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GEOMETRICAL ASPECTS IN THE RIGID BODY DYNAMICS WITH THREE QUADRATIC CONTROLS

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Abstract. The dynamics of the rigid body with three quadratic controls is discussed and some of its geometrical and dynamical properties are pointed out.

1. Introduction

The problem of geometrical study of the rigid body dynamics with controls has received a great deal of interest in recent years. We can remind here the papers of Brockett [5], Aeyels [1], Krishnaprasad [11], Crouch [8], Aeyels and Szafranski [2], Bloch and Marsden [3], Bloch, Krishnaprasad and Sanchez de Alvarez [4], Holm and Marsden [9], Byrnes and Isidori [6], Posberg and Zhao [14], Puta [15–20], Puta and Craioveanu [21], Puta and Ivan [22], Puta and Comânescu [23] and Puta and Casu [25].

We shall consider here a class of feedback laws that depends on a parameter matrix W which is nonsingular and symmetric and we shall study its Hamiltonian and Lagrangian picture, its Lax formulation, its numerical integration via Kahan's integrator, its stability via the energy-Casimir method and its geometric prequantization.

2. The Lie Group $SO(3)$ and Its Lie Algebra $so(3)$

The configuration of a rigid body free to rotate about a fixed point in space is described by an element of $SO(3)$, the set of all 3×3 orthogonal and real matrices with determinant one, i. e.

$$SO(3) = \{A \in \mathcal{M}_{3\times 3}(\mathbb{R});\ A^t A = I_3,\ \det A = 1\}\,.$$

Proposition 2.1. $SO(3)$ is a 3-dimensional Lie group.

Proof: Indeed, $SO(3)$ is the kernel of the map

$$\det : O(3) \to \{-1, 1\}\,,$$

i. e.,

$$SO(3) = \det^{-1}(\{1\})\,.$$

Therefore $SO(3)$ is a closed subgroup of the Lie group $O(3)$, so it is a Lie group. It is clear also that

$$\dim(SO(3)) = 3\,.$$

□

Proposition 2.2. $SO(3)$ is a compact Lie group.

Proof: It is clear that $SO(3)$ is a closed set of $\mathcal{M}_{3\times 3}(\mathbb{R}) \simeq \mathbb{R}^9$. Hence $SO(3)$ is compact if and only if it is bounded. But for each $A \in SO(3)$ we have succesively:

$$\|A\|^2 = \langle A, A\rangle = \operatorname{trace}(A^t A) = \operatorname{trace} I_3 = 3\,,$$

and then our assertion follows imediately. □

Proposition 2.3. The Lie algebra of $SO(3)$ is the set of all real 3×3 skew-symmetric matrices, i. e.

$$so(3) = \{A \in \mathcal{M}_{3\times 3};\ A^t = -A\}\,.$$

Proof: Let us consider the rotations $R_1(\alpha)$, $R_2(\beta)$, $R_3(\gamma) \in SO(3)$, given by:

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$R_2(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$R_3(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

They are curves in $SO(3)$ and:

$$R_1(0) = R_2(0) = R_3(0) = I_3\,.$$

It follows that their derivatives at $\alpha = 0, \beta = 0$ and respectively $\gamma = 0$, belong to $so(3)$, i. e.,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in so(3).$$

Moreover these elements are linearly independent and so,

$$so(3) = \left\{ \begin{pmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{pmatrix} ; \ a, b, c \in \mathbb{R} \right\}.$$

Proposition 2.4. The Lie algebra $(so(3), [\cdot, \cdot])$ can be identified with the Lie algebra $(\mathbb{R}^3, \times)$, where "$\times$" is the cross product.

Proof: Indeed, an easy computation shows us that the map "$\wedge$" given by:

$$\wedge : \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \in \mathbb{R}^3 \mapsto \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so(3)$$

is an isomorphism of Lie algebras, and then we obtain the desired result. □

Proposition 2.5. (Rodrigues) The exponential map:

$$\exp : so(3) \to SO(3)$$

is given by the formula:

$$\exp(\hat{v}) = I_3 + \frac{\sin \|v\|}{\|v\|} \hat{v} + \frac{1}{2} \left(\frac{\sin \frac{\|v\|}{2}}{\frac{\|v\|}{2}} \right)^2 \hat{v}^2 .$$

Proof: Indeed, we have the recurrence relations:

$$\hat{v}^3 = -\|v\|^2 \hat{v}\,, \quad \hat{v}^4 = -\|v\|^2 \hat{v}^2\,, \quad \hat{v}^5 = \|v\|^4 \hat{v}\,, \quad \hat{v}^6 = \|v\|^4 \hat{v}^2\,, \quad \ldots$$

So,

$$\begin{aligned} \exp(\hat{v}) &= \sum_{n=0}^{\infty} \frac{\hat{v}^n}{n!} \\ &= I_3 + \frac{\hat{v}}{1!} + \frac{\hat{v}^2}{2!} + \frac{\hat{v}^3}{3!} + \frac{\hat{v}^4}{4!} + \cdots \\ &= I_3 + \frac{\hat{v}}{1!} + \frac{\hat{v}^2}{2!} - \frac{\|v\|^2}{3!} \hat{v} - \frac{\|v\|^2}{4!} \hat{v}^2 + \cdots \end{aligned}$$

$$= I_3 + \left[I_3 - \frac{\|v\|^2}{3!} + \frac{\|v\|^4}{5!} + \cdots \right] \hat{v}$$

$$+ \left[\frac{1}{2!} I_3 - \frac{\|v\|^2}{4!} + \cdots \right] \hat{v}^2$$

$$= I_3 + \frac{\sin \|v\|}{\|v\|} \hat{v} + \frac{1 - \cos \|v\|}{\|v\|^2} \hat{v}^2$$

$$= I_3 + \frac{\sin \|v\|}{\|v\|} \hat{v} + \frac{1}{2} \left(\frac{\sin \frac{\|v\|}{2}}{\frac{\|v\|}{2}} \right)^2 \hat{v}^2 ,$$

as required. □

Remark 2.1. It is not hard to see also that the exponential map is onto. □

3. The Rigid Body with Three Particular Controls

Consider the classical Euler equations of a free rigid body on $so(3) \simeq \mathbb{R}^3$, i. e. in terms on angular velocity:

$$J\dot{\Omega} = J\Omega \times \Omega + N \tag{3.1}$$

or in terms of angular momentum, i. e. on $(so(3))^* \simeq \mathbb{R}^3$:

$$\dot{m} = m \times J^{-1} m + N \tag{3.2}$$

where Ω is the angular velocity in body coordinates, m is the angular momentum in body coordinates, J is a constant diagonalized inertia matrix and N is the applied torque or control. Let us now add to our control system an input control U. Then it becomes:

$$\dot{m} = m \times J^{-1} m + N + U \tag{3.3}$$

In all that follows we shall concentrate to the particular case:

$$U = m \times (J_c^{-1} - J^{-1}) m - N$$

where W is a constant nonsingular symmetric matrix, $WJ^{-1} + J^{-1}W$ is invertible and

$$J_c^{-1} = \frac{1}{2}(WJ^{-1} + J^{-1}W) .$$

Under this feedback law our closed loop system becomes

$$\dot{m} = m \times J_c^{-1} m . \tag{3.4}$$

If we take now:

$$J_c^{-1} = \begin{pmatrix} a & a_1 & b_1 \\ a_1 & b & c_1 \\ b_1 & c_1 & c \end{pmatrix},$$

then our system (3.4) can be written in the equivalent form:

$$\begin{aligned} \dot{m}_1 &= (c-b)m_2m_3 + b_1m_1m_2 - a_1m_1m_3 + c_1(m_2^2 - m_3^2) \\ \dot{m}_2 &= (a-c)m_1m_3 - c_1m_1m_2 + a_1m_2m_3 + b_1(m_3^2 - m_1^2) \\ \dot{m}_3 &= (b-a)m_1m_2 + c_1m_1m_3 - b_1m_2m_3 + a_1(m_1^2 - m_2^2) \end{aligned} \tag{3.5}$$

Theorem 3.1. ([4]) The system (3.5) is a Hamilton-Poisson system with the phase space $(so(3))^* \simeq \mathbb{R}^3$, the Poisson structure given by the matrix:

$$\Pi_- = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}, \tag{3.6}$$

is in fact the minus-Lie–Poisson structure on $(so(3))^*$, and the Hamiltonian H given by:

$$\begin{aligned} H(m_1, m_2, m_3) = \frac{1}{2}[&am_1^2 + bm_2^2 + cm_3^2 \\ &+ 2a_1m_1m_2 + 2b_1m_1m_3 + 2c_1m_2m_3]. \end{aligned} \tag{3.7}$$

Proof: One readily checks that:

$$\dot{m} = \Pi \cdot \nabla H,$$

and then our assertion follows easily. □

Remark 3.1. It is easy to see that the function C given by:

$$C(m_1, m_2, m_3) = \frac{1}{2}[m_1^2 + m_2^2 + m_3^2] \tag{3.8}$$

is a Casimir of our configuration. □

Remark 3.2. The trajectories of the motion are intersections of the sphere

$$C = \mathrm{const}$$

with the quadric

$$H = \mathrm{const}.$$

□

Let us observe now that the equations of motion (3.5) can be put in the equivalent form:

$$\dot{m} = \nabla C \times \nabla H\,.$$

Then we can prove:

Theorem 3.2. The system (3.5) may be realized as a Hamilton–Poisson system in an infinite number of different ways, i. e. there exist infinitely many different (in general non-isomorphic) Poisson structures on $\mathbb{R}^3$ such that the system (3.5) is induced by an appropriate Hamiltonian.

Proof: An easy computation shows us that the system (3.5) may be realized as a Hamilton- Poisson system with the phase space $\mathbb{R}^3$, the Poisson structure $\{\cdot\,,\cdot\}_{ab}$ given by:

$$\{f, g\}_{ab} = -\nabla C' \cdot (\nabla f \times \nabla g)\,,$$

where $a, b \in \mathbb{R}$,

$$C' = aC + bH,$$

and the Hamiltonian H' given by:

$$H' = cC + dH,$$

where $c, d \in \mathbb{R}$, $ad - bc = 1.$ $\square$

Let us finish this section with the following result:

Theorem 3.3. The equations (3.5) have a Lax formulation.

Proof: Let us take:

$$L = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -cm_3 - b_1m_1 - c_1m_2 & bm_2 + a_1m_1 + c_1m_3 \\ cm_3 + b_1m_1 + c_1m_2 & 0 & -am_1 - a_1m_2 - b_1m_3 \\ -bm_2 - a_1m_1 - c_1m_3 & am_1 + a_1m_2 + b_1m_3 & 0 \end{pmatrix}$$

Then a long but straightforward computation shows us that the system (3.5) can be put in the equivalent form:

$$\dot{L} = [L, B]\,,$$

as required. $\square$

Remark 3.3. As a consequence of the above result we can conclude that the flow of the system (3.5) is isospectral.

4. Variational Formulation of the Angular Velocity Equations

We have seen in the previous section that the angular momentum equations (3.4) have a Hamilton–Poisson formulation. Therefore it is natural to ask if their angular velocity counterpart, i. e., the equations:

$$J_c\dot{\Omega} = J_c\Omega \times \Omega \tag{4.1}$$

can be formulated via a variational principle?

For the beginning let us fix some notations. Let $R = R(t) \in SO(3)$ be a time dependent matrix, δR its variation and $\widehat{\Sigma}$ the skew-symmetric matrix given by:

$$\hat{\Sigma} = R^{-1} \cdot \delta R\,.$$

It defines naturally the vector Σ by the equality:

$$\hat{\Sigma}v = \Sigma \times v,$$

for each $v \in \mathbb{R}^3$.

Then we can prove:

Theorem 4.1. The angular velocity equations (4.1) are equivalent to the constrained variational principle:

$$\delta_c \int_a^b l(\Omega)\,\mathrm{d}t = 0\,,$$

where

$$\begin{aligned} \delta_c\Omega &= \dot{\Sigma} + \Omega \times \Sigma\,, \\ \Sigma(a) &= \Sigma(b) = 0\,, \\ l(\Omega) &= \frac{1}{2}(J_c\Omega) \cdot \Omega\,. \end{aligned}$$

Proof: Since J_c is symmetric we get:

$$\begin{aligned} \delta_c \int_a^b l(\Omega)dt &= \int_a^b (J_c\Omega) \cdot \delta_c\Omega\,\mathrm{d}t \\ &= \int_a^b (J_c\Omega)(\dot{\Sigma} + \Omega \times \Sigma)\,\mathrm{d}t \\ &= \int_a^b [-\frac{d}{dt}(J_c\Omega)\Sigma + (J_c\Omega) \cdot (\Omega \times \Sigma)]\,\mathrm{d}t \end{aligned}$$

$$= \int_a^b [-\frac{d}{dt}(J_c\Omega) + J_c\Omega \times \Omega]\Sigma\, \mathrm{d}t$$

where we have integrated by parts and used the boundary conditions:

$$\Sigma(a) = \Sigma(b) = 0\,.$$

Since Σ is otherwise arbitrary,

$$\delta_c \int_a^b l(\Omega) dt = 0$$

is equivalent to:

$$J_c\dot{\Omega} = J_c\Omega \times \Omega\,,$$

as required. □

5. Prequantization

Let us consider the following diagram:

$$\begin{pmatrix} (so(3))^* \simeq \mathbb{R}^3 \\ \{\cdot,\cdot\}_- \end{pmatrix} \xrightarrow{\text{prequantization}} \begin{pmatrix} \mathcal{H} \\ \delta \end{pmatrix}$$

where in the left hand $(so(3))^*$ is the dual of the Lie-algebra $so(3)$ which can be canonically identified with $\mathbb{R}^3$ and $\{\cdot,\cdot\}_-$ is the minus-Lie–Poisson structure on $(so(3))^* \simeq \mathbb{R}^3$. In the right hand $\mathcal{H}$ is a Hilbert space and δ is a map which assigns to each $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$ a self adjoint operator $\delta_f : \mathcal{H} \to \mathcal{H}$. The arrow from left to right is called prequantization, i. e., a procedure to derive from classical data $(\mathbb{R}^3, \{\cdot,\cdot\}_-)$ the quantum data $(\mathcal{H}, \delta)$ such that the following conditions, called Dirac conditions, to be satisfied:

$$\begin{aligned} &(D1) \quad \delta_{f+g} = \delta_f + \delta_g\,, \\ &(D2) \quad \delta_{\alpha f} = \alpha \cdot \delta_f\,, \\ &(D3) \quad \delta_{id_{\mathbb{R}^3}} = Id_{\mathcal{H}}\,, \\ &(D4) \quad [\delta_f, \delta_g] = i\hbar\delta_{\{f,g\}_-}\,, \end{aligned}$$

for each $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$ and for each $\alpha \in \mathbb{R}$, and where $\hbar$ is the Planck constant divided by 2π.

The problem is now to prove the existence of such a prequantization. For this we must establish an auxiliary result. Let $T^*SO(3)$ be the cotangent bundle of $SO(3)$ and

$$\lambda : T^*SO(3) \to (so(3))^*$$

is the map defined by:

$$(\lambda(\alpha_g))(\xi) = \alpha_g(TL_g(\xi)),$$

i. e., left translation of covectors to the identity.

Proposition 5.1. λ is a Poisson map.

Proof: For the proof it is enough to show that λ is in fact the momentum map associated to the right translations of $SO(3)$ on $T^*SO(3)$. For to see this, let

$$\Lambda : SO(3) \times SO(3) \to SO(3)$$

be the action of $SO(3)$ on itself by right translations, that is

$$\Lambda_g = R_g \,,$$

for all $g \in SO(3)$. Consider the induced action Λ^{T^*} on $T^*SO(3)$. Then the momentum map of this action

$$J : T^*SO(3) \to (so(3))^*$$

is given by:

$$(J(\alpha_g))(\xi) = \alpha_g(\xi_{SO(3)}(g)) = \alpha_g(TL_g(\xi)) \,,$$

and so

$$\lambda = J \,,$$

as required. □

Let us take now:

$$\mathcal{H} = L^2(T^*SO(3), \mathbb{C}) \,, \tag{5.1}$$

and for each $f \in C^\infty((so(3))^*, \mathbb{R})$

$$\delta_f = -\mathrm{i}\hbar \left[X_{f\circ\lambda} \frac{\mathrm{i}}{\hbar} \theta(X_{f\circ\lambda}) \right] + f \circ \lambda \,, \tag{5.2}$$

where

$$\omega = \mathrm{d}\theta$$

is the canonical symplectic structure on $T^*SO(3)$. Then an easy computation leads us to:

Theorem 5.1. The pair $(\mathcal{H}, \delta)$ given by (5.1) and (5.2) gives rise to a prequantization of the Poisson manifold $((so(3))^*, \{\cdot,\cdot\}_-)$.

Using now the same arguments as in [7] (with obvious modifications) we can prove also:

Theorem 5.2. Let $O(L^2(T^*SO(3),\mathbb{C}))$ be the space of self-adjoint operators on the Hilbert space $L^2((so(3))^*,\mathbb{C})$. Then the map:

$$f \in C^\infty((so(3))^*,\mathbb{R}) \mapsto \delta_f \in O(L^2(T^*SO(3),\mathbb{C}))$$

gives rise to an irreducible representation of $C^\infty((so(3))^*,\mathbb{R})$ onto the Hilbert space $L^2(T^*SO(3),\mathbb{C})$.

6. Stability

In this section we shall study the nonlinear stability of the equilibrium states of the system (3.5) under the restrictions:

$$a_1 = 0\,, \quad b_1 = 0\,, \quad c_1 \neq 0\,, \quad a < b < c \tag{6.1}$$

or equivalent, the nonlinear stability of the equilibrium states of the system:

$$\begin{aligned} \dot{m}_1 &= (c-b)m_2m_3 + c_1(m_2^2 - m_3^2) \\ \dot{m}_2 &= (a-c)m_1m_3 - c_1m_1m_2 \\ \dot{m}_3 &= (b-a)m_1m_2 + c_1m_1m_3 \end{aligned} \tag{6.2}$$

under the restriction

$$a < b < c\,. \tag{6.3}$$

Recall that an equilibrium state m_e is nonlinearly stable if trajectories starting close to m_e stay close to m_e. In other words, a neighborhood of m_e must be flow invariant.

An easy and direct computation shows that the equilibrium states of our system (6.2), (6.3) are:

$$\begin{aligned} e_1 &= (0,0,0)\,, \\ e_2 &= (M,0,0)\,, \quad M \neq 0\,, \\ e_3 &= \left(0\,,\ \frac{1}{2}\,\frac{-c+b+\sqrt{(c-b)^2+4c_1^2}}{c_1}M\,,\ M\right),\ M \neq 0\,, \\ e_4 &= \left(0\,,\ \frac{1}{2}\,\frac{-c+b-\sqrt{(c-b)^2+4c_1^2}}{c_1}M\,,\ M\right),\ M \neq 0\,, \\ e_5 &= (M\,,\ \frac{a-c}{c_1}\alpha,\alpha)\,,\ M \neq 0\,,\ c_1^2 = (a-b)(a-c)\,. \end{aligned}$$

Then we have:

Theorem 6.1. The equilibrium state e_1 is nonlinearly stable.

Proof: An easy computation shows us that the function C given by (3.8) (in fact the Casimir) is a Lyapunov function and then the assertion is a consequence of the Lyapunov theorem. $\square$

We can also prove the following spectral stability result.

Theorem 6.2.

i) The equilibrium state e_2 is spectrally stable if

$$c_1^2 \leq (a-b)(a-v)\,.$$

ii) The equilibrium state e_3 is spectrally stable.

iii) The equilibrium state e_4 is spectrally stable if

$$c_1^2 \geq (a-b)(a-c)\,.$$

iv) The equilibrium state e_5 is spectrally stable.

Proof:

i) The linearized system around the state e_2 has the characteristic polynomial

$$p_2(t) = t[t^2 - M^2(c_1^2 - (b-a)(c-a))]\,.$$

It is then obvious that e_2 is spectrally stable iff $c_1^2 \leq (a-b)(a-c)$.

ii) The linearized system around e_3 has the characteristic polynomial

$$p_3(t) = t(t^2 + \lambda_1)\,,$$

where

$$\lambda_1 = -\frac{M^2}{4c_1^2}(b+c-2a+u)((c-b)^3 - u(c-b)^2 + 4c_1^2(c-b-u))$$

and $u = \sqrt{(c-b)^2 + 4c_1^2}$.

It is not hard to see that λ_1 can be put into the following form:

$$\lambda_1 = -\frac{M^2}{4c_1^2}u^2[-u+(c-b)](b+c-2a+u)\,.$$

Then it is obvious that e_3 is spectrally stable iff $u \geq -(b+c-2a)$, which is always true, because $u \geq 0$ and $a < b < c$.

iii) The linearized system around the equilibrium state e_4 has the characteristic polynomial

$$p_4(t) = t(t^2 + \lambda_2)\,,$$

where

$$\lambda_2 = -\frac{M^2}{4c_1^2}(b+c-2a-u)((c-b)^3 + u(c-b)^2 + 4c_1^2(c-b+u))\,.$$

It is not hard to see that

$$\lambda_2 = -\frac{M^2}{4c_1^2}u^2[u+(c-b)](b+c-2a-u)\,.$$

Then it is obvious that e_4 is spectrally stable iff $u \geq b+c-2a$, which is equivalent to $c_1^2 \geq (a-b)(a-c)$.

iv) An easy computation shows that the linearized system around the states e_5 has only the null solution when $c_1^2 = (b-a)(c-a)$. □

Theorem 6.3. The equilibrium state e_2 is nonlinearly stable iff:

$$(a-b)(a-c) > c_1^2$$

and unstable iff

$$(a-b)(a-c) < c_1^2\,.$$

Proof: The second assertion follows directly from the Theorem 6.2, (i).
If

$$c_1^2 < (a-b)(a-c)$$

then the equilibrium state e_2 is spectrally stable. Is it nonlinearly stable? We shall prove that it is via the energy-Casimir method. Recall that the energy-Casimir method (see [10], [12], [13] or [18]) requires finding a constant of motion for the system, say H, usually the energy, and a family of constants of motion $\mathcal{C}$ such that for some $C \in \mathcal{C}$, $C+H$ has a critical point at the equilibrium of interest. $\mathcal{C}$'s are often taken to be Casimirs. Definiteness of $\delta^2(H+C)$, the second variation of $H+C$ at the critical point is sufficient to prove the stability, if the phase space of the system is finite dimensional.
Let us consider the energy-Casimir function

$$H_\varphi = H + \varphi(C)\,,$$

where H and C are given by the relations (3.7), (3.8), (6.1), respectively and

$$\varphi : \mathbb{R} \to \mathbb{R}$$

is an arbitrary smooth function. Now, the first variation of H_φ is given by:

$$\begin{aligned}\delta H_\varphi &= am_1\delta m_1 + bm_2\delta m_2 + cm_3\delta m_3 \\ &\quad + c_1m_2\delta m_2 + c_1m_3\delta m_3 \\ &\quad + \dot{\varphi}[m_1\delta m_1 + m_2\delta m_2 + m_3\delta m_3]\,.\end{aligned}$$

This equals zero at the equilibrium of interest if and only if:

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = -a\,. \tag{6.4}$$

Then

$$\begin{aligned}\delta^2 H_\varphi &= a(\delta m_1)^2 + b(\delta m_2)^2 + c(\delta m_3)^2 \\ &\quad + \ddot{\varphi}[m_1\delta m_1 + m_2\delta m_2 + m_3\delta m_3]^2 \\ &\quad + \dot{\varphi}[(\delta m_1)^2 + (\delta m_2)^2 + (\delta m_3)^2] \\ &\quad + 2c_1\delta m_2\delta m_3\,.\end{aligned}$$

At the equilibrium of interest e_2 we have via (6.4):

$$\begin{aligned}\delta^2 H_\varphi(e_2) &= (b-a)(\delta m_2)^2 + (c-a)(\delta m_3)^2 \\ &\quad + \ddot{\varphi}\left(\frac{1}{2}M^2\right)M^2(\delta m_1)^2 + 2c_1\delta m_2\delta m_3\,.\end{aligned}$$

If we choose φ such that:

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) > 0$$

then it is not hard to see that the quadratic form $\delta^2 H_\varphi(e_2)$ is positive definite and so the equilibrium state e_2 is nonlinearly stable. □

Remark 6.1. The result of our last theorem has also been obtained independently by Posberg and Zhao [14] using the energy-momentum method.

7. Numerical Integration of the Equations (6.2), (6.3)

In this last section we shall discuss the numerical integration of the equations (6.2), (6.3) via the Kahan's integrator and we shall point out some of its geometrical properties from the Poisson geometry point of view.

For the equations (6.2), (6.3) Kahan's integrator can be written in the following form:

$$\begin{aligned}m_1^{n+1} - m_1^n &= \frac{h}{2}[(c-b)(m_2^{n+1}m_3^n + m_3^{n+1}m_2^n) \\ &\qquad + 2c_1(m_2^{n+1}m_2^n - m_3^{n+1}m_3^n)] \\ m_2^{n+1} - m_2^n &= \frac{h}{2}[(a-c)(m_1^{n+1}m_3^n + m_3^{n+1}m_1^n) \\ &\qquad - c_1(m_1^{n+1}m_2^n + m_2^{n+1}m_1^n] \\ m_3^{n+1} - m_3^n &= \frac{h}{2}[(b-a)(m_1^{n+1}m_2^n + m_2^{n+1}m_1^n) \\ &\qquad + c_1(m_1^{n+1}m_3^n + m_3^{n+1}m_1^n\end{aligned} \tag{7.1}$$

Theorem 7.1. If

$$c_1^2 = (a-b)(a-c),$$

then the following statements hold:

i) Kahan's integrator (7.1) is a Poisson integrator;
ii) Kahan's integrator (7.1) is energy preserving;
iii) Kahan's integrator (7.1) is Casimir preserving.

Proof:

i) The first statement can be easily obtained by a long and straightforward computation or using eventually the computer algebra system MAPLE V.

ii) If we denote

$$H_n = \frac{1}{2}\left[a(m_1^n)^2 + b(m_2^n)^2 + c(m_3^n)^2 + 2c_1 m_2^n m_3^n\right]$$

then we have for each $n \in N$:

$$H_{n+1} - H_n = h^3[c_1^2 - (a-b)(a-c)]H_n \cdot P_n, \tag{7.2}$$

where P_n is a rational functions of variables h, a, b, c, c_1, m_1^n,m_2^n, m_3^n. Using now the hypothesis our assertion follows immediately via the relation (7.2).

iii) Using the same technique as in the previous statement, let

$$C_n = \frac{1}{2}\left[(m_1^n)^2 + (m_2^n)^2 + (m_3^n)^2\right]$$

Then we have for each $n \in \mathbb{N}$:

$$C_{n+1} - C_n = h^3[c_1^2 - (a-b)(a-c)]C_n \cdot Q_n, \tag{7.3}$$

where Q_n is a rational functions of variables h, a, b, c,c_1, m_1^n,m_2^n, m_3^n. Using now the hypothesis our assertion follows immediately via the relation (7.2). □

Remark 7.1. In the particular case

$$a = \frac{1}{I_1}; \quad b = \frac{1}{I_2}; \quad c = \frac{1}{I_3}; \quad I_1 > I_2 > I_3 > 0; \quad c_1 = 0$$

we refind our main result from [24]. □

Finally we can make a comparison between Kahan's integrator and the 4th order Runge–Kutta integrator. It is clear that both algorithms lead to the same picture. However, Kahan's integrator has the advantage that it is more convenient for implementation.

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