and not, as is often thought[1], from the laws of negation which have not yet been stated. We shall see that these laws possess the same property and consequently preserve the duality, but they do not originate it; and duality would exist even if the idea of negation were not introduced. For instance, the equality (§ 12)

$$ab + ac + bc = (a + b)\ (a + c)\ (b + c)$$

is its own reciprocal by duality, for its two members are transformed into each other by duality.

It is worth remarking that the law of duality is only applicable to primary propositions. We call [after BOOLE] those propositions *primary* which contain but one copula ($<$ or $=$). We call those propositions *secondary* of which both members (connected by the copula $<$ or $=$) are primary propositions, and so on. For instance, the principle of identity and the principle of simplification are primary propositions, while the principle of the syllogism and the principle of composition are secondary propositions.

15. Definition of Negation.—The introduction of the terms 0 and 1 makes it possible for us to define *negation*. This is a "uni-nary" operation which transforms a single term into another term called its *negative*.[2] The negative of a is called not-a and is written a'.[3] Its formal definition implies the following postulate of existence[4]:

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1 [BOOLE thus derives it (*Laws of Thought*, London 1854, Chap. III, Prop. IV).]

2 [In French] the same word *negation* denotes both the operation and its result, which becomes equivocal. The result ought to be denoted by another word, like [the English] "negative". Some authors say, "supplementary" or "supplement", [e. g. BOOLE and HUNTINGTON]. Classical logic makes use of the term "contradictory" especially for propositions.

3 We adopt here the notation of MACCOLL; SCHRÖDER indicates not-a by a_1 which prevents the use of indices and obliges us to express them as exponents. The notation a' has the advantage of excluding neither indices nor exponents. The notation $\bar{a}$ employed by many authors is inconvenient for typographical reasons. When the negative affects a proposition written in an explicit form (with a copula) it is applied to the copula ($<$ or $=$) by a vertical bar ($\not<$ or $\neq$). The accent can be considered as the indication of a vertical bar applied to letters.

4 [BOOLE follows Aristotle in usually calling the law of duality the

(Ax. VIII.) Whatever the term a may be, there is also a term a' such that we have at the same time

$$aa' = 0, \quad a + a' = 1.$$

It can be proved by means of the following *lemma* that if a term so defined exists it is unique:

If at the same time

$$ac = bc, \quad a + c = b + c,$$

then

$$a = b.$$

Demonstration.—Multiplying both members of the second premise by a, we have

$$a + ac = ab + ac.$$

Multiplying both members by b,

$$ab + bc = b + bc.$$

By the first premise,

$$ab + ac = ab + bc.$$

Hence

$$a + ac = b + bc,$$

which by the law of absorption may be reduced to

$$a = b.$$

Remark.—This demonstration rests upon the direct distributive law. This law cannot, then, be demonstrated by means of negation, at least in the system of principles which we are adopting, without reasoning in a circle.

This lemma being established, let us suppose that the same term a has two negatives; in other words, let a'_1 and a'_2 be two terms each of which by itself satisfies the conditions of

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principle of contradiction "which affirms that it is impossible for any being to possess a quality and at the same time not to possess it". He writes it in the form of an equation of the second degree, $x - x^2 = 0$, or $x (1 - x) = 0$ in which $1 - x$ expresses the universe less x, or not x. Thus he regards the law of duality as derived from negation as stated in note 1 above.]

the definition. We will prove that they are equal. Since, by hypothesis,

$$aa'_1 = 0, \quad a + a'_1 = 1,$$
$$aa'_2 = 0, \quad a + a'_2 = 1,$$

we have

$$aa'_1 = aa'_2, \quad a + a'_1 = a + a'_2;$$

whence we conclude, by the preceding lemma, that

$$a'_1 = a'_2.$$

We can now speak of *the* negative of a term as of a unique and well-defined term.

The *uniformity* of the operation of negation may be expressed in the following manner:

If $a = b$, then also $a' = b'$. By this proposition, both members of an equality in the logical calculus may be "denied".

16. The Principles of Contradiction and of Excluded Middle.—By definition, a term and its negative verify the two formulas

$$aa' = 0, \quad a + a' = 1,$$

which represent respectively the *principle of contradiction* and the *principle of excluded middle.*[1]

C. I.: 1. The classes a and a' have nothing in common; in other words, no element can be at the same time both a and not-a.

2. The classes a and a' combined form the whole; in other words, every element is either a or not-a.

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[1] As Mrs. LADD-FRANKLIN has truly remarked (BALDWIN, *Dictionary of Philosophy and Psychology*, article "Laws of Thought"), the principle of *contradiction* is not sufficient to define *contradictories;* the principle of excluded middle must be added which equally deserves the name of principle of contradiction. This is why Mrs. LADD-FRANKLIN proposes to call them respectively the *principle of exclusion* and the *principle of exhaustion,* inasmuch as, according to the first, two contradictory terms are *exclusive* (the one of the other); and, according to the second, they are *exhaustive* (of the universe of discourse).