Fifteenth International Conference on
Geometry, Integrability and Quantization
June 7–12, 2013, Varna, Bulgaria
Ivaïlo M. Mladenov, Andrei Ludu
and Akira Yoshioka, Editors
Avangard Prima, Sofia 2014, pp 249–258
doi: 10.7546/giq-15-2014-249-258

A CONSTRUCTION OF A RECURSION OPERATOR FOR SOME SOLUTIONS OF EINSTEIN FIELD EQUATIONS

TSUKASA TAKEUCHI

Department of Mathematics, Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

Abstract. The $(1, 1)$-tensor field on symplectic manifold that satisfies some integrability conditions is called a recursion operator. It is known the recursion operator is a characterization for integrable systems, and gives constants of motion for integrable systems. We construct recursion operators for the geodesic flows of some solutions of Einstein equation like Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman metrics.

1. Introduction

Liouville proved that when a Hamiltonian system with n degrees of freedom on a $2n$-dimensional phase space has n independent first integrals in involution the system is integrable by quadratures (cf [1]).

On the other hand, de Filippo, Marmo, Salerno and Vilasi (see e.g. [2, 3, 6, 10] and [11]) proposed a new characterization of integrable systems. Let us consider a vector field on $\mathcal{M}^{2n}$.

Theorem 1 ([11])**.** *A vector field X is separable, integrable and Hamiltonian for certain symplectic structure when X admits an invariant, mixed, diagonalizable $(1, 1)$-tensor field T with vanishing Nijenhuis torsion and doubly degenerate eigenvalues without stationary points. Then, the vector field X is a separable and completely integrable Hamiltonian system with respect to the symplectic structure in the sense of Liouville.*

Now, the operator T in Theorem 1 is called a **recursion operator**. Several examples of recursion operators e.g., the harmonic oscillator and the Kepler dynamics, are given in [6] and [11]. In this paper we consider geodesic flows for the

Minkowski and the Kerr-Newman metrics. We construct recursion operators for these metrics, and moreover, we obtain constants of motion.

2. The Geodesic Flow for the Minkowski Metric

In this section, our aim is to construct a recursion operator for the geodesic flow for the Minkowski metric.

Now, we construct a vector field X on the phase space for the geodesic flow for the **Minkowski metric**. A matrix g_{ij} of the Minkowski metric is

$$g_{ij} = g^{ij} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and the equation of geodesics is

$$\frac{\mathrm{d}^2 q^\kappa}{\mathrm{d}t^2} + \Gamma^{\,\kappa}_{\mu\nu} \frac{\mathrm{d}q^\mu}{\mathrm{d}t}\frac{\mathrm{d}q^\mu}{\mathrm{d}t} = \frac{\mathrm{d}^2 q^\kappa}{\mathrm{d}t^2} = 0, \qquad \kappa = 1, 2, 3, 4.$$

If we put

$$v^\kappa = \frac{\mathrm{d}q^\kappa}{\mathrm{d}t}$$

then we have a system of first order differential equations on TM

$$\dot{q}^\kappa = v^\kappa, \qquad \dot{v}^\kappa = -\Gamma^{\,\kappa}_{\mu\nu} v^\mu v^\nu = 0.$$

From the above equations, we get a geodesic spray

$$X = v^\kappa \frac{\partial}{\partial q^\kappa} - \Gamma^{\,\kappa}_{\mu\nu} v^\mu v^\nu \frac{\partial}{\partial v^\kappa} = v^\kappa \frac{\partial}{\partial q^\kappa}\cdot$$

By setting $p_\kappa = g_{\kappa\varepsilon} v^\varepsilon$, the vector field X is equivalently transformed to the vector field X on T^*M such that

$$X = \sum_{k=1}^{4} \left(\dot{q}_k \frac{\partial}{\partial q_k} - \dot{p}_k \frac{\partial}{\partial p_k} \right) = -p_1 \frac{\partial}{\partial q_1} + \sum_{k=2}^{4} p_k \frac{\partial}{\partial q_k}\cdot$$

The vector field X is a Hamiltonian vector field of a certain Hamiltonian function. Taking the canonical symplectic form ω

$$\omega = \sum_{k=1}^{4} \mathrm{d}p_k \wedge \mathrm{d}q_k$$

the function H is found to be

$$H = \frac{1}{2} \left(-p_1^2 + \sum_{k=2}^{4} p_k^2 \right). \tag{1}$$

Then, we see that (as should be) we have

$$i_X\omega = -\mathrm{d}H.$$

A vector field X is called a **Hamiltonian vector field** of the Hamiltonian function H which will be denoted by X_H. Next, we consider the Hamilton-Jacobi equation by the Hamiltonian function (1). The function (1) does not include q_k, $k = 1,2,3,4$, therefore $p_k,\ k = 2,3,4$ are circular coordinates. We consider the **Hamilton-Jacobi equation**

$$E = H\left(q, \frac{\partial W}{\partial q}\right)$$

where E is a constant. We set the **generating function** in the flow

$$W = \sum_{i=1}^{4} W_i(q_i) = \sum_{i=1}^{4} W_i.$$

Since $p_k = \dfrac{\partial W_k}{\partial q_k},\ k = 2,3,4$ are first integrals, we set $a_k = \dfrac{\partial W_k}{\partial q_k}$. Then we see

$$2E = -\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \sum_{k=2}^{4} a_k^2.$$

Thus, the generating function W is

$$W = \sqrt{\sum_{k=2}^{4} a_k^2 - 2E}\ q_1 + \sum_{k=2}^{4} a_k q_k.$$

In addition, we determine canonical coordinates using the generating function W. We put

$$Q_1 = H, \qquad Q_k = \frac{\partial W_k}{\partial q_k}, \qquad k = 2,3,4$$

and then the canonical coordinates (P,Q) are given by

$$Q_1 = H,\ \ Q_k = \frac{\partial W_k}{\partial q_k},\ \ P_1 = -\frac{\partial W}{\partial Q_1} = \frac{q_1}{p_1},\ \ P_k = -\frac{\partial W}{\partial Q_k} = -\frac{q_1 p_k}{p_1} - q_k.$$

Here $Q_k,\ k = 1,2,3,4$ are constants, but we consider that they are variables. Hence, the relationship between the canonical coordinates (P,Q) and the original coordinates (p,q) is

$$p_1 = \sqrt{\sum_{k=2}^{4} Q_k^2 - 2Q_1},\ q_1 = P_1\sqrt{\sum_{k=2}^{4} Q_k^2 - 2Q_1},\ p_k = Q_k,\ q_k = -P_k - Q_k P_1.$$

Let us introduce a tensor field T of $(1,1)$ type of the form

$$T = \sum_{i=1}^{4} Q_i \left(\frac{\partial}{\partial P_i} \otimes \mathrm{d}P_i + \frac{\partial}{\partial Q_i} \otimes \mathrm{d}Q_i \right). \tag{2}$$

We state also the general result as

Lemma 2. *Let* $M = \sum_{i,j=1}^{2n} m^i{}_j \frac{\partial}{\partial x_j} \otimes \mathrm{d}x_i$ *be a* $(1,1)$*-tensor field on* $\mathbb{R}^{2n}$ *such that* $(m^i{}_j) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, $A = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ *and let us consider the vector fields* $X_j = -\frac{\partial}{\partial x_{n+j}}$, $j = 1, \ldots, n$ *on* $\mathbb{R}^{2n}$. *Then we have vanishing Nijenhuis torsion* $\mathcal{N}_M = 0$ *and* $\mathcal{L}_{X_j} M = 0$.

Then we see that $\mathcal{L}_{X_H} T = 0$, $\mathcal{N}_T = 0$ and $\deg Q_i = 2$ for the equation (2). Thus, the $(1,1)$-tensor field T is a recursion operator for X_H.
It is known that the traces $\mathrm{Tr}(T)$, $\mathrm{Tr}(T^2)$, $\mathrm{Tr}(T^3)$ and $\mathrm{Tr}(T^4)$ are constants of motion (see, [10] and [2]). If we express T and $\mathrm{Tr}(T^\ell)$ in the original coordinates (q,p), they are written respectively as

$$T = \sum_{i,j=1}^{4} \left(\left({}^tA\right)^i{}_j \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_j + B^i{}_j \frac{\partial}{\partial q_i} \otimes \mathrm{d}p_j + A^i{}_j \frac{\partial}{\partial q_i} \otimes \mathrm{d}q_j \right)$$

$$\mathrm{Tr}(T^\ell) = \frac{1}{2^{\ell-1}} \left(-p_1^2 + p_2^2 + p_3^2 + p_4^2\right)^\ell + 2\left(p_2^\ell + p_3^\ell + p_4^\ell\right), \qquad \ell = 1,2,3,4$$

$$\text{where} \quad A = \begin{pmatrix} H & & & \\ \frac{p_2}{p_1}(p_2 - H) & p_2 & & \\ \frac{p_3}{p_1}(p_3 - H) & & p_3 & \\ \frac{p_4}{p_1}(p_4 - H) & & & p_4 \end{pmatrix}, \qquad B = \frac{q_1}{p_1}({}^tA - A).$$

Here, we introduce a vector field Γ following the method of [11]. In the coordinate system (Q,P) T is represented by the matrix

$$T = \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & Q_3 & \\ & & & Q_4 \end{pmatrix}.$$

We define also a two-form ω_1 and a vector field Γ by the formulas

$$\omega_1 := \sum_{i=1}^{4} \mathrm{d}K_i \wedge \mathrm{d}Q_i, \qquad \Gamma := \sum_{i=1}^{4} K_i \frac{\partial}{\partial P_i}$$

where $K_i = Q_i P_i,\ i = 1, 2, 3, 4$, respectively. Then ω_1 is a symplectic form and satisfies

$$\omega_1 = \mathcal{L}_\Gamma \omega.$$

The symplectic form ω_1 is the Lie derivative of the symplectic form ω with respect to the vector field Γ. We construct vector fields recursively by using X_H, Γ such that $X_{k+1} := [X_k, \Gamma]$, $(X_0 = X_H)$. Then we have

$$X_0 = -\frac{\partial}{\partial P_1}, \qquad X_1 = -Q_1 \frac{\partial}{\partial P_1}, \qquad X_2 = -Q_1^2 \frac{\partial}{\partial P_1}, \qquad X_3 = -Q_1^3 \frac{\partial}{\partial P_1}.$$

In this case, we only consider X_0, X_1, X_2 and X_3.
We define the following Poisson bracket $\{\cdot\,,\cdot\}_1$ of the symplectic form ω_1

$$\{f, g\}_1 := \sum_{i,j=1}^{n} \left(\mathcal{S}^{-1}\right)^i{}_j \left(\frac{\partial f}{\partial P_j} \frac{\partial g}{\partial Q_i} - \frac{\partial f}{\partial Q_i} \frac{\partial g}{\partial P_j} \right)$$

where f and g are functions. Hence, we see that these vector fields are Hamiltonian vector fields such that

$$X_k = \{H_k, \cdot\,\} = \{H_{k+1}, \cdot\,\}_1\,, \qquad k = 0, 1, 2$$

and

$$H = Q_1, \qquad H_1 = \frac{1}{2} Q_1^2, \qquad H_2 = \frac{1}{3} Q_1^3, \qquad H_3 = \frac{1}{4} Q_1^4.$$

In particular, we consider the Hamiltonian function H_1 and the vector field X_1. Then a recursion operator T_1 corresponding to the vector field X_1 is given by

$$T_1 = \sum_{i=1}^{4} Q_i \left(\frac{\partial}{\partial P_i} \otimes \mathrm{d}P_i + \frac{\partial}{\partial Q_i} \otimes \mathrm{d}Q_i \right).$$

Thus, the $(1,1)$-tensor field T_1 coincides with T, and hence T_1 is a recursion operator not only for X_1 but also for the original vector field X_H. In the same way, for X_2, we have a recursion operator T_2 which coincides with T. Similarly, we have that T_3 coincides with T. Therefore, the $(1,1)$-tensor field T is a recursion operator for $X_k,\ k = 1, 2, 3$.

3. The Geodesic Flow for Some Solutions of Einstein Field Equations

In this section, we consider geodesic flows for some solutions of Einstein field equations, and we construct recursion operators. We describe the construction of a recursion operator for the solution of Einstein equations. In the first subsection, we consider the Schwarzschild metric. Then in the next subsection, we consider the Kerr-Newman metric. By this discussion, we can construct recursion operators for the solutions of the Einstein field equations of four types respectively. And we get constants of motions with recursion operators.

3.1. The Geodesic Flow for the Schwarzschild Metric

The Einstein field equation has several solutions. For example, the Schwarzschild, the Reissner-Nordström, the Kerr and Kerr-Newman metrics. We consider recursion operators for these solutions of the Einstein equation in detail. The Schwarzschild metric is the simplest solution among the four solutions, and the Kerr-Newman metric is the most complex one. Now, we consider the Schwarzschild metric. For a spherically symmetric gravitational field outside a massive non-rotating body in vacuum, the line element becomes the Schwarzchild metric given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

where M is the mass of the black hole, $t \in (-\infty, \infty)$, $r \in (2M, \infty)$, $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$. Here, for simplicity of notation, we put $t = q_1$, $r = q_2$, $\theta = q_3$ and $\phi = q_4$

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{q_2}\right)\mathrm{d}q_1^2 + \left(1 - \frac{2M}{q_2}\right)^{-1}\mathrm{d}q_2^2 + q_2^2\mathrm{d}q_3^2 + q_2^2\sin^2 q_3\mathrm{d}q_4^2.$$

For the canonical symplectic structure, we have the Hamiltonian vector field X_H of the geodesic flow

$$\begin{aligned} X_H =& -\left(1 - \frac{2M}{q_2}\right)^{-1} p_1\frac{\partial}{\partial q_1} + \left(1 - \frac{2M}{q_2}\right)^{-1} p_2\frac{\partial}{\partial q_2} + \frac{p_3}{q_2^2}\frac{\partial}{\partial q_3} + \frac{p_4}{q_2^2\sin^2 q_3}\frac{\partial}{\partial q_4} \\ &+ \left(-\frac{M}{q_2^2}\left(1 - \frac{2M}{q_2}\right)^{-2} p_1^2 - \frac{M}{q_2^2}p_2^2 + \frac{p_3^2}{q_2^3} + \frac{p_4^2}{q_2^3\sin^2 q_3}\right)\frac{\partial}{\partial p_2} + \frac{p_4^2\cos q_3}{q_2^2\sin^3 q_3}\frac{\partial}{\partial p_3} \end{aligned}$$

where the Hamiltonian function is

$$H = \frac{1}{2}\left(-\left(1 - \frac{2M}{q_2}\right)^{-1} p_1^2 + \left(1 - \frac{2M}{q_2}\right)p_2^2 + q_2^{-2}p_3^2 + \left(q_2^2\sin^2 q_3\right)^{-1} p_4^2\right).$$

We consider the Hamilton-Jacobi equation for H. We point out that the Hamiltonian function does not include q_1 and q_4. Thus, we set $p_1 = \alpha$ and $p_4 = \beta$ where α and β are constants. Hence, we consider the Hamilton-Jacobi equation for W

$$\left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 + \frac{\beta^2}{\sin^2 q_3} \\ = 2Eq_2^2 + \alpha^2\left(1 - \frac{2M}{q_2}\right)^{-1} q_2^2 - \left(1 - \frac{2M}{q_2}\right) q_2^2 \left(\frac{\mathrm{d}W_2}{\mathrm{d}q_2}\right)^2 \tag{3}$$

where $W = \sum_{k=1}^{4} W_k(q_k)$ is the generating function. The equation (3) allows a separation of variables and we introduce K as

$$K = \left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 + \frac{\beta^2}{\sin^2 q_3}$$

where K is the third integral. Thus, we get the generating function

$$W = \alpha q_1 + \int \frac{\mathrm{d}W_2}{\mathrm{d}q_2}\mathrm{d}q_2 + \int \frac{\mathrm{d}W_3}{\mathrm{d}q_3} dq_3 + \beta q_4 = \alpha q_1 + W_2 + W_3 + \beta q_4.$$

Now, we determine a new canonical coordinate system (P, Q) using the generating function W. We found out that

$$Q_1 = H, \qquad Q_2 = K, \qquad Q_3 = \frac{\mathrm{d}W_1}{\mathrm{d}q_1}, \qquad Q_4 = \frac{\mathrm{d}W_4}{\mathrm{d}q_4}$$

$$P_1 = -\frac{\partial W_2}{\partial Q_1} - \frac{\partial W_3}{\partial Q_1},\ P_2 = -\frac{\partial W_2}{\partial Q_2} - \frac{\partial W_3}{\partial Q_2},\ P_3 = -q_1 - \frac{\partial W_2}{\partial Q_3},\ P_4 = -\frac{\partial W_3}{\partial Q_4} - q_4$$

by considering a canonical coordinates in the same manner as in Section 2. Here Q_k, $k = 1, 2, 3, 4$ are constant, but we consider that they are variables. In terms of the canonical coordinate system, the Hamiltonian vector field X_H is written as

$$X_H = \{H, \cdot\,\} = \{Q_1, \cdot\,\} = -\frac{\partial}{\partial P_1}\cdot$$

Hence, the recursion operator T and the constants of motion $\mathrm{Tr}(T^\ell)$ $(\ell = 1, 2, 3, 4)$ of the geodesic flow for the Schwarzschild metric are respectively

$$T = \sum_{k=1}^{4} Q_k \left(\frac{\partial}{\partial P_k} \otimes \mathrm{d}P_k + \frac{\partial}{\partial Q_k} \otimes \mathrm{d}Q_k\right)$$
$$\mathrm{Tr}(T^\ell) = 2\left(E^\ell + K^\ell + \alpha^\ell + \beta^\ell\right), \qquad \ell = 1, 2, 3, 4.$$

From Lemma 2, we get also $\mathcal{L}_{X_H} T = 0, \mathcal{N}_T = 0$ and $\deg Q_i = 2$.

3.2. The Geodesic Flow for the Kerr-Newman Metric

For the case of the Kerr metric, many results are already known. For example, at very large radii, the curvature and dragging effects of the central object are negligible, so the Kerr metric becomes **flat** as can be seen by letting $q_1 \to \infty$ (see, [7] and [9]). Of the several forms of the Kerr metric, the most useful expression for our purpose is given by the Boyer-Lindquist coordinates. If the charge is equal to zero, the Kerr-Newman metric is the Kerr metric.

We consider the Kerr-Newman metric in the Boyer-Lindquist coordinates

$$\begin{aligned} \mathrm{d}s^2 = & -\frac{1}{\rho^2}\left(\kappa - a^2\sin^2 q_3\right)\mathrm{d}q_1^2 + \frac{2a\sin^2 q_3}{\rho^2}\left(Q^2 - 2Mq_2\right)\mathrm{d}q_1\mathrm{d}q_4 \\ & + \frac{\rho^2}{\kappa}\mathrm{d}q_2^2 + \rho^2\mathrm{d}q_3^2 + \frac{\sin^2 q_3}{\rho^2}\left(\left(q_2^2 + a^2\right)^2 - a^2\kappa\sin^2 q_3\right)\mathrm{d}q_4^2 \end{aligned} \tag{4}$$

where $\kappa = q_2^2 - 2q_2M + a^2 + Q^2$, $\rho^2 = q_2^2 + a^2\cos^2 q_3$, $aM = J$. Here M is the mass of the black hole, Q is the electric charge and J is the angular momentum.

So, the vector field of the geodesic flow for the Kerr-Newman metric is

$$X_H = \sum_{k=1}^{4}\left(U_k\frac{\partial}{\partial q_k} + V_k\frac{\partial}{\partial p_k}\right)$$

where

$$U_1 = \frac{2}{\rho^2}\left(aB\sin q_3 - \frac{A}{\kappa}(Ap_1 + ap_4)\right), \qquad U_2 = \frac{2\kappa p_2}{\rho^2}$$

$$U_3 = \frac{2p_2}{\rho^2}, \qquad U_4 = \frac{2}{\kappa}\left(\frac{C(\kappa - \rho^2 + A)}{\rho^2} - ap_1\right)$$

$$V_1 = 0, \qquad V_2 = \frac{2q_2}{\rho^4}(C^2 - q_3^2 + \kappa p_2^2) - \frac{4q_2p_1}{\kappa\rho^2}(Ap_1 + ap_4) - (M - q_2)p_2^2$$

$$V_3 = \frac{\sin 2q_3}{\rho^2}\left(\frac{a^2}{\rho^2}\left(B^2 + \kappa p_2^2 - \frac{2p_4^2}{\sin^2 q_3}\right) + B^2 - \frac{2ap_1p_4}{\sin^2 q_3}\right), \qquad V_4 = 0$$

$$A = a^2 + q_2^2, \qquad B = ap_1\sin q_3 + \frac{p_4}{\sin q_3}, \qquad C = ap_1\sin q_3 + \frac{p_4}{\sin q_3}\cdot$$

The Hamiltonian function H of the vector field X_H is respectively

$$\begin{aligned} H = \frac{1}{2}\Bigg[& \left(\frac{a^2}{\rho^2}\sin^2 q_3 - \frac{(q_2^2 + a^2)^2}{\kappa\rho^2}\right)p_1^2 + \frac{\kappa}{\rho^2}p_2^2 \\ & + \frac{1}{\rho^2}p_3^2 + \left(\frac{a^2}{\kappa\rho^2} - \frac{1}{\rho^2\sin^2 q_3}\right)p_4^2 + 2\left(\frac{a}{\rho^2} - \frac{a(q_2^2 + a^2)}{\kappa\rho^2}\right)p_1p_4\Bigg]. \end{aligned}$$

Now, if $Q = 0$, (4) is the **Kerr metric**. If $J = 0$, (4) is the **Reissner-Nordström metric**. And if $Q = 0$ and $J = 0$, (4) is the **Schwarzschild metric**. Thus, if we can construct a recursion operator for the Kerr-Newman metric, then it enables us to get the other three respective recursion operators for the Kerr, the Reissner-Nordström and the Schwarzschild metrics.

We see that the Hamiltonian function H does not include q_1 and q_4. Hence, p_1 and p_4 are first integrals, and we put $p_1 = \alpha$, $p_4 = \beta$. Then, we consider the Hamilton-Jacobi equation

$$2Eq_2^2 + \frac{(q_2^2+a^2)^2}{\kappa}\alpha^2 - \kappa\left(\frac{\mathrm{d}W_2}{\mathrm{d}q_2}\right)^2 + \frac{a^2}{\kappa}\beta^2 + \frac{2a(q_2^2+a^2)}{\kappa}\alpha\beta$$
$$= -2Ea^2\cos^2 q_3 + a^2\alpha^2\sin^2 q_3 + \left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 - \frac{\beta^2}{\sin^2 q_3} + 2a\alpha\beta. \quad (5)$$

Since the equation (5) allows a separation of the variables, we put K as

$$K = -2Ea^2\cos^2 q_3 + a^2\alpha^2\sin^2 q_3 + \left(\frac{\mathrm{d}W_3}{\mathrm{d}q_3}\right)^2 - \frac{\beta^2}{\sin^2 q_3} + 2a\alpha\beta$$

where K is the third integral. Therefore, we have the generating function

$$W = \alpha q_1 + \int \frac{\mathrm{d}W_2}{\mathrm{d}q_2}\mathrm{d}q_2 + \int \frac{\mathrm{d}W_3}{\mathrm{d}q_3}\mathrm{d}q_3 + \beta q_4 = \alpha q_1 + W_2 + W_3 + \beta q_4.$$

Next, we determine the canonical coordinate system (P, Q) using the generating function W. Thus, we get

$$Q_1 = E, \qquad Q_2 = K, \qquad Q_3 = \frac{\mathrm{d}W_1}{\mathrm{d}q_1}, \qquad Q_4 = \frac{\mathrm{d}W_4}{\mathrm{d}q_4}$$
$$P_1 = -\frac{\partial W_2}{\partial Q_1} - \frac{\partial W_3}{\partial Q_1}, \qquad P_2 = -\frac{\partial W_2}{\partial Q_2} - \frac{\partial W_3}{\partial Q_2}$$
$$P_3 = -q_1 - \frac{\partial W_2}{\partial Q_3} - \frac{\partial W_3}{\partial Q_3}, \qquad P_4 = -\frac{\partial W_2}{\partial Q_4} - \frac{\partial W_3}{\partial Q_4} - q_4.$$

In this case, the vector field X_H and the symplectic form ω are written as follows

$$X_H = \{H, \cdot\,\} = -\frac{\partial}{\partial P_1}, \qquad \omega = \sum_{k=1}^{4} \mathrm{d}P_k \wedge \mathrm{d}Q_k.$$

Hence, the recursion operator T and the constants of motion $\mathrm{Tr}(T^\ell)$ of the geodesic flow of the Kerr-Newman metric are

$$T = \sum_{i=1}^{4} Q_i \left(\frac{\partial}{\partial P_i} \otimes \mathrm{d}P_i + \frac{\partial}{\partial Q_i} \otimes \mathrm{d}Q_i\right)$$
$$\mathrm{Tr}(T^\ell) = 2\left(E^\ell + K^\ell + \alpha^\ell + \beta^\ell\right), \qquad \ell = 1, 2, 3, 4.$$

Therefore, we obtain recursion operators for the solutions of the Einstein field equation of four types. Of course, (3) and (5) are integrable systems and they allow separation of variables. In addition, we are able to construct constants of motion explicitly using the traces of recursion operators.

These results provide new examples of recursion operators for the geodesic flow of pseudo-Riemannian metrics.

Acknowledgements

I would like to express sincere gratitude to Professor Gaetano Vilasi for giving his invaluable comments and suggestions. Also, the author wish to thank Professor Akira Yoshioka for his comments.

References

[1] Arnold V., *Mathematical Methods of Classical Mechanics*, Springer Graduate Texts in Mechanics **60**, Springer, New York 1978.

[2] de Filippo S., Marmo G., Salerno M. and Vilasi G., *A New Characterization of Completely Integrable Systems*, Nuovo Cimento B **83** (1984) 97–112.

[3] de Filippo S., Marmo G. and Vilasi G., *A Geometrical Setting for the Lax Representation*, Phys. Lett. B **117** (1982) 418–422.

[4] Hosokawa K. and Takeuchi T., *About the Construction and Characteristic of Concrete Recursion Operator*, The Mathematical Society of Japan 2013 Annual Meeting (Titles and Short Summaries of the Talks).

[5] Hosokawa K. and Takeuchi T., *A Construction for the Concrete Example of a Recursion Operator*, submitted.

[6] Marmo G. and Vilasi G., *When Do Recursion Operators Generate New Conservation Laws?*, Phys. Lett. B **277** (1992) 137–140.

[7] Misner W., Thorne S. and Wheeler J., *Gravitation*, W. H. Freeman, San Francisco 1973.

[8] Mladenov I. and Tsanov V., *Geometric Quantization of the Multidimensional Kepler Problem*, J. Geom. Phys. **2** (1985) 17–24.

[9] Mould R., *Basic Relativity*, Springer, New York 1994.

[10] Vilasi G., *On the Hamiltonian Structures of the Korteweg-de Vries and Sine-Gordon Theories*, Phys. Lett. B **94** (1980) 195–198.

[11] Vilasi G., *Hamiltonian Dynamics*, World Scientific, River Edge 2001.