Demonstration.—First multiplying by x both members of the given equality [which is the first member of the entire secondary equality], we have

$$x = ax,$$

which, as we know, is equivalent to the inclusion

$$x < a.$$

Now multiplying both members by x', we have

$$0 = bx',$$

which, as we know, is equivalent to the inclusion

$$b < x.$$

Summing up, we have

$$(x = ax + bx') < (b < x < a).$$

Conversely,

$$(b < x < a) < (x = ax + bx').$$

For

$$(x < a) = (x = ax),$$
$$(b < x) = (bx' = 0).$$

Adding these two equalities member to member [the second members of the two larger equalities],

$$(x = ax)\ (0 = bx) < (x = ax + bx').$$

Therefore

$$(b < x < a) < (x = ax + bx'),$$

and thus the equivalence is proved.

30. Schröder's Theorem.[1]—The equality

$$ax + bx' = 0$$

signifies that x lies between a' and b.

Demonstration:

$$(ax + bx' = 0) = (ax = 0)\ (bx' = 0),$$
$$(ax = 0) = (x < a'),$$
$$(bx' = 0) = (b < x).$$

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[1] SCHRÖDER, *Operationskreis des Logikkalkuls* (1877), Theorem 20.

Hence

$$(ax + bx' = 0) = (b < x < a').$$

Comparing this theorem with the formula of PORETSKY, we obtain at once the equality

$$(ax + bx' = 0) = (x = a'x + bx'),$$

which may be directly proved by reducing the formula of PORETSKY to an equality whose second member is 0, thus:

$$\begin{aligned}(x = a'x + bx') &= [x(ax + b'x') + x'(a'x + bx') = 0]\\ &= (ax + bx' = 0).\end{aligned}$$

If we consider the given equality as an *equation* in which x is the unknown quantity, PORETSKY's formula will be its solution.

From the double inclusion

$$b < x < a'$$

we conclude, by the principle of the syllogism, that

$$b < a'.$$

This is a consequence of the given equality and is independent of the unknown quantity x. It is called the *resultant of the elimination* of x in the given equation. It is equivalent to the equality

$$ab = 0.$$

Therefore we have the implication

$$(ax + bx' = 0) < (ab = 0).$$

Taking this consequence into consideration, the solution may be simplified, for

$$(ab = 0) = (b = a'b).$$

Therefore

$$\begin{aligned}x &= a'x + bx' = a'x + a'bx'\\ &= a'bx + a'b'x + a'bx' = a'b + a'b'x\\ &= b + a'b'x = b + a'x.\end{aligned}$$

This form of the solution conforms most closely to common sense: since x' contains b and is contained in a', it is natural that x should be equal to the sum of b and a part of a'

(that is to say, the part common to a' and x). The solution is generally indeterminate (between the limits a' and b); it is determinate only when the limits are equal,

$$a' = b,$$

for then

$$x = b + a'x = b + bx = b = a'.$$

Then the equation assumes the form

$$(ax + a'x' = 0) = (a' = x)$$

and is equivalent to the double inclusion

$$(a' < x < a') = (x = a').$$

31. The Resultant of Elimination.—When ab is not zero, the equation is impossible (always false), because it has a false consequence. It is for this reason that SCHRÖDER considers the resultant of the elimination as a *condition* of the equation. But we must not be misled by this equivocal word. The resultant of the elimination of x is not a *cause* of the equation, it is a *consequence* of it; it is not a *sufficient* but a *necessary* condition.

The same conclusion may be reached by observing that ab is the inferior limit of the function $ax + bx'$, and that consequently the function can not vanish unless this limit is 0.

$$(ab < ax + bx')\ (ax + bx' = 0) < (ab = 0).$$

We can express the resultant of elimination in other equivalent forms; for instance, if we write the equation in the form

$$(a + x')\ (b + x) = 0,$$

we observe that the resultant

$$ab = 0$$

is obtained simply by dropping the unknown quantity (by suppressing the terms x and x'). Again the equation may be written:

$$a'x + b'x' = 1$$

and the resultant of elimination:

$$a' + b' = 1.$$