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AN APPLICATION OF PROLONGATION ALGEBRAS TO DETERMINE BÄCKLUND TRANSFORMATIONS FOR NONLINEAR EQUATIONS

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Abstract. Prolongation algebras which are determined by applying a version of the Wahlquist-Estabrook method to three different nonlinear partial differential equations can be employed to obtain not only Lax pairs but Bäcklund transformations as well. By solving Maurer-Cartan equations for the related group specified by the prolongation algebra, a set of differential forms is obtained which can lead directly to these kinds of results. Although specific equations are studied, the approach should be applicable to large classes of partial differential equations.

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1. Introduction

Surfaces which have constant Gaussian curvature are of great interest for a variety of reasons. There is a correspondence between solutions of certain nonlinear partial differential equations and manifolds of constant Gaussian curvature. The types of equations which pertain to constant Gaussian curvature are in fact $\sinh$ and sine-Gordon type equations. These are nonlinear partial differential equations and possess solutions which will have a solitonic character in general [1, 9, 11]. If a particular solution for one of these equations can be obtained, it can then be used to obtain a surface in, for example, three space by means of the structure equations for a two-dimensional manifold [10]. Moreover, by formulating the equations with the Gaussian curvature kept as an explicit parameter, it can be thought that a deformation parameter has been introduced into the picture. Therefore, the prolongation algebras will contain a deformation parameter which is directly related to the Gaussian curvature.

These equations can be analyzed by using a version of the method proposed first by Wahlquist and Estabrook, and it will simply be summarized at the start [13, 14]. The method has been discussed at length already [3,5], so an overview will suffice. To study the integrability and Bäcklund properties of nonlinear equations, a closed differential ideal of two-forms is proposed [4, 6]. It has to be shown that the ideal is closed and it must produce the equation of interest on transversal integral manifolds. These forms are then used to obtain Wahlquist-Estabrook prolongations and the associated prolongation algebra. Based on this differential system, a nontrivial prolongation structure for the equation is obtained, and it is immediately possible to write down a Lax pair for the equation as well. What is done here is very similar to that done in [4], however, the procedure here is modified to a more systematic form, and new results are obtained as well. Both approaches could be regarded as applications of Cartan-cocycles [5].

Once the algebra has been formulated, the next step is to give an explicit matrix representation for the elements of the algebra. In terms of these matrices, a Maurer-Cartan form for the group is defined by means of a set of one-forms to be determined. This matrix of one-forms is required to satisfy the Maurer-Cartan equation for the associated group. This yields a differential system for the component forms, and it is required to obtain a solution for these component forms in the subsequent step of the procedure. Any representation which satisfies the prolongation algebra should suffice. Different representations should lead to other Bäcklund transformations in the end. However, it is advisable to obtain matrices as concise as possible so that the resulting differential system is not difficult to solve. For the cases considered here, solutions to the Maurer-Cartan system can be constructed explicitly.

The final step is to use these one-forms to create a Bäcklund transformation for the equation. Based on this Bäcklund transformation, the theorem of permutability [7] can be formulated for the equation. This is a kind of algebraic, nonlinear superposition principle which is used to determine a new solution to the equation from given solutions. The permutability theorem serves to provide a framework for generating new multi-soliton solutions for the equation. These can in turn be used to generate multisoliton surfaces by integrating the corresponding structure equations [8, Chapter 6]. Although this procedure is illustrated by applying it to the sine-Gordon equation, it is quite general provided the requisite one-forms can be produced. It will be shown that it gives a systematic method of obtaining Lax pairs and Bäcklund transformations for nonlinear equations [2].

2. Generating Prolongation Algebras

The analysis done here will be relevant to equations which can be written in the form

$$u_{xt} = Q(u). \tag{1}$$

Let M be the differentiable manifold $M = \mathbb{R}^5(x,t,u,p,q)$ and, in terms of the coordinates of M, define the following differential system $\{\alpha^i\}$ on this manifold

$$\begin{aligned} \alpha^1 &= \mathrm{d}u \wedge \mathrm{d}x + q\,\mathrm{d}x \wedge \mathrm{d}t, \qquad & \alpha^2 &= \mathrm{d}u \wedge \mathrm{d}t - p\,\mathrm{d}x \wedge \mathrm{d}t \\ \alpha^3 &= \mathrm{d}p \wedge \mathrm{d}x + \mathrm{d}q \wedge \mathrm{d}t, \qquad & \alpha^4 &= \mathrm{d}p \wedge \mathrm{d}x + Q(u)\,\mathrm{d}x \wedge \mathrm{d}t. \end{aligned} \tag{2}$$

For the cases investigated here, the function $Q(u)$ will be continuously differentiable with respect to u. The exterior derivatives of the forms α^i in (2) can be calculated and expressed as

$$\begin{aligned} &\mathrm{d}\alpha^1 = -\mathrm{d}x \wedge \mathrm{d}q \wedge \mathrm{d}t = -\alpha^3 \wedge \mathrm{d}x, \qquad \mathrm{d}\alpha^2 = -\mathrm{d}p \wedge \mathrm{d}x \wedge \mathrm{d}t = -\alpha^3 \wedge \mathrm{d}t \\ &\mathrm{d}\alpha^3 = 0, \qquad \mathrm{d}\alpha^4 = Q'(u)\,\mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}t = Q'(u)\,\alpha^1 \wedge \mathrm{d}t. \end{aligned} \tag{3}$$

Therefore, the ideal is closed and by the Frobenius theorem, $\{\alpha^i\}$ is integrable. Let S be a section of the projection $\pi : M \to \mathbb{R}^2$ so that $\pi(x,t,u,p,q) = (x,t)$. The transversal integral manifolds are given by

$$S(x,t) = (x,t,u(x,t),p(x,t),q(x,t)). \tag{4}$$

By working out $S^*\alpha^i$, the expression $S^*\alpha^i = 0$ implies that the relations $u_t = q$, $u_x = p$ and $p_t = Q(u)$ hold on transverse integral manifolds specifying solutions to (1). To introduce prolongation variables $\mathbf{y}$, define the fiber bundle $\tilde{M} = M \times \mathbb{R}^k$, where $\mathbb{R}^k$ carries coordinates $\mathbf{y} = (y^1, \cdots, y^k)$. A vertical valued one-form is then defined as

$$\eta^i = F^i(x,t,u,p,q,\mathbf{y})\,\mathrm{d}x + G^i(x,t,u,p,q,\mathbf{y})\,\mathrm{d}t. \tag{5}$$

The prolongation condition requires that the expression $(\mathrm{d}\eta + \frac{1}{2}[\eta,\eta])^i$ be expressible as a linear combination of the forms in (2) so that

$$(\mathrm{d}F \wedge \mathrm{d}x + \mathrm{d}G \wedge \mathrm{d}t + [F,G]\,\mathrm{d}x \wedge \mathrm{d}t) = \sum_{j=1}^{4} \tau_j \alpha^j. \tag{6}$$

Substituting (2) and then comparing the coefficients of the two-forms on both sides of (6), the following system of differential constraints is obtained

$$\begin{gathered} F_u = \tau_1, \quad F_p = \tau_3 + \tau_4, \quad F_q = 0, \quad G_u = \tau_2, \quad G_p = 0, \quad G_q = \tau_3 \\ -F_t + G_x + [F,G] = \tau_1 q - \tau_2 p + \tau_4 Q(u). \end{gathered} \tag{7}$$

The subscripts indicate partial differentiation with respect to the indicated variables. Eliminating the set of τ_i from equations (7), they are found to reduce to

$$F_q = 0, \qquad G_p = 0, \qquad [F, G] = F_u q - G_u p + (F_p - G_q) Q(u). \tag{8}$$

The first two equations imply that $F = F(u, p, \mathbf{y})$ and $G = G(u, q, \mathbf{y})$. The dependence on the y_i variables can be pushed into the noncommuting variables. Let $G = G(u)$, so that the left-hand side of the last equation in (8) is independent of q. This forces $F_u = 0$ and it suffices to take

$$F = X_1 + pX_2, \qquad G = G(u) \tag{9}$$

where the X_i will be found to satisfy some set of brackets that constitute an algebra. The remaining equation of (8) then becomes

$$[X_1 + pX_2, G] = -G_u\, p + Q(u)\, X_2. \tag{10}$$

Equating coefficients of the variable p on both sides of (10), it is found that the following pair of brackets must hold

$$[X_1, G] = Q(u) X_2, \qquad [X_2, G] = -G_u. \tag{11}$$

To make further progress in completing the definition of an algebra, it is necessary to specify a form for the function $Q(u)$ in (11).

3. Exponential Equation

First, an equation which is specified by taking $Q(u) = f\mathrm{e}^u$ in (11) will be discussed. Here, f is a nonzero real constant. It suffices to take $G = \mathrm{e}^u X_3$, and (11) consists of an algebra which is made up of two brackets

$$[X_1, X_3] = f X_2, \qquad [X_2, X_3] = -X_3. \tag{12}$$

Substituting G into (9), the following result holds [5].

Theorem 1. *A Lax pair for the equation*

$$u_{xt} = f\mathrm{e}^u \tag{13}$$

can be expressed in terms of the elements X_i of algebra (12) *as*

$$\mathbf{y}_x = -(X_1 + u_x X_2)\mathbf{y}, \qquad \mathbf{y}_t = -\mathrm{e}^u X_3 \mathbf{y}. \tag{14}$$

The system in (14) *holds irrespective of the particular representation taken for the X_i provided they satisfy the algebra* (12).

Proof: The compatibility condition for (14) is given by

$$-u_{xt} X_2 + \mathrm{e}^u [X_1, X_3] + u_x \mathrm{e}^u [X_2, X_3] + u_x \mathrm{e}^u X_3 = 0.$$

Substituting (12), this expression holds provided that u is a solution of equation (13). ■

Following the outline in the introduction, let us write down a specific representation of (12) in terms of 2×2 matrices. This need not be unique and the following set will suffice

$$X_1 = \begin{pmatrix} 1 & 0 \\ \frac{f}{2} & 1 \end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{15}$$

A Maurer-Cartan form corresponding to (14) can now be written down. Substituting representation (15) we obtain

$$\underline{\omega} = \omega_i X_i = \begin{pmatrix} \omega_1 - \frac{1}{2}\omega_2 & \omega_3 \\ \frac{f}{2}\omega_1 & \omega_1 + \frac{1}{2}\omega_2 \end{pmatrix}. \tag{16}$$

The forms ω_i can be determined, by requiring that the forms $\underline{\omega}$ satisfy the Maurer-Cartan equation

$$\mathrm{d}\underline{\omega} + \underline{\omega} \wedge \underline{\omega} = 0. \tag{17}$$

This equation is equivalent to the following differential system

$$\begin{aligned} &\mathrm{d}\omega_1 - \frac{1}{2}\mathrm{d}\omega_2 + \frac{f}{2}\omega_3 \wedge \omega_1 = 0, \qquad && \mathrm{d}\omega_3 + \omega_3 \wedge \omega_2 = 0 \\ &\mathrm{d}\omega_1 + \frac{1}{2}\mathrm{d}\omega_2 + \frac{f}{2}\omega_1 \wedge \omega_3 = 0, \qquad && \mathrm{d}\omega_1 - \omega_1 \wedge \omega_2 = 0. \end{aligned} \tag{18}$$

Only one solution to (18) is required to produce a Bäcklund tranformation. It can be shown by substituting that the following set of one-forms constitutes a solution to (18)

$$\omega_1 = \mathrm{d}x, \qquad \omega_2 = u_x\,\mathrm{d}x, \qquad \omega_3 = \mathrm{d}x + \mathrm{e}^u\,\mathrm{d}t. \tag{19}$$

It will be shown that a Bäcklund transformation for the equation can be constructed based on a form with a Riccatti structure, namely

$$\mathrm{d}\psi = \omega_3 + \omega_2\psi - \frac{1}{2}f\omega_1\psi^2. \tag{20}$$

Theorem 2. *There exists a Bäcklund transformation for the equation* $u_{xt} = f\mathrm{e}^u$ *which has the following form*

$$\psi_x = 1 + u_x\psi - \frac{\lambda}{2}f\psi^2, \qquad \psi_t = \frac{1}{\lambda}\mathrm{e}^u. \tag{21}$$

In (21), λ *is a spectral parameter.*

Proof: Substituting the forms (19) into (20) and writing $\mathrm{d}\psi = \psi_x\,\mathrm{d}x + \psi_t\,\mathrm{d}t$, system (21) appears immediately without the spectral parameter. The spectral parameter can be included by placing it near the constant f, which is not a spectral parameter. To verify the compatibility condition, the required derivatives are given by

$$\psi_{xt} = u_{xt}\psi + \frac{1}{\lambda}u_x \mathrm{e}^u - \lambda f \psi \frac{1}{\lambda}\mathrm{e}^u, \qquad \psi_{tx} = \frac{1}{\lambda}\mathrm{e}^u u_x. \tag{22}$$

Thus the compatibility condition reduces to $u_{xt}\psi + \frac{1}{\lambda}u_x\mathrm{e}^u - f\mathrm{e}^u\psi = \frac{1}{\lambda}\mathrm{e}^u u_x$. This is independent of the spectral parameter and is proportional to the desired equation, so the compatibility condition holds provided u satisfies equation (13). ■

4. The Sine-Gordon Equation

The next equation to be considered is obtained by taking $Q(u) = f\sin u$ in (1), where $f > 0$ is a real constant. This gives the sine-Gordon equation

$$u_{xt} - f\sin u = 0. \tag{23}$$

In an application of (23) to surfaces, f can be related to the absolute value of the Gaussian curvature of the surface [4]. This equation has already been considered [4], and a Lax pair was presented, however here the results will be obtained by means of a different approach. A closed algebra for the X_i can be obtained by taking $G = X_1\cos u + f\,X_3\sin u$ in (11). Substituting $Q(u)$ and $G(u)$ into (11) and identifying the coefficients of $\sin u$ and $\cos u$ on both sides, the following algebra is obtained

$$[X_1, X_3] = X_2, \qquad [X_1, X_2] = fX_3, \qquad f[X_2, X_3] = X_1. \tag{24}$$

The X_i in (24) can be said to make up a deformed $\mathfrak{sl}(2,\mathbb{R})$ algebra. A representation in terms of the following 2×2 matrices exists of the form

$$X_1 = \frac{\sqrt{f}}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \frac{1}{2\sqrt{f}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

At this point, three one-forms ω_i are introduced in order to define a Maurer-Cartan form for the group of (24). It is described by the following matrix of one-forms

$$\underline{\omega} = \frac{1}{2}\begin{pmatrix} \dfrac{1}{\sqrt{f}}\omega_1 & -\sqrt{f}\omega_2 + \omega_3 \\ -\sqrt{f}\omega_2 - \omega_3 & -\dfrac{1}{\sqrt{f}}\omega_1 \end{pmatrix}. \tag{26}$$

It is now required that matrix (26) be required to satisfy (17). The matrix obtained from (17) is composed of the following collection of equations

$$\frac{1}{\sqrt{f}}\mathrm{d}\omega_1 + \frac{1}{2}(\sqrt{f}\omega_2 - \omega_3)\wedge(\sqrt{f}\omega_2 + \omega_3) = 0$$

$$-\sqrt{f}\mathrm{d}\omega_2 + \mathrm{d}\omega_3 + \frac{1}{2\sqrt{f}}\omega_1 \wedge (-\sqrt{f}\omega_2 + \omega_3) + \frac{1}{2\sqrt{f}}(\sqrt{f}\omega_2 - \omega_3) = 0 \tag{27}$$

$$-\sqrt{f}\mathrm{d}\omega_2 - \mathrm{d}\omega_3 - \frac{1}{2\sqrt{f}}(\sqrt{f}\omega_2 + \omega_3) \wedge \omega_1 + \frac{1}{2\sqrt{f}}\omega_1 \wedge (\sqrt{f}\omega_2 + \omega_3) = 0.$$

Simplifying and taking linear combinations of the second and third equations in (27), system (27) simplifies to the following differential system

$$\mathrm{d}\omega_1 + f\omega_2 \wedge \omega_3 = 0, \qquad -f\mathrm{d}\omega_2 + \omega_1 \wedge \omega_3 = 0, \qquad \mathrm{d}\omega_3 - \omega_1 \wedge \omega_2 = 0. \quad (28)$$

It suffices to obtain one nontrivial solution to these equations. One solution which is not complicated is given by

$$\omega_1 = \sqrt{f}\sin u\,\mathrm{d}t, \qquad \omega_2 = \sqrt{f}\,\mathrm{d}x + \frac{1}{\sqrt{f}}\cos u\,\mathrm{d}t, \qquad \omega_3 = u_x\,\mathrm{d}x. \quad (29)$$

This solution can be verified by calculating the derivatives and substituting into system (28).

A Bäcklund transformation for (23) can be obtained using solution (29) by defining a one-form $d\psi$ as a linear combination of the ω_i in (29)

$$\mathrm{d}\psi = a\omega_3 - b\sin\psi\,\omega_1 - c\cos\psi\omega_2. \quad (30)$$

The real constants a, b, c appearing in (30) can be determined by working out the compatibility condition $\psi_{xt} - \psi_{tx} = 0$. The presence of the parameter f in the solution makes it very clear how to include a Bäcklund parameter into the results.

Theorem 3. *A Bäcklund transformation for* (23) *exists of the form*

$$\psi_x = u_x - \lambda f\cos\psi, \qquad \psi_t = -\frac{1}{\lambda}\sin u\sin\psi - \frac{1}{\lambda}\cos u\cos\psi \quad (31)$$

where λ is a spectral parameter.

Proof: Differentiating both expressions in (31) there results the second derivatives

$$\psi_{xt} = u_{xt} + \lambda f\sin\psi\psi_t$$

$$\psi_{tx} = -\frac{1}{\lambda}u_x\cos u\sin\psi - \frac{1}{\lambda}\sin u\cos\psi\psi_x + \frac{1}{\lambda}u_x\sin u\cos\psi + \frac{1}{\lambda}\cos u\sin\psi\psi_x.$$

The compatibility condition then reduces to

$$\begin{aligned} u_{xt} & - f\sin u\sin^2\psi - f\cos u\cos\psi\sin\psi + \frac{1}{\lambda}u_x\cos u\sin\psi + \frac{1}{\lambda}\sin u\cos\psi u_x \\ & - f\sin u\cos^2\psi - \frac{1}{\lambda}u_x\sin u\cos\psi - \frac{1}{\lambda}u_x\cos u\sin\psi + f\cos u\sin\psi\cos\psi \\ = {} & u_{xt} - f\sin u. \end{aligned}$$

The spectral parameter is not present at the end and what remains is exactly the equation (23) which vanishes on solutions. ■

5. The Sinh-Gordon Equation

The last equation to be considered is the one in which $Q(u) = g\sinh u$. In (1) this gives

$$u_{xt} - g\sinh u = 0. \tag{32}$$

Here, g is a positive real constant which can be related to the Gaussian curvature of a surface in a geometric application. It was also determined in [4] that (42) admits the following prolongation algebra given as

$$[X_1, X_3] = X_2, \qquad [X_1, X_2] = gX_3, \qquad g[X_2, X_3] = -X_1. \tag{33}$$

An explicit representaion for the algebra in (33) in terms of 2×2 matrices is given as

$$X_1 = \frac{\sqrt{g}}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad X_2 = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \frac{\mathrm{i}}{2\sqrt{g}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{34}$$

Introducing again a set of one-forms ω_i, the following Maurer-Cartan form $\underline{\omega}$ can be constructed

$$\underline{\omega} = \frac{1}{2}\begin{pmatrix} \dfrac{\mathrm{i}}{\sqrt{g}}\omega_1 & -\sqrt{g}\omega_2 + \mathrm{i}\omega_3 \\ -\sqrt{g}\omega_2 - \mathrm{i}\omega_3 & -\dfrac{\mathrm{i}}{\sqrt{g}}\omega_1 \end{pmatrix}. \tag{35}$$

The Maurer-Cartan equation (17) can be written down by substituting the matrix (35). By taking suitable linear combinations of the matrix elements, the following differential system results

$$\mathrm{d}\omega_1 + g\omega_2\wedge\omega_3 = 0, \qquad g\mathrm{d}\omega_2 + \omega_1\wedge\omega_3 = 0, \qquad \mathrm{d}\omega_3 - \omega_1\wedge\omega_2 = 0. \tag{36}$$

To illustrate a procedure for obtaining a solution to (36) which need not be unique, it will be shown that there exists classes of solutions of the following form

$$\begin{aligned} \omega_1 &= b_1\sinh u\,\mathrm{d}t, \qquad \omega_2 = (b_2 + b_3u_x)\mathrm{d}x + b_4\cosh u\,\mathrm{d}t \\ \omega_3 &= (b_5 + b_6u_x)\,\mathrm{d}x + b_7\cosh u\,\mathrm{d}t. \end{aligned} \tag{37}$$

To obtain the unknown coefficients b_i in (37), substitute the forms (37) into the differential system (36). The equations in (36) hold provided the b_i satisfy the following equations

$$\begin{aligned} b_1 + gb_3b_7 - gb_4b_6 = 0, \qquad b_2b_7 - b_4b_5 = 0, \qquad b_1b_5 - g^2b_3 = 0 \\ gb_4 - b_1b_6 = 0, \qquad b_1b_2 - gb_6 = 0, \qquad b_7 + b_1b_3 = 0. \end{aligned} \tag{38}$$

There are several different general solutions to (38), however, only one is given $b_1, b_2 = \pm\frac{g}{b_1}\sqrt{1-gb_3^2}$, $b_3, b_4 = \frac{b_1}{g}\sqrt{1-gb_3^2}$, $b_5 = -\frac{b_3}{b_1}g^2$, $b_6 = \pm\sqrt{1-gb_3^2}$,

$b_7 = -b_1 b_3$. Setting $b_1 = 1$ and $b_3 = 0$ in this solution leads to the following system of one-forms of the form (37)

$$\omega_1 = \sinh u \, \mathrm{d}t, \qquad \omega_2 = g \, \mathrm{d}x + \frac{1}{g} \cosh u \, \mathrm{d}t, \qquad \omega_3 = u_x \, \mathrm{d}x. \tag{39}$$

Finally, a Bäcklund transformation for (32) will be obtained by means of solution (39) by means of the following linear combination of ω_i

$$\mathrm{d}\psi = a\omega_3 + b \sinh \psi \omega_1 + c \cosh \psi \, \omega_2 \tag{40}$$

where again a, b and c are real constants to be determined by the compatibility relation.

Theorem 4. *Equation* (32) *admits a Bäcklund transformation of the form*

$$\psi_x = u_x - \lambda g^{3/2} \cosh \psi, \qquad \psi_t = -\frac{1}{\lambda\sqrt{g}} \sinh \psi \sinh u + \frac{1}{\lambda\sqrt{g}} \cosh \psi \cosh u \tag{41}$$

where λ is a spectral parameter.

Proof: The compatibility condition for system (41) takes the form

$$\begin{aligned} u_{xt} &+ g \sinh^2 \psi \sinh u - g \cosh^2 \psi \sinh u - g \sinh \psi \cosh \psi \cosh u \\ &+ \frac{1}{\lambda\sqrt{g}} u_x \cosh u \sinh u + \frac{1}{\lambda\sqrt{g}} u_x \sinh \psi \cosh u - \frac{1}{\lambda\sqrt{g}} u_x \sinh \psi \cosh u \\ &+ g \sinh \psi \cosh \psi \cosh u - \frac{1}{\lambda\sqrt{g}} u_x \cosh \psi \sinh u = 0. \end{aligned}$$

The left-hand side simplifies to exactly (32) and clearly holds on its solutions. ■

6. Formulation of the Theorem of Permutability

The theorem of permutability will be formulated for each of the equations presented, and for clarity the details will be shown for the first two.

1) Using Bäcklund transformation (21), from Theorem 3.2, two solutions (u_0, u_1) of equation (13) can be related by using a Bäcklund parameter λ_1 by means of the first equation in (21). Next solution u_1 can be related to another solution u_{12} by means of Bäcklund parameter λ_2. This produces the following set of equations

$$u_{1,x} = 1 + u_{0,x} u_1 - \frac{1}{2} \lambda_1 f u_1^2, \qquad u_{12,x} = 1 + u_{1,x} u_{12} - \frac{1}{2} \lambda_2 f u_{12}^2. \tag{42}$$

This procedure is repeated with the same Bäcklund parameters but in reverse order beginning with u_0 and arriving at a solution u_{21} giving

$$u_{2,x} = 1 + u_{0,x} u_2 - \frac{1}{2} \lambda_2 f u_2^2, \qquad u_{21,x} = 1 + u_{2,x} u_{21} - \frac{1}{2} \lambda_1 f u_{21}^2. \tag{43}$$

The theorem of permutability requires that the functions satisfy $u_{12} = u_{21} = U$. Solving (42) and (43) for U_x and equating, we obtain

$$(1 + u_{0,x}u_1 - \frac{1}{2}\lambda_1 f u_1^2)U - \frac{1}{2}\lambda_2 f U^2 = (1 + u_{0,x}u_2 - \frac{1}{2}\lambda_2 f u_2^2)U - \frac{1}{2}\lambda_1 f U^2.$$

The derivatives $u_{1,x}$ and $u_{2,x}$ have been substituted as well. Finally, solving this for the function U, the following transform is obtained

$$U = \frac{1}{\lambda_1 - \lambda_2}\big(\frac{2}{f}(u_2 - u_1)u_{0,x} - (\lambda_2 u_2^2 - \lambda_1 u_1^2)\big). \tag{44}$$

2) From Bäcklund transformation (31) in Theorem 4.1, two solutions of equation (23) (u_0, u_1) can be linked by means of a Bäcklund parameter λ_1 using the first equation in (31). Next, solution u_1 can be linked to a solution u_{12} using the Bäcklund parameter λ_2. The following sequence $u_0 \to u_1 \to u_{12}$ results, as well as the equations

$$(u_1 - u_0)_x = -\lambda_1 f \cos u_1, \qquad (u_{12} - u_1)_x = -\lambda_2 f \cos u_{12}. \tag{45}$$

Repeating this procedure with these same parameters but in reverse order starting from u_0, the solution u_{21} is obtained corresponding to the sequence $u_0 \to u_2 \to u_{21}$. This gives the differential relations

$$(u_2 - u_0)_x = -\lambda_2 f \cos u_2, \qquad (u_{21} - u_2)_x = -\lambda_1 f \cos u_{21}. \tag{46}$$

The theorem of permutability demands again that $u_{12} = u_{21} = U$ hold. Adding (45) and (46) pairwise yields the following system with $\mu_i = f\lambda_i$,

$$(U - u_0)_x = -\mu_1 \cos u_1 - \mu_2 \cos U, \qquad (U - u_0)_x = -\mu_1 \cos U - \mu_2 \cos u_2. \tag{47}$$

Since the left-hand sides of (47) are identical, the two expressions on the right can be equated producing an equation for $\cos U$. Solving the result for $\cos U$, it is found that

$$\cos U = \frac{\lambda_2 \cos u_2 - \lambda_1 \cos u_1}{\lambda_2 - \lambda_1}. \tag{48}$$

3) The transformations for the remaining equation (32) can be calculated following the same steps as in 2) using (41) in Theorem 5.1. The result is summarized as follows

$$\cosh U = \frac{\lambda_2 \cosh u_2 - \lambda_1 \cosh u_1}{\lambda_2 - \lambda_1}. \tag{49}$$

Results (44), (48) and (49) relate corresponding solutions of their respective non-linear equations.

7. Summary

A brief review of a method for producing prolongation algebras has been presented. The procedure starts by picking an exterior differential system such that it is closed and the specific equation is produced on transverse integral manifolds. It is shown in detail how this prolongation algebra can be used to construct a Maurer-Cartan form for the associated group. By selecting a specific representation for the algebra, a differential system can be obtained which is then solved for a set of basic one-forms. It is this part that may present most difficulties since the resulting differential system must be concise enough to admit a reasonable solution for the set of one-forms. Finally, a procedure which makes use of these forms to determine a Bäcklund transformation for the equation has been given.

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