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A NATURAL GEOMETRIC FRAMEWORK FOR THE SPACE OF INITIAL DATA OF NONLINEAR PDES

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Abstract. The modern geometrical approach to nonlinear PDEs is the outcome of a nontrivial synthesis of differential calculus over commutative algebras and cohomological algebra in the context of infinite jet spaces. In this paper we propose a very natural generalization of the notion of a jet space, which allows to treat the space of initial data of a nonlinear PDE on the same footing as the space of its solutions.

1. Introduction

In spite of its age, Henneaux & Teitelboim's book "Quantization of Gauge System" still stands as a sort of Bible for modern theoretical physicists. In the middle of it, the reader meets three simple prescriptions (ten lines overall) which allow to recast in a field-theoretical context the results of BRST formalism obtained so far. But soon comes a warning: "Although useful, this approach to the field theoretical case remains, however, rather formal as long as one does not specify precisely the functional space to which the relevant functionals should belong. This turns out, in general, to be a complicated task" ([2], Chapter 12). The fact that deep field-theoretical results keep coming, regardless of the lack of solid mathematical foundations, might be taken as an evidence that any effort to find such foundations is, in fact, superfluous.

What theoretical physicists seem to ignore is that a robust geometrical language for nonlinear field theories *does exists*, but its comprehension is – to use the same words as the authors above – "a complicated task". We will call such a language *Secondary Calculus*, following Vinogradov (see [10, 11] and references therein), but it should be stressed that, in spite of its remarkable achievements in the covariant description of nonlinear Lagrangian theories and PDEs, it is still a young theory which needs to be duly revisited and expanded. In this perspective, we use term *Secondary Calculus* in a purely tentative way.

In the first section of this paper we try to explain why "the functional space to which the relevant functionals should belong" the authors above speak about, cannot be thought of as a functional space in the sense of functional analysis, both rather as a (duly generalized) leaf space in the sense of involutive distributions. This point of view allows to see the "relevant functionals" as a cohomological feature of an infinite-dimensional manifold equipped with a finite-dimensional involutive distribution (the $\mathcal{C}$-spectral sequence associated with a so-called *diffiety*).

In the second section we show that a small modification in the construction of jet spaces allows to frame simultaneously leaves of different dimensions in the same geometric context. The dimension of the leaves is an important constant in nonlinear Lagrangian formalism and in the geometrical theory of nonlinear PDEs, since it coincides precisely with the *number of independent variables*. However, there are remarkable circumstances in which this number is no longer a constant, e.g., when one passes from the Lagrangian to the Hamiltonian description of a field theory (the so-called "time-slicing", see [1]), or when one needs to define functionals on the space of initial datas (see, for instance, [9]).

Throughout this paper, E is a fixed smooth manifold. Any number n such that $0 < n \leq \dim E$ will be referred to as the *number of independent variables*. We also set $m = \dim E - n$.

2. Review of Jet Spaces and Their Natural Structures

2.1. Introduction

In a (classical) field theory, E arises as a product $M \times T$, where

- M is the space-time, i.e., the space where the fields are defined
- T is the target space, i.e., the space where the fields take their values.

Depending on the approach, the value of n can be three or four, according to the role of time. This example should motivate why the formalism we are going to introduce is conceived in such a way that the value of n does not have to be fixed.

A (nonlinear) Lagrangian (classical) field theory consists in certain conditions imposed on the set $\Gamma(\pi)$, where $\pi : E \to M$ is the natural projection. More precisely, $\Gamma(\pi)$ is the *space of histories* of the theory, or the set of *field configurations*, and one looks for the histories which are critical with respect to an action integral. The subset $\boldsymbol{P} \subseteq \Gamma(\pi)$ of such "critical histories" is the so-called **covariant phase space** of the theory (see [12] and references therein). Functional analysis fails here, since neither $\Gamma(\pi)$, nor $\boldsymbol{P}$ possess, in general, a linear structure.

It might be said that jet space has been introduced to overcome this failure, and give $\Gamma(\pi)$ a nice geometrical description, which in its turn allows to recast the notion of an action integral in a differential-geometric and cohomological framework. This perspective is briefly reviewed below (see the recent review [5] and references therein for more extensive information).

2.2. First Order Jet Spaces

Morally, the first order jet space of E, usually denoted by $J^1(E,n)$, should be understood as "the smallest and smoothest container" of all first order approximations of all n-dimensional submanifolds $L \subseteq E$. Rigorously, there are (at least) three equivalent ways to define $J^1(E,n)$. It is convenient to recall them all, since they will help understanding the generalization of jet spaces which will be introduced in the next section. In practice, one can

- identify a "first order approximation" with an n-dimensional linear subspace of TE, and put $J^1(E,n) = \mathrm{Gr}\,(TE,n)$
- introduce an equivalence $\sim_y^1$ between n-dimensional submanifolds tangent to each other at $y \in E$, and put

$$J^1(E,n) = \coprod_{y\in E} \frac{\{\text{all submanifolds of dimension } n\}}{\sim_y^1}$$

- just add to the coordinates $(x^1,\dots,x^n,u^1,\dots,u^m)$ of E new coordinates u_i^j, $i=1,2,\dots,n$, $j=1,2,\dots,m$, and get once again $J^1(E,n)$.

Example 1. If x^μ is a point of the space-time, and ϕ^j are the values of the field, the action of a first order Lagrangian on ϕ is usually written as

$$S[\phi] = \int_M \mathrm{d}^n x L(x^\mu, \phi^j, \phi^j_{(\mu)})$$

where $(x^\mu, \phi^j, \phi^j_{(\mu)})$ are precisely the coordinates of $J^1(E,n)$.

2.3. Jet Projection

Let us recall that any vector bundle η over E with fiber V can be defined by means of a covering $\{U_\alpha\}$ of E and **transition functions** $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}\,(V)$. By using the standard representation of $\mathrm{GL}\,(V)$ on the Grassmann manifold $\mathrm{Gr}\,(V,k)$, one can use the same $g_{\alpha\beta}$'s to define a smooth bundle $\mathrm{Gr}\,(\eta,k) \to E$, with abstract fiber $\mathrm{Gr}\,(V,k)$. When $\eta : TE \to E$ is the tangent bundle, $\mathrm{Gr}\,(TE,n)$ is precisely $J^1(E,n)$.

So, the canonical projection $\pi_{1,0} : J^1(E,n) \to E$ can be seen equivalently as

- the bundle projection of $\mathrm{Gr}\,(TE,n)$ over E

- the point of tangency of two submanifolds
- the first $n+m$ coordinates of $J^1(E,n)$.

According to the commonly adopted notation, $L\sim_y^1 L'$ is the tangency relation, and $[L]_y^1$ is the equivalence class of L.

2.4. The Relative Distribution on *E*

Notice that any point $\theta = [L]_y^1 \in J^1(E,n)$ defines the linear subspace $R_\theta = T_yL \leq T_yE$.

Definition 1. *Assignment $R : \theta \mapsto R_\theta$ is the* canonical relative distribution *on E.*

Notice that R is not a true distribution, but a relative one (with respect to $\pi_{1,0}$). So, the concept of an integral manifold of R is ill-defined (passing to infinite jet spaces will make it well-defined). The closest thing we have to an integral manifold is the so-called **ray manifold** (ray manifolds play a key role in the proof of the Lie-Bäcklund theorem, see [4]).

Given a vector subspace $W \subseteq T_yE$, the *ray* of W is the submanifold $\ell(W) \subseteq \pi_{1,0}^{-1}(y)$ made of points θ such that $R_\theta \supseteq W$. The *ray manifold* $\ell(N)$ of a submanifold $N \subseteq E$ is the union of the rays of all the tangent subspaces to N. N might be called *integral* of R iff its ray manifold $\ell(N)$ projects diffeomorphically on N. In this sense, any n-dimensional submanifold $L \subseteq E$ is integral. Ray manifolds $\ell(L)$ deserve a special attention.

The embedding $L \subseteq E$ is canonically lifted to an embedding $j_1(L) : L \to J^1(E,n)$, where $j_1(L)(y) \stackrel{\text{def}}{=} [L]_y^1$. Its image is denoted by $L^{(1)}$, and is precisely $\ell(L)$.

Definition 2. *Embedding $j_1(L)$ (or its image $L^{(1)}$) is called the first* jet-prolongation *of L.*

2.5. Differential Equations

Let $\{F^\alpha\}$ be a set of functions on $J^1(E,n)$, and suppose that their zero locus $\mathcal{E} : F^\alpha = 0$ is a smooth submanifold $\mathcal{E} \subseteq J^1(E,n)$. Then $\mathcal{E}$ is interpreted as a (system of) first order nonlinear PDE(s).

Definition 3. *L is a solution of $\mathcal{E}$ iff*

$$L^{(1)} \subseteq \mathcal{E}$$

or, equivalently, if $j_1(L)^*(F^\alpha) = 0$, for all α's.

2.6. Iterated Jets and Higher-Order Jets

As noticed in Subsection 2.2, the passage from $J^1(E,n)$ to $J^1(J^1(E,n),n)$ might be seen as the addition of new coordinates $(u^j)_i$ and $(u^j_i)_k$ to those of $J^1(E,n)$. In this perspective, it is easy to recognize in $J^1(J^1(E,n),n)$ a special subset $J^2(E,n)$, which may be thought of as

- the equation $(u^j)_i = u^j_i$, $(u^j_i)_k = (u^j_k)_i$
- the set of jets of submanifolds of $J^1(E,n)$ which are of the form $L^{(1)}$
- the sub-bundle of the Grassmann bundle $\mathrm{Gr}\,(J^1(E,n),n)$ made of R-*horizontal* elements $\Theta_\theta \in T_\theta J^1(E,n)$, i.e., such that $d\pi_{1,0}(\Theta_\theta) = R_\theta$.

Definition 4. *$J^2(E,n)$ is called the* ***second-order jet space*** *of E.*

Obviously, $J^2(E,n)$ identifies with the quotient space of all submanifolds modulo second-order tangency, and the canonical projection $\pi_{2,1}$ is just the passage from a finer quotient space to a coarser one.

2.7. Cartan Distribution

The relative to $\pi_{2,1}$ distribution R can be defined much as in Subsection 2.4. However, in view of the subsequent generalization, we rather define $\pi_{2,1}$ as restriction, $\pi_{2,1} = \pi_{1,0}|_{J^2(E,n)}$, and R as the restriction of the relative distribution on $J^1(E,n)$.

Definition 5. *An R-*plane *is a subspace belonging to the relative distribution on $J^1(E,n)$.*

All the R-planes passing trough θ generate the **Cartan plane** (or **contact plane**) $\mathcal{C}_\theta$. It is easy to see that $\mathcal{C}_\theta = R_\theta \oplus T(\pi^{-1}_{1,0}(\pi_{1,0}(\theta)))$. This is the standard way to pass from a relative distribution to a true one. The drawback is an increased dimension, due to the vertical part, which makes it impossible to use the Cartan distribution on $J^1(E,n)$ to distinguish submanifolds of the form $L^{(1)}$ (called **olonomic** by some authors).

The theorem below, whose proof can be found in [4], shows that olonomic submanifolds are "contaminated" by higher-dimensional submanifolds.

Theorem 1. *Jet-prolongations $L^{(1)}$ are precisely the maximal $\pi_{(1,0)}$-horizontal integral submanifolds of $\mathcal{C}$. Other integral submanifolds are ray manifolds.*

2.8. Infinite Jet Spaces

The trick to pass from relative distributions along jet projections to a true distribution *without increasing its dimension*, is to introduce infinite jet spaces. Very roughly speaking, $J^\infty(E,n)$ is the limit of the tower of projections

$$\pi_{k,k-1} : J^k(E,n) \longrightarrow J^{k-1}(E,n)$$

and the Cartan distribution $\mathcal{C}$ on $J^\infty(E,n)$ is involutive and n-dimensional, due to the "lack of vertical directions". The analog of Theorem 1 for $J^\infty(E,n)$ reads

Theorem 2. *Jet-prolongations $L^{(\infty)}$ are precisely the maximal integral submanifolds of $\mathcal{C}$.*

In coordinates, the distribution $\mathcal{C}$ can be thought of

- as an infinite Pfaff system $\omega^j_\sigma = 0$ (in field theory the ω^j_σ's look like $\omega^j_\mu = d_V\phi^j_{(\mu)}$)
- as generated by the total derivatives

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\sigma,j} u^j_{\sigma+1_i}\frac{\partial}{\partial u^j_\sigma} \quad 1_i = (0,\ldots,\underset{i\text{-th place}}{1},\ldots,0).$$

As promised in Section 1, the space of histories $\Gamma(\pi)$ is now identified with the set of leaves of $\mathcal{C}$. It should be stressed that, due to infinite dimensionality of $J^\infty(E,n)$, the Cartan distribution $\mathcal{C}$ does not fulfill the Frobenius theorem. In this sense, we cannot speak about *the* leaf of $\mathcal{C}$ through some point of $J^\infty(E,n)$, but we retain the term "leaf", since it is more suggestive.
In this perspective, solutions to a PDE should be some sort of "special leaves" of $\mathcal{C}$.

2.9. Infinitely Prolonged Equations

Notice that the restricted distribution $\mathcal{C}|_\mathcal{E}$ is not an involutive n-dimensional one, since, in general, $\mathcal{C}$ is not tangent to $\mathcal{E}$. The biggest submanifold of $\mathcal{E}$ to which $\mathcal{C}$ is tangent is called the **infinite prolongation** of $\mathcal{E}$ and denoted by $\mathcal{E}^{(\infty)}$. Algebraically the latter is obtained from the former by adding to the F^α's all their differential consequences (i.e., the total derivatives).

Example 2. In field theory the differential consequences of $F = 0$ are denoted by $\partial_{(\mu)}F = 0$. For example,

$$\partial_{(\mu)}\frac{\delta L}{\delta\phi^i} = 0$$

represent the infinitely prolonged Euler-Lagrange equations associated with the Lagrangian $L\mathrm{d}^n x$, i.e., the covariant phase space $\boldsymbol{P}$ associated with L (see Section 1). So, $\boldsymbol{P}$ can now be though of the set of leaves of $\mathcal{C}|_{\mathcal{E}^{(\infty)}}$.

These considerations pushed towards the introduction of the so-called *diffieties*.

2.10. Diffieties

A **diffiety** (from *diff*erential var*iety*) is a couple $(\mathcal{O}, \mathcal{C})$ were $\mathcal{O}$ is the geometrical object corresponding to a filtered smooth algebra, and $\mathcal{C}$ is a finite-dimensional involutive distribution on it. Leaves of $\mathcal{C}$ are called the **secondary points** of the diffiety, and their totality can be denoted by $\boldsymbol{M}$.

Example 3. There follow simple examples of diffieties.

- If $\mathcal{O}$ is a fiber bundle, and $\mathcal{C}$ is the vertical distribution on it, then $\boldsymbol{M}$ is just the base of the bundle (i.e., the manifold of all the fibers).
- $(\mathcal{E}^{(\infty)}, \mathcal{C}|_{\mathcal{E}^{(\infty)}})$ is a diffiety, and $\boldsymbol{M}$ is precisely the set of solutions of $\mathcal{E}$.

To better fit the circumstances, modifier *secondary* can be replaced by *variational* or *functional*. The main advantage of the above point of view, is that the most important notions of differential calculus can be straightforwardly generalized to diffieties, thus obtaining the formalism mentioned in the introduction – the so-called *secondary calculus*.

2.11. Elements of Secondary Calculus

One of the most fundamental notion of differential calculus is that of a *vector field*. Without going into details (see [3, 11] for more information), we claim that secondary vector fields on $\mathcal{E}^{(\infty)}$ are nontrivial infinitesimal symmetries of $\mathcal{E}$. More precisely, we have to take the vector fields which "respect" $\mathcal{C}|_{\mathcal{E}^{(\infty)}}$, also known as **contact fields**

$$D_{\mathcal{C}}(\mathcal{E}) = \{X \text{ vector field on } \mathcal{E}^{(\infty)};\ [X, \overline{D}_i] = \sum \phi_j \overline{D}_j\}$$

where $\overline{D}_i$ is the restriction of D_i to $\mathcal{E}^{(\infty)}$, and $X, Y \in D_{\mathcal{E}}$ should be thought of as equivalent if they generate the same flow in the space of solutions of $\mathcal{E}$. Obviously, trivial contact fields are $\mathcal{C}D(\mathcal{E}) = \{X = \sum f_i \overline{D}_i\}$, and if we identify $X \sim Y \Leftrightarrow X - Y \in \mathcal{C}D(\mathcal{E})$ we obtain the higher symmetries of $\mathcal{E}$

$$\operatorname{sym} \mathcal{E} = \frac{D_{\mathcal{C}}(\mathcal{E})}{\mathcal{C}D(\mathcal{E})}.$$

The same result can be obtained cohomologically (see, for instance, [12])

$$\operatorname{sym} \mathcal{E} = H^0(\text{Horizontal Jet Spencer Complex on } \mathcal{E}^{(\infty)})$$

Example 4. In the simple case $\mathcal{E}^{(\infty)} = J^\infty(\pi)$, it is easy to see that

$$\varkappa = \operatorname{sym} J^\infty = \{\ni_\varphi;\ \varphi = (\varphi_1, \ldots, \varphi_m), \quad \varphi_i \in C^\infty(J^\infty)\}$$

where

$$\ni_\varphi \stackrel{\text{def}}{=} \sum_{\sigma,i} D_\sigma(\varphi_i)\frac{\partial}{\partial u^i_\sigma}, \qquad D_\sigma = D_1^{\sigma_1}\circ\cdots D_n^{\sigma_n}$$

and φ is referred to as the *generating function* of $\chi = \ni_\varphi \mod \mathcal{C}D(J^\infty)$.

Dually to vector fields, we can introduce the secondary version of functions (horizontal cohomology), differential forms ($\mathcal{C}$-spectral sequence) and de Rham differential (d_1 differential of the $\mathcal{C}$-spectral sequence). We suggest that the interested reader consult the book [11] for an exhaustive explanation.

Basically, we define first the horizontal complex of $\mathcal{E}^{(\infty)}$

$$0 \to \overline{\Lambda}^0(\mathcal{E}^{(\infty)}) = C^\infty(\mathcal{E}^{(\infty)}) \xrightarrow{\overline{d}} \overline{\Lambda}^1(\mathcal{E}^{(\infty)}) \xrightarrow{\overline{d}} \cdots \xrightarrow{\overline{d}} \overline{\Lambda}^n(\mathcal{E}^{(\infty)}) \to 0$$

where

$$\overline{\Lambda}^i(\mathcal{E}^{(\infty)}) \stackrel{\text{def}}{=} \frac{\Lambda^i(\mathcal{E}^{(\infty)})}{\mathcal{C}\Lambda^i(\mathcal{E}^{(\infty)})}, \qquad \overline{d}:\overline{\Lambda}^i \to \overline{\Lambda}^{i+1}$$

and $\mathcal{C}\Lambda(\mathcal{E}^{(\infty)})$ is the ideal of the differential forms vanishing on the Cartan distribution.

Then, we take its cohomologies $\overline{H}^i(\mathcal{E}^{(\infty)})$, called **horizontal**.

Example 5. Among horizontal cohomologies we find

- the action functionals: $\overline{H}^n(J^\infty(E,n))$
- the conservation laws: $\overline{H}^{n-1}(\mathcal{E}^{(\infty)})$.

Take now the powers of $\mathcal{C}\Lambda^*(\mathcal{E}^{(\infty)})$, and the corresponding filtered complex

$$\Lambda^*(\mathcal{E}^{(\infty)}) \supset \mathcal{C}\Lambda^*(\mathcal{E}^{(\infty)}) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}^{(\infty)}) \supset \cdots .$$

Then the associated spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ is called $\mathcal{C}$-spectral

Example 6. Among terms and differentials of the $\mathcal{C}$-spectral sequence we find

- the Euler operator: $d_1^{0,n}$
- the LHS of E-L equations: $E_1^{1,n}$
- the Helmholtz conditions: $E_1^{2,n}$.

3. Jet Spaces of Pairs of Manifolds

3.1. Introduction

Now we are going to relax the hypothesis that n is fixed, and rather consider two integers $n_2 \geq n_1$. Thinking about our previous discussion, n was linked to a very intrinsic property of submanifolds L, their dimension. So, in the context of jet spaces, what might it mean to allow n to vary? Apparently, it should mean to allow L to be of various dimensions, but if one tries to implement such an idea, for instance in the definition of first order jets spaces (see Subsection 2.2), immediately realizes that the result is not smooth. In a deeper perspective, one is actually forced to consider more general objects than submanifolds, namely *pair of submanifolds*. As the analogy with linear algebra

$$\text{Grassmann manifold } \mathrm{Gr}\,(V,n) \longrightarrow \text{flag manifold } \mathrm{Gr}\,(V,n_2,n_1)$$

brightly confirms, the second point of view is the good one, since there is no way to find any geometrical structure whatsoever on the set of all integral submanifolds of an involutive distribution, if we allow their dimension to take, for instance, two different values.

To begin with, let $k \geq l$. The main definition of this section is the following

Definition 6. *$J^{k,l}(E,n_2,n_1)$ is the subset of $J^k(E,n_2) \times_E J^l(E,n_1)$ made by those elements $([L_2]^k_y, [L_1]^l_y)$ such that $L_2 \sim^l_y L_1$.*

The reader should have noticed that the tangency relation $\sim^l_y$ above is different than the one used in Subsection 2.2, since it involves submanifolds of different dimensions. Condition $L_2 \sim^l_y L_1$ means that L_2 contains an n_1-dimensional submanifold L_2' such that $L_2' \sim^l_y L_1$ in the standard sense.

So, in the pair $\theta = ([L_2]^k_y, [L_1]^l_y)$ one can always assume that L_1 is contained into L_2, at least locally around y. In other words, θ carries a local information on the *pair* (L_2, L_1). Understanding jets as Taylor series, θ contains a k-th order approximation of the function in n_2 variable which locally defines L_2 in E, plus an l-th order approximation of the function in 1_2 variable which locally defines L_1 in L_2. These remarks motivate the name **jet of pairs** given to θ.

3.2. Flag Bundles and Flag Projections

Definition 6 becomes more clear when $k = l = 1$. In this case we meet a well-known construction of linear algebra, namely the *flag manifold*.

Lemma 1. $J^{1,1}(E,n_2,n_1) = \mathrm{Gr}\,(TE,n_2,n_1)$.

Proof: Much as in Subsection 2.3, notice that to any vector bundle η we can associate a smooth bundle $\mathrm{Gr}\,(\eta,n_2,n_1)$, by replacing the linear space $\eta^{-1}(y)$ with the flag manifold $\mathrm{Gr}\,(\eta^{-1}(y),n_2,n_1)$, for all $y \in E$. Then, it is immediate to see that in the case when η is the tangent bundle over E, we get precisely $J^{1,1}(E,n_2,n_1)$. ■

Lemma 1 above explains the appearance of flag manifolds in the theory of jets of submanifolds of various dimensions. Indeed, the flag manifold $\mathrm{Gr}\,(V,n_2,n_1)$ is a "smooth envelope" of both Grassmann manifolds $\mathrm{Gr}\,(V,n_2)$ and $\mathrm{Gr}\,(V,n_1)$, i.e., the minimal (in a category-theoretic sense) object which is smooth and also keeps the information about both Grassmann manifolds. In particular, we have the canonical flag projections

$$\begin{tikzcd} & \mathrm{Gr}\,(V, n_2, n_1) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Gr}\,(V, n_2) & & \mathrm{Gr}\,(V, n_1) \end{tikzcd}$$

whose fibers are in their turn Grassmann manifolds

- $p^{-1}(L_2) = \mathrm{Gr}\,(L_2, n_1)$
- $q^{-1}(L_1) = \mathrm{Gr}\,\left(\left(\frac{V}{L_1}\right)^*, n_2 - n_1\right).$

So, the next result is straightforward.

Lemma 2. *Fibers of canonical projections*

$$\begin{tikzcd} & J^1(E, n_2, n_1) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ J^1(E, n_2) & & J^1(E, n_1) \end{tikzcd}$$

are

- $p^{-1}(\Theta) = \mathrm{Gr}\,(R_\Theta, n_1)$
- $q^{-1}(\theta) = \mathrm{Gr}\,\left(\left(\frac{T_{\theta_0}E}{r_\theta}\right)^*, n_2 - n_1\right)$

where $R : \Theta \mapsto R_\Theta$ (*respectively,* $r : \theta \mapsto r_\theta$) *is the relative distribution on* E (*See* Subsection 2.4) *with respect to jet projection* $J^1(E, n_2) \to E$ (*respectively,* $J^1(E, n_1) \to E$).

3.3. The Normal Bundle

When $n_1 = n$ and $n_2 = n - 1$ we are considering the so-called *codimension one* case. It is remarkable in that it naturally encodes the notion of *normal jets*. Indeed, let $N^1 = J^1(E, n, n-1)$, and $\nu^1 = q$. Then $N^1 \xrightarrow{\nu_1} J^1(E, n_1)$ is the **first order normal bundle**

$$\nu_1^{-1}(\theta) = N^1_\theta \overset{\mathrm{def}}{=} \left(\frac{T_{\theta_0}E}{r_\theta}\right)^* = \mathrm{Ann}(r_\theta) \leq T^*_{\theta_0}E.$$

Similarly, when $k = \infty$ and $l = 1$, the natural projection q reads

$$\nu_\infty : J^{\infty,1}(E, n, n-1) \longrightarrow J^1(E, n-1)$$

ν_∞ is naturally interpreted as the **infinite-order normal bundle**, which plays a prominent role in the approach to *natural boundary conditions* ([6–8]).

3.4. Canonical Embedding into Iterated Jet Spaces

Besides the case $k = l = 1$, the case $k = l = \infty$ is the simplest one, due to the fact that on J^∞ the relative distribution is a true distribution. Intermediate cases will be examined somewhere else.

The next result, which generalizes the discussion made in Subsection 2.6, casts a bridge between jets of pairs of manifolds and iterated jet spaces.

Lemma 3. *$J^{\infty,\infty}(E, n_2, n_1)$ is embedded canonically into $J^\infty(J^\infty(E, n_2), n_1)$.*

Proof: For any point $([L_2]^\infty_y, [L_1]^\infty_y)$ the jet prolongation

$$j_\infty(L_2) : L_2 \longrightarrow J^\infty(E, n_2)$$

can be used to lift L_1 inside $J^\infty(E, n_2)$.

So we obtain the n_1-dimensional submanifold $(j_\infty(L_2))(L_1)$, of which we can take the ∞-jet at the point $[L_2]^\infty_y$

$$([L_2]^\infty_y, [L_1]^\infty_y) \longmapsto [(j_\infty(L_2))(L_1)]_{[L_2]^\infty_y}.$$

■

3.5. The Space of Initial Data

In the codimension one case, $J^{\infty,\infty}(E, n, n-1)$ can be easily describes as an equation, namely

$$\mathcal{E} = \{(\phi^j_{(\sigma+l_n)})_{(\mu)} = \phi^j_{(\sigma+1_\mu+l_n)} + \phi^j_{\sigma+(l+1)_n} t_{(\mu)}\}_\infty.$$

Definition 7. *$J^{\infty,\infty}(E, n, n-1)$ is the diffiety of initial data.*

The space of initial data can be found in literature (see, for instance, [9]), but its nature of infinitely prolonged nonlinear PDEs has never been recognized before. This result has been independently found by L. Vitagliano.

3.6. Structure of $J^{\infty,\infty}(E, n, n-1)$

Lemma 3 allow to frame $J^{\infty,\infty}(E, n, n-1)$ into the following diagram

$$\begin{tikzcd}
& J^\infty(J^\infty(E,n),n-1) & \\
& J^{\infty,\infty}(E,n,n-1) \arrow[u] \arrow[dl, "p"'] \arrow[dr, "q"] & \\
J^\infty(E,n) & & J^\infty(E,n-1)
\end{tikzcd}$$

where the projections are infinite-dimensional analogues of the ones described in Lemma 2. So, it looks evident that $J^{\infty,\infty}(E, n_2, n_1)$ possesses an inherited $(n-1)$-dimensional distribution $\mathcal{D}$, and two infinite-dimensional distributions. Denote by $\widetilde{\mathcal{C}}$ the one induced by p^* and by $\widetilde{\mathcal{C}}'$ the one induced by q^*.

Lemma 4. *Leaves of $\widetilde{\mathcal{C}}$ are precisely the embedded jet spaces $J^\infty(L, n_1)$, and as such are in one-to-one correspondence with the leaves of $\mathcal{C}$ in $J^\infty(E, n)$.*

Proof: Any n-dimensional manifold L produces the embedding $\widetilde{j}_\infty(L)$ of $J^\infty(L, n-1)$ into $J^{\infty,\infty}(E, n, n-1)$, which closes the diagram

$$\begin{tikzcd} J^\infty(L,n-1) \arrow[r, "\widetilde{j}_\infty(L)"] \arrow[d, "\pi_{\infty,0}"] & J^{\infty,\infty}(E,n,n-1) \arrow[d, "p"] \\ L \arrow[r, "j_\infty(L)"] & J^\infty(E,n) \end{tikzcd}$$

■

Cirillary 1. Any equation $\mathcal{E}$ is equivalent to its own lifting $\widetilde{\mathcal{E}} = p^*(\mathcal{E})$.

3.7. Further Developments

In order to develop differential calculus over the diffiety of initial data along the lines described in Subsection 2.11, nontrivial results have to be proved:

- the leaves of $\widetilde{\mathcal{C}}'$ are jet spaces of the infinite-order normal bundle, restricted to n_1-dimensional submanifolds
- the $\mathcal{D}$-spectral sequence is one-line
- the term E_1 of the $\widetilde{\mathcal{C}}$-spectral sequence is trivial above the line $q = n$.

To the author's opinion, the results above looks intuitively true, however a rigorous proof might be hard due to the infinite-dimensionality of the objects under consideration.

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