\mathcal{Q}_{\lambda_0, \lambda_0}$ denotes the natural and conformally invariant quantization introduced in Theorem 7 whereas the isomorphism $\Gamma(TM \otimes F_{\frac{2}{n}}) \cong \Gamma(T^*M)$ provided by the metric is denoted by \flat .

Theorem 9. *The second order (conformal) symmetries of Δ_Y are exactly the operators*

$$\mathcal{Q}_{\lambda_0, \lambda_0}(K + X) + f$$

where X is a (conformal) Killing vector field, K is a (conformal) Killing two-tensor such that $\mathbf{Obs}(K)^\flat$ is an exact one-form and $f \in C^\infty(M)$ is defined up to a constant by $\mathbf{Obs}(K)^\flat = -2df$.

3.2.1. Remarks

- On a conformally flat manifold, $\mathbf{Obs}(K)$ vanishes identically. For all (conformal) Killing tensor K , the second-order differential operator $\mathcal{Q}_{\lambda_0, \lambda_0}(K)$ is then a (conformal) symmetry of Δ_Y . We recover in this way the results exposed in [13] and [23].
- If (M, g) is Einstein and if K is a Killing two-tensor, it turns out that $\mathbf{Obs}(K)^b$ is exact. Using our method, it is then easy to see that $\nabla_a K^{ab} \nabla_b$ is a symmetry of Δ_Y . We recover in this way the result exposed in [8].
- A classification of the symmetries of $\Delta + V$, where Δ denotes the Laplace-Beltrami operator and where $V \in C^\infty(M)$, was already obtained in [2]. This classification is formulated in the following way

Theorem 10 ([2]). *Let K be a Killing two-tensor and let us put $\mathbf{I}(K)^{ab} = K^{ac} \text{Ric}_c^b - \text{Ric}^{ac} K_c^b$. Then, we have*

$$[\nabla_a K^{ab} \nabla_b + f, \Delta + V] = 0 \iff K^{ab} (\nabla_a V) - \frac{1}{3} (\nabla_b \mathbf{I}(K)^{ab}) = \nabla^a f$$

where $\Delta = \nabla_a g^{ab} \nabla_b$ and $f, V \in C^\infty(M)$.

4. Examples in Dimension 3

In this section, we consider the space \mathbb{R}^3 endowed successively with two types of metrics: the conformal Stäckel metrics and the Di Pirro metrics.

The conformal Stäckel metrics are those for which the Hamilton-Jacobi equation

$$g^{ij} (\partial_i W) (\partial_j W) = E$$

admits additive separation in an orthogonal coordinate system for $E = 0$ (see [5] and references therein). They are conformally related to the Stäckel metrics, for which the additive separation of the Hamilton-Jacobi equation holds for all $E \in \mathbb{R}$. Moreover, the separating coordinates, called (conformal) Stäckel coordinates are characterized by two commuting (conformal) Killing two-tensors.

Except for the Stäckel metrics, every diagonal metric on \mathbb{R}^3 admitting a diagonal Killing tensor is a di Pirro metric g (see [26], page 113), whose corresponding Hamiltonian is (see e.g. [12])

$$H = g^{-1} = \frac{1}{2(\gamma(x_1, x_2) + c(x_3))} (a(x_1, x_2)p_1^2 + b(x_1, x_2)p_2^2 + p_3^2) \quad (15)$$

where a, b, c and γ are arbitrary functions and (x^i, p_i) are canonical coordinates on $T^*\mathbb{R}^3$.

4.1. An Example of Second Order Symmetry

The di Pirro metrics defined via equation (15) admit diagonal Killing tensors K given by

$$K = \frac{1}{\gamma(x_1, x_2) + c(x_3)} (c(x_3)a(x_1, x_2)p_1^2 + c(x_3)b(x_1, x_2)p_2^2 - \gamma(x_1, x_2)p_3^2).$$

For generic functions a, b, c and γ , the vector space of Killing 2-tensors is generated by H and K . However, for some choices of functions, this metric can admit other Killing tensors. For example, if (r, θ) denote the polar coordinates in the plane with coordinates (x_1, x_2) , if the functions a, b, γ depend only on r and if $a = b$, then the metric is Stäckel and admits p_θ^2 as additional Killing tensor.

Proposition 11. *On the space \mathbb{R}^3 , endowed with the metric g defined by (15), there exists a symmetry D of Δ_Y whose principal symbol is equal to the Killing tensor K . In terms of the conformally related metric*

$$\hat{g} := \frac{1}{2(\gamma(x_1, x_2) + c(x_3))} g$$

this symmetry is given by $D = \mathcal{Q}_{\lambda_0, \lambda_0}(K) + \frac{1}{16}(3\widehat{\text{Ric}}_{ab} - \widehat{\text{Sc}}\hat{g}_{ab})K^{ab}$, i.e., by

$$D = \widehat{\nabla}_a K^{ab} \widehat{\nabla}_b - \frac{1}{16}(\widehat{\nabla}_a \widehat{\nabla}_b K^{ab}) - \frac{1}{8}\widehat{\text{Ric}}_{ab} K^{ab}$$

where $\widehat{\nabla}$, $\widehat{\text{Ric}}$ and $\widehat{\text{Sc}}$ represent respectively the Levi-Civita connection, the Ricci tensor and the scalar curvature associated with the metric \hat{g} .

Proof: We use Theorem 9. In order to compute the obstruction $\text{Obs}(K)^\flat$, we used a Mathematica package called “Riemannian Geometry and Tensor Calculus”, by Bonanos [4].

This obstruction turns out to be an exact one-form equal to

$$d\left(-\frac{1}{8}(3\widehat{\text{Ric}}_{ab} - \widehat{\text{Sc}}\hat{g}_{ab})K^{ab}\right).$$

The first expression of the symmetry D follows, the second one is deduced from (14), giving $\mathcal{Q}_{\lambda_0, \lambda_0}(K)$. ■

4.2. An Example of Obstructions to Symmetries

If written in conformal Stäckel coordinates, the conformal Stäckel metrics g on \mathbb{R}^3 admit four possible normal forms, depending on the numbers of ignorable coordinates (see [5]). A coordinate x is ignorable if ∂_x is a conformal Killing vector field of the metric.

Thus, if x_1 is an ignorable coordinate, the conformal Stäckel metrics g read as

$$g = Q \left((dx_1)^2 + (u(x_2) + v(x_3))((dx_2)^2 + (dx_3)^2) \right) \quad (16)$$

where $Q \in C^\infty(\mathbb{R}^3)$ is the conformal factor and where u and v are functions depending respectively on the coordinates x_2 and x_3 . Such metrics admit ∂_{x_1} as conformal Killing vector field and

$$K = (u(x_2) + v(x_3))^{-1}(v(x_3)p_2^2 - u(x_2)p_3^2) \tag{17}$$

as conformal Killing two-tensor.

Proposition 12. *On \mathbb{R}^3 , there exist metrics g as in (16) whose conformal Laplacian Δ_Y admits no conformal symmetry with principal symbol K .*

Proof: Indeed, the obstruction associated with K , $\mathbf{Obs}(K)^\flat$, is generally not closed. Thanks to the Mathematica package ‘‘Riemannian Geometry and Tensor Calculus’’, by Bonanos [4], we can actually compute that $d\mathbf{Obs}(K)^\flat$ is equal to

$$\frac{v'(6u'^3 - 6u'(-v'^2 + (u+v)(u'' + v'')) + (u+v)^2u''') + (u+v)^2u'v'''}{4(u+v)^4} dx_2 \wedge dx_3$$

where the symbol $'$ denotes the derivatives with respect to the coordinates x_2 and x_3 . This expression does not vanish e.g. for the functions $u(x_2) = x_2$ and $v(x_3) = x_3$.

We conclude then using Theorem 9. ■

An example of a metric of the form (16) is provided by the Minkowski metric on \mathbb{R}^4 reduced along the Killing vector field $X = x_3\partial_t + t\partial_{x_3} + a(x_1\partial_{x_2} - x_2\partial_{x_1})$, $a \in \mathbb{R}$ (see [15]). In the time-like region of X and in appropriate coordinates (r, ϕ, z) , the reduced metric is equal to

$$g = dr^2 + \frac{r^2z^2}{z^2 - a^2r^2}d\phi^2 + dz^2$$

and admits ∂_ϕ as Killing vector field. Moreover, after reduction, the Killing tensor $p_{x_1}^2 + p_{x_2}^2$ is equal to

$$K = p_r^2 + \frac{1}{r^2}p_\phi^2.$$

Notice that the metric g is a Stäckel metric with one ignorable coordinate. Indeed, the metric takes the form (16), with $Q(r, z) = \frac{r^2z^2}{z^2 - a^2r^2}$, $u(r) = 1/r^2$ and $v(z) = -a^2/z^2$, whereas the conformal Killing tensor $K - \frac{z^2}{z^2 - a^2r^2}H$ can be written as in (17). Here, $H = g^{-1}$ is the metric Hamiltonian.

In this situation, there is no conformal symmetry of Δ_Y with principal symbol K if $a \neq 0$. Indeed, the one-form $\mathbf{Obs}(K)^\flat$ is then non-exact, as shown by Mathematica computations

$$d\mathbf{Obs}(K)^\flat = \frac{3}{2}(a + a^3) \left(\frac{1}{(z + ar)^4} - \frac{1}{(z - ar)^4} \right) dr \wedge dz.$$

5. Application to the R -Separation of the Schrödinger Equation

Following [17, 19], we provide an intrinsic characterization for R -separation of the Schrödinger equation and of the Schrödinger equation at fixed energy in terms of second order (conformal) symmetries of the operator $\Delta_Y + V$, where $V \in C^\infty(M)$. Resorting to our previous results, this leads to a new criterion for having R -separation of the equations $\Delta_Y \psi = 0$ and $\Delta_Y \psi = E\psi$, where $E \in \mathbb{R}$.

5.1. Definition of R -Separation

The Schrödinger equation, with fixed potential $V \in C^\infty(M)$, reads as

$$(\Delta_Y + V)\psi = E\psi \quad (18)$$

where $\psi \in C^\infty(M)$ is the unknown and $E \in \mathbb{R}$ is called the energy. Solving the Schrödinger equation means to determine the solutions for all $E \in \mathbb{R}$.

We consider also the Schrödinger equation at fixed energy, i.e., up to changing V

$$(\Delta_Y + V)\psi = 0. \quad (19)$$

We restrict ourself to separation along orthogonal coordinates (x^i) , i.e., coordinates such that $g_{ij} = 0$ if $i \neq j$. The concept of R -separation of the equations (18) and (19) can be defined in the following way

Definition 13 ([17, 19]). *Equation (19) is R -separable in an orthogonal coordinate system (x^i) if there exist $n + 1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that*

$$R^{-1}(\Delta_Y + V)R = \sum_{i=1}^n h_i L_i.$$

The Schrödinger equation (18) is also R -separable in the coordinate system (x^i) if, for all $E \in \mathbb{R}$, there exist $n + 1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R - E = \sum_{i=1}^n h_i L_i.$$

The existence of a R -separating coordinates system for the equation (18) or the equation (19) allows one to reduce the resolution of these partial differential equations to the resolution of a system of ordinary differential equations. Indeed, one can show that ψ , defined by

$$\psi(x) = R(x) \prod_{i=1}^n \phi_i(x^i)$$

is a solution of the Schrödinger equation or of the equation (19) if and only if $L_i\phi_i = 0$ for all i .

5.2. Intrinsic Characterizations for R -Separation

In order to characterize in an intrinsic way the R -separation of the equation (18) (respectively (19)), we need the concept of (respectively conformal) Killing-Stäckel algebra. These notions are defined e.g. in [1, 18] and [3, 16].

Recall first that, if (M, g) is a pseudo-Riemannian manifold and if (x^i, p_i) is a canonical coordinate system on T^*M , then the Hamiltonian H is equal to $H = g^{ij}p_i p_j \in \mathcal{S}^2$. We can notice that a quadratic symmetric tensor $K \in \mathcal{S}^2$ identifies via the metric with a symmetric endomorphism of TM . In that way, H identifies with the identity.

Definition 14. A (respectively conformal) Killing-Stäckel algebra is an n -dimensional linear space \mathcal{I} of (respectively conformal) Killing two-tensors which satisfy the three following properties

- i) they commute as linear operators
- ii) they are diagonalizable as linear operators
- iii) they are in (respectively conformal) involution: $\{K_1, K_2\} = 0$ (respectively $\{K_1, K_2\} \in (H)$) for all $K_1, K_2 \in \mathcal{I}$.

In [17, 19], Kalnins and Miller provide a characterization for R -separation of the equations $\Delta\psi = 0$ and $\Delta\psi = E\psi$ in terms of (conformal) symmetries of the Laplace-Beltrami operator Δ . In the presence of potentials, their results extend easily, leading to the two following theorems.

Theorem 15. The equation (18) (respectively (19)) R -separates in an orthogonal coordinate system if and only if

- a) there exists a (respectively conformal) Killing-Stäckel algebra \mathcal{I}
- b) for all $K \in \mathcal{I}$, there exists $D \in \mathcal{D}^2(M)$ with principal symbol $\sigma_2(D) = K$ such that $[\Delta_Y + V, D] = 0$ (respectively $[\Delta_Y + V, D] = A \circ (\Delta_Y + V)$ for some $A \in \mathcal{D}(M)$).

The proofs of the latter theorems are simply straightforward adaptations of the proofs of the corresponding theorems in [17, 19]. It is the reason for which we do not give them here. Let us just say that, in order to prove Theorem 15, we prove first as in [17, 19] a characterization of the R -separation of (18) and (19) by means of a condition on the metric g and of a condition on the functions V and R . This characterization was already given in [9] for the equation (19). It reads as follows:

Theorem 16. The equation (19) R -separates in the orthogonal coordinate system (x^i) if and only if the two following conditions hold

- i) g is a conformal Stäckel metric with conformal Stäckel coordinates (x^i)
- ii) $V + \frac{\Delta_Y R}{R}$ is a pseudo-Stäckel multiplier.

A pseudo-Stäckel multiplier for a conformal Stäckel metric g is a function Q such that Qg is a Stäckel metric (i.e., a metric for which there exists a Killing-Stäckel algebra).

The link between the R -separating coordinates system in Theorem 15 and the (conformal) Killing-Stäckel algebra \mathcal{I} is given as follows: actually, if $K \in \mathcal{I}$, then K , viewed as an endomorphism, admits n principal directions which integrate in the R -separating orthogonal coordinate system.

Finally, from our results on the second order (respectively conformal) symmetries of the conformal Laplacian we can deduce the following characterization for the R -separation of the equation $\Delta_Y \psi = E\psi$ (respectively $\Delta_Y \psi = 0$)

Corollary 17. *The equation $\Delta_Y \psi = E\psi$ (respectively $\Delta_Y \psi = 0$) R -separates in orthogonal coordinates if and only if there exists a (respectively conformal) Killing-Stäckel algebra \mathcal{I} such that $\text{Obs}(K)^p$ is exact for all $K \in \mathcal{I}$.*

Acknowledgements

This research has been partially funded by the Interuniversity Attraction Poles Program initiated by the Belgian Science Policy Office. J. Šilhan was supported by the grant agency of the Czech republic under the grant P201/12/G028.

References

- [1] Benenti S., Chanu C. and Rastelli G., *Remarks on the Connection Between the Additive Separation of the Hamilton-Jacobi Equation and the Multiplicative Separation of the Schrödinger Equation. I. The Completeness and Robertson Conditions*, J. Math. Phys. **43** (2002) 5183-5222.
- [2] Benenti S., Chanu C. and Rastelli G., *Remarks on the Connection Between the Additive Separation of the Hamilton-Jacobi Equation and the Multiplicative Separation of the Schrödinger Equation. II. First Integrals and Symmetry Operators*, J. Math. Phys. **43** (2002) 5223-5253.
- [3] Benenti S., Chanu C. and Rastelli G., *Variable-Separation Theory for the Null Hamilton-Jacobi Equation*, J. Math. Phys. **46** (2005) 042901, 29 pp.
- [4] Bonanos S., *Riemannian Geometry and Tensor Calculus (Mathematica Package)*, version 3.8.5 (2012), <http://www.inp.demokritos.gr/~sbonano/RGTC/>.
- [5] Boyer C., Kalnins E. and Miller W., *R-Separable Coordinates for Three Dimensional Complex Riemannian Spaces*, Trans. Amer. Math. Soc. **242** (1978) 355-376.
- [6] Boyer C., Kalnins E. and Miller W., *Symmetry and Separation of Variables for the Helmholtz and Laplace Equations*, Nagoya Math. J. **60** (1976) 35-80.
- [7] Čap A. and Šilhan J., *Equivariant Quantizations for AHS-Structures*, Adv. Math. **224** (2010) 1717-1734.

-
- [8] Carter B., *Killing Tensor Quantum Numbers and Conserved Currents in Curved Space*, Phys. Rev. D **16** (1977) 3395-3414.
- [9] Chanu C. and Rastelli G., *Fixed Energy R-Separation for Schrödinger Equation*, Int. J. Geom. Methods Mod. Phys. **3** (2006) 489-508.
- [10] Duval C., Lecomte P. and Ovsienko V., *Conformally Equivariant Quantization: Existence and Uniqueness*, Ann. Inst. Fourier (Grenoble) **49** (1999) 1999-2029.
- [11] Duval C. and Ovsienko V., *Conformally Equivariant Quantum Hamiltonians*, Selecta Math. (N.S.) **7** (2001) 291-320.
- [12] Duval C. and Valent G., *Quantum Integrability of Quadratic Killing Tensors*, J. Math. Phys. **46** (2005) 053516, 22 pp.
- [13] Eastwood M., *Higher Symmetries of the Laplacian*, Ann. of Math. (2) **161** (2005) 1645-1665.
- [14] Gover A. and Šilhan J., *Higher Symmetries of the Conformal Powers of the Laplacian on Conformally Flat Manifolds*, J. Math. Phys. **53** (2012) 032301-1-26.
- [15] Ishihara H., Koike T. and Kozaki H., *Exactly Solvable Strings in the Minkowski Spacetime*, Class. Quant. Grav. **27** (2010) 105006, 10 pp.
- [16] Kalnins E. and Miller W., *Conformal Killing Tensors and Variable Separation for Hamilton-Jacobi Equations*, SIAM J. on Math. Analysis **14** (1983) 126-137.
- [17] Kalnins E. and Miller W., *Intrinsic Characterisation of Orthogonal R-Separation for Laplace Equations*, J. Phys. A **15** (1982) 2699-2709.
- [18] Kalnins E. and Miller W., *Killing Tensors and Variable Separation for Hamilton-Jacobi and Helmholtz Equations*, SIAM J. Math. Analysis **11** (1980) 1011-1026.
- [19] Kalnins E. and Miller W., *The Theory of Orthogonal R-Separation for Helmholtz Equations*, Adv. Math. **51** (1984) 91-106.
- [20] Kolář, I., Michor P. and Slovák, J., *Natural Operations in Differential Geometry*, Springer, Berlin 1993.
- [21] Lecomte P., *Towards Projectively Equivariant Quantization*, Progr. Theoret. Phys. Suppl. **144** (2001) 125-132.
- [22] Mathonet P. and Radoux F., *Existence of Natural and Conformally Invariant Quantizations of Arbitrary Symbols*, J. Nonlinear Math. Phys. **17** (2010) 539-556.
- [23] Michel J.-P., *Higher Symmetries of Laplacian via Quantization*, to appear in Annales de l'Institut Fourier 2011.
- [24] Michel J.-P., Radoux F. and Šilhan J., *Second Order Symmetries of the Conformal Laplacian*, SIGMA **10** (2014) 016-1-26.
- [25] Penrose R. and Rindler W., *Spinors and Space-Time vol. 1*, Cambridge Monographs on Mathematical Physics, Two-Spinor Calculus and Relativistic Fields, Cambridge University Press, Cambridge 1984.
- [26] Perelomov A., *Integrable Systems of Classical Mechanics and Lie Algebras vol. 1*, Translated from the Russian by A. Reyman, Birkhäuser, Basel, 1990.
- [27] Šilhan J., *Conformally Invariant Quantization - Towards Complete Classification*, Differ. Geom. Appl. **33** (2014), 162-176.