Chapter 2

Probability measure P on $\mathbb{R}^{\mathbb{B}}$

Topics of this chapter come from Fukuyama [11]. To tell the truth, Chapter 1 was prepared, since we wanted the fact that every almost periodic function always has the mean value. Based on these mean values, we define a probability measure **P** on the space $\mathbb{R}^{\mathbb{B}}$ of large volume.

2.1 Definition of the probability measure P

Definition 2.1 $\mathbb{B} := AP(\mathbb{R}) \cap C(\mathbb{R};\mathbb{R})$, i.e., $\mathbb{B}$ is the set of all real-valued almost periodic functions.

Definition 2.2 For $T > 0$, we define a probability measure P_T on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

$$P_T(E) := \frac{1}{2T}\mu\big([-T, T] \cap E\big), \quad E \in \mathcal{B}(\mathbb{R}).$$

Here μ is the 1-dimensional Lebesgue measure.

Lemma 2.1 *For $f_1, \ldots, f_n \in \mathbb{B}$, let $P_T^{(f_1,\ldots,f_n)}$ be an image measure of P_T by the continuous mapping*

$$\mathbb{R} \ni t \mapsto (f_1, \ldots, f_n)(t) := \big(f_1(t), \ldots, f_n(t)\big) \in \mathbb{R}^n.$$

($P_T^{(f_1,\ldots,f_n)}$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.) Then

$$\begin{aligned} &{}^{\exists}P\text{: a probability measure on } \big(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\big)\\ &\text{s.t. } P_T^{(f_1,\ldots,f_n)} \to P \ \text{ weakly as } T \to \infty. \end{aligned}$$

Proof. Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. By Claim 1.3 and Claim 1.1(iii), $e^{\sqrt{-1}\sum_{i=1}^n \xi_i f_i(\cdot)} \in AP(\mathbb{R})$. Thus, by Theorem 1.1,

$$\begin{aligned} \widehat{P_T^{(f_1,\ldots,f_n)}}(\xi)^{\dagger 1} &= \int_{\mathbb{R}^n} e^{\sqrt{-1}\langle \xi, x\rangle} P_T^{(f_1,\ldots,f_n)}(dx)\\ &= \int_{\mathbb{R}} e^{\sqrt{-1}\sum_{i=1}^n \xi_i f_i(t)} P_T(dt) \end{aligned}$$

$$= \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\sum_{i=1}^{n}\xi_i f_i(t)}dt$$
$$\to M\big(e^{\sqrt{-1}\sum_{i=1}^{n}\xi_i f_i}\big) \quad \text{as } T \to \infty.$$

On the other hand,

$$\begin{aligned}
\Big|M\big(e^{\sqrt{-1}\sum_{i=1}^{n}\xi_i f_i}\big) - 1\Big| &= \lim_{T\to\infty}\Big|\frac{1}{2T}\int_{-T}^{T}\big(e^{\sqrt{-1}\sum_{i=1}^{n}\xi_i f_i(t)} - 1\big)dt\Big| \\
&\le \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big|e^{\sqrt{-1}\sum_{i=1}^{n}\xi_i f_i(t)} - 1\Big|dt \\
&\le \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big|\sum_{i=1}^{n}\xi_i f_i(t)\Big|dt \\
&\qquad \big[\because\ |e^{\sqrt{-1}u} - e^{\sqrt{-1}v}| \le |u-v|\ (u, v \in \mathbb{R})\big] \\
&\le |\xi|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sqrt{\sum_{i=1}^{n} f_i^2(t)}\,dt \\
&= |\xi| M\Bigg(\sqrt{\sum_{i=1}^{n} f_i^2(\cdot)}\Bigg) \\
&\to 0 \quad \text{as } \xi \to 0.
\end{aligned}$$

The assertion of the lemma follows from Lévy's continuity theorem [cf. Claim A.3]. ■

We denote the P in Lemma 2.1 by $\mathbf{P}^{(f_1,\dots,f_n)}$.

Lemma 2.2 *For* $1 \le i \le n+1$*, let*

$$\begin{array}{cccc}
\pi_i{}^{\dagger 2}: & \mathbb{R}^{n+1} & \to & \mathbb{R}^n \\
 & \Cup & & \Cup \\
 & (x_1, \dots, x_{n+1}) & \mapsto & (x_1, \overset{i}{\check{\dots}}, x_{n+1}).
\end{array}$$

Then, for $^\forall f_1, \dots, {}^\forall f_{n+1} \in \mathbb{B}$,

$$\mathbf{P}^{(f_1,\dots,f_{n+1})} \circ \pi_i^{-1} = \mathbf{P}^{(f_1,\overset{i}{\check{\dots}},f_{n+1})}.$$

Proof. For $^\forall \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$,

$$\begin{aligned}
&\widehat{\mathbf{P}^{(f_1,\dots,f_{n+1})} \circ \pi_i^{-1}}(\xi) \\
&= \int_{\mathbb{R}^n} e^{\sqrt{-1}\langle \xi, y\rangle}\mathbf{P}^{(f_1,\dots,f_{n+1})} \circ \pi_i^{-1}(dy) \\
&= \int_{\mathbb{R}^{n+1}} e^{\sqrt{-1}\langle \xi, \pi_i(x)\rangle}\mathbf{P}^{(f_1,\dots,f_{n+1})}(dx)
\end{aligned}$$

†1 For a probability measure ν on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $\widehat{\nu}$ is the characteristic function of ν [cf. Definition A.2]. When ν is a complicated expression and its width $(= \underset{\leftrightarrow}{\nu})$ is wide, we write its characteristic function as $\nu\hat{}$ instead of $\widehat{\nu}$.

†2 We denote this mapping by π_i temporarily, which is used only in this lemma.

$$
\begin{aligned}
&= \int_{\mathbb{R}^{n+1}} e^{\sqrt{-1}\langle \xi, (x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})\rangle} \mathbf{P}^{(f_1,\dots,f_{n+1})}(dx_1\cdots dx_{n+1}) \\
&= \int_{\mathbb{R}^{n+1}} e^{\sqrt{-1}(\sum_{j<i}\xi_j x_j + \sum_{i\le j\le n}\xi_j x_{j+1})} \mathbf{P}^{(f_1,\dots,f_{n+1})}(dx_1\cdots dx_{n+1}) \\
&= \int_{\mathbb{R}^{n+1}} e^{\sqrt{-1}(\sum_{j<i}\xi_j x_j + 0\cdot x_i + \sum_{i\le j\le n}\xi_j x_{j+1})} \mathbf{P}^{(f_1,\dots,f_{n+1})}(dx_1\cdots dx_{n+1}) \\
&= M\big(e^{\sqrt{-1}(\sum_{j<i}\xi_j f_j + 0\cdot f_i + \sum_{i\le j\le n}\xi_j f_{j+1})}\big) \\
&= M\big(e^{\sqrt{-1}(\sum_{j<i}\xi_j f_j + \sum_{i\le j\le n}\xi_j f_{j+1})}\big) \\
&= \widehat{\mathbf{P}^{(f_1,\dots,f_{i-1},f_{i+1},\dots,f_{n+1})}}(\xi).
\end{aligned}
$$

■

Theorem 2.1 *$^{\exists 1}\mathbf{P}$: a probability measure on $\big(\mathbb{R}^{\mathbb{B}}, \mathcal{B}_{\mathrm{K}}(\mathbb{R}^{\mathbb{B}})\big)$ s.t. for $^{\forall}n \in \mathbb{N}$ and $^{\forall}f_1, \dots, {}^{\forall}f_n \in \mathbb{B}$ with $f_i \neq f_j$ $(i \neq j)$*

$$\mathbf{P} \circ \pi^{-1}_{(f_1,\dots,f_n)} = \mathbf{P}^{(f_1,\dots,f_n)}.$$

Here

$$
\begin{array}{cccc}
\pi_{(f_1,\dots,f_n)}: & \mathbb{R}^{\mathbb{B}} & \to & \mathbb{R}^n \\
& \Cup & & \Cup \\
& (x_f)_{f\in\mathbb{B}} & \mapsto & (x_{f_1},\dots,x_{f_n})
\end{array}
$$

and

$$\mathcal{B}_{\mathrm{K}}(\mathbb{R}^{\mathbb{B}})^{\dagger 3} = \sigma(\pi_f; f \in \mathbb{B}).$$

Proof. For $\Lambda \subset \mathbb{B}$ with $\#\Lambda = n$ $(\in \mathbb{N})$, let $\Lambda = \{f_1, \dots, f_n\}$, and define $\varphi_{(f_1,\dots,f_n)}$: $\mathbb{R}^n \to \mathbb{R}^{\Lambda}$ by $\big(\varphi_{(f_1,\dots,f_n)}(x_1,\dots,x_n)\big)_f := x_i$ if $f = f_i$. $\varphi_{(f_1,\dots,f_n)}$ is bijective and the inverse $\varphi^{-1}_{(f_1,\dots,f_n)}$ is $\varphi^{-1}_{(f_1,\dots,f_n)}\big((x_f)_{f\in\Lambda}\big) = (x_{f_1},\dots,x_{f_n})$. Let μ_Λ be a probability measure on $\big(\mathbb{R}^{\Lambda}, \mathcal{B}(\mathbb{R}^{\Lambda})\big)$ defined by

$$\mu_\Lambda := \mathbf{P}^{(f_1,\dots,f_n)} \circ \varphi^{-1}_{(f_1,\dots,f_n)}.$$

In the following, we divide the proof into three steps:

<u>1°</u> μ_Λ is well-defined.

∵ Let $\Lambda = \{f_1, \dots, f_n\}$ and $\sigma \in \mathfrak{S}_n$. σ is a permutation of $\{1, \dots, n\}$. For $^{\forall}\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, it follows from Lemma 2.1 that

$$
\begin{aligned}
&\big(\mathbf{P}^{(f_{\sigma(1)},\dots,f_{\sigma(n)})} \circ \varphi^{-1}_{(f_{\sigma(1)},\dots,f_{\sigma(n)})} \circ (\varphi^{-1}_{(f_1,\dots,f_n)})^{-1}\big)\widehat{\ }(\xi) \\
&= \big(\mathbf{P}^{(f_{\sigma(1)},\dots,f_{\sigma(n)})} \circ (\varphi^{-1}_{(f_1,\dots,f_n)} \circ \varphi_{(f_{\sigma(1)},\dots,f_{\sigma(n)})})^{-1}\big)\widehat{\ }(\xi) \\
&= \int_{\mathbb{R}^n} e^{\sqrt{-1}\langle \xi, y\rangle} \mathbf{P}^{(f_{\sigma(1)},\dots,f_{\sigma(n)})} \circ (\varphi^{-1}_{(f_1,\dots,f_n)} \circ \varphi_{(f_{\sigma(1)},\dots,f_{\sigma(n)})})^{-1}(dy) \\
&= \int_{\mathbb{R}^n} e^{\sqrt{-1}\big\langle \xi, \varphi^{-1}_{(f_1,\dots,f_n)}\circ\varphi_{(f_{\sigma(1)},\dots,f_{\sigma(n)})}(x)\big\rangle} \mathbf{P}^{(f_{\sigma(1)},\dots,f_{\sigma(n)})}(dx)
\end{aligned}
$$

[†3] $\mathcal{B}_{\mathrm{K}}(\mathbb{R}^{\mathbb{B}})$ is called the *Kolmogorov σ-algebra* on $\mathbb{R}^{\mathbb{B}}$. Of course the subscript 'K' comes from the initial of Kolmogorov.

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} e^{\sqrt{-1}\left\langle \xi, \varphi^{-1}_{(f_1,\ldots,f_n)}(\varphi_{(f_{\sigma(1)},\ldots,f_{\sigma(n)})}(x))\right\rangle} \mathbf{P}^{(f_{\sigma(1)},\ldots,f_{\sigma(n)})}(dx)\\
&= \int_{\mathbb{R}^n} e^{\sqrt{-1}\sum_{i=1}^n \xi_i (\varphi_{(f_{\sigma(1)},\ldots,f_{\sigma(n)})}(x))_{f_i}} \mathbf{P}^{(f_{\sigma(1)},\ldots,f_{\sigma(n)})}(dx)\\
&= \int_{\mathbb{R}^n} e^{\sqrt{-1}\sum_{i=1}^n \xi_i x_{\sigma^{-1}(i)}} \mathbf{P}^{(f_{\sigma(1)},\ldots,f_{\sigma(n)})}(dx_1\cdots dx_n)\\
&\qquad \left[\because \text{ Since } f_i = f_{\sigma(\sigma^{-1}(i))},\ \left(\varphi_{(f_{\sigma(1)},\ldots,f_{\sigma(n)})}(x)\right)_{f_i} = x_{\sigma^{-1}(i)}\right]\\
&= \int_{\mathbb{R}^n} e^{\sqrt{-1}\sum_{i=1}^n \xi_{\sigma(i)} x_i} \mathbf{P}^{(f_{\sigma(1)},\ldots,f_{\sigma(n)})}(dx_1\cdots dx_n)\\
&= \widehat{\mathbf{P}^{(f_{\sigma(1)},\ldots,f_{\sigma(n)})}}\big((\xi_{\sigma(1)},\ldots,\xi_{\sigma(n)})\big)\\
&= M\left(e^{\sqrt{-1}\sum_{i=1}^n \xi_{\sigma(i)} f_{\sigma(i)}}\right)\\
&= M\left(e^{\sqrt{-1}\sum_{i=1}^n \xi_i f_i}\right)\\
&= \widehat{\mathbf{P}^{(f_1,\ldots,f_n)}}(\xi).
\end{aligned}
$$

This implies that

$$\mathbf{P}^{(f_{\sigma(1)},\ldots,f_{\sigma(n)})} \circ \varphi^{-1}_{(f_{\sigma(1)},\ldots,f_{\sigma(n)})} \circ (\varphi^{-1}_{(f_1,\ldots,f_n)})^{-1} = \mathbf{P}^{(f_1,\ldots,f_n)},$$

so that we have

$$\mathbf{P}^{(f_{\sigma(1)},\ldots,f_{\sigma(n)})} \circ \varphi^{-1}_{(f_{\sigma(1)},\ldots,f_{\sigma(n)})} = \mathbf{P}^{(f_1,\ldots,f_n)} \circ \varphi^{-1}_{(f_1,\ldots,f_n)},$$

which shows the well-definedness of μ_Λ.

$\underline{2^{\circ}}$ For $\emptyset \subsetneq \Lambda_1 \subset \Lambda_2 \subset \mathbb{B}$, we define $\pi_{\Lambda_1,\Lambda_2} : \mathbb{R}^{\Lambda_2} \to \mathbb{R}^{\Lambda_1}$ by

$$\pi_{\Lambda_1,\Lambda_2}\big((x_f)_{f\in\Lambda_2}\big) := (x_f)_{f\in\Lambda_1}, \quad (x_f)_{f\in\Lambda_2} \in \mathbb{R}^{\Lambda_2}.$$

When Λ_2 is finite, it holds that

$$\mu_{\Lambda_2} \circ \pi^{-1}_{\Lambda_1,\Lambda_2} = \mu_{\Lambda_1}.$$

(Thus $\{\mu_\Lambda; \Lambda \subset \mathbb{B} \text{ is finite}\}$ satisfies the consistency condition [cf. Definition A.5].)

$\because$ Let $\Lambda_1 \subset \Lambda_2 \subset \mathbb{B}$ be finite and $\#\Lambda_1 = m$, $\#\Lambda_2 = n$ ($m \le n$). In case $m = n$, $\Lambda_1 = \Lambda_2 = \Lambda$, thus $\pi_{\Lambda_1,\Lambda_2} = \mathrm{id}_\Lambda$. In this case, the assertion of 2° is trivial. Let $m < n$, $\Lambda_2 = \{f_1,\ldots,f_m,f_{m+1},\ldots,f_n\}$ and $\Lambda_1 = \{f_1,\ldots,f_m\}$. For $0 \le j \le n-m-1$, put $\pi^{(j)} : \mathbb{R}^{n-j} \to \mathbb{R}^{n-j-1}$ as

$$\pi^{(j)}(x_1,\ldots,x_{n-j}) := (x_1,\ldots,x_{n-j-1}).$$

Since, by Lemma 2.2,

$$\mathbf{P}^{(f_1,\ldots,f_{n-j})} \circ {\pi^{(j)}}^{-1} = \mathbf{P}^{(f_1,\ldots,f_{n-j-1})}, \quad 0 \le j \le n-m-1,$$

it follows that

$$\mathbf{P}^{(f_1,\ldots,f_n)} \circ {\pi^{(0)}}^{-1} \circ {\pi^{(1)}}^{-1} \circ \cdots \circ {\pi^{(n-m-1)}}^{-1}$$

$$
\begin{aligned}
&= \mathbf{P}^{(f_1,\ldots,f_{n-1})} \circ \pi^{(1)^{-1}} \circ \pi^{(2)^{-1}} \circ \cdots \circ \pi^{(n-m-1)^{-1}} \\
&= \mathbf{P}^{(f_1,\ldots,f_{n-2})} \circ \pi^{(2)^{-1}} \circ \pi^{(3)^{-1}} \circ \cdots \circ \pi^{(n-m-1)^{-1}} \\
&= \cdots \\
&= \mathbf{P}^{(f_1,\ldots,f_{m+1})} \circ \pi^{(n-m-1)^{-1}} \\
&= \mathbf{P}^{(f_1,\ldots,f_m)}.
\end{aligned}
$$

Here, noting that

$$
\begin{aligned}
&\varphi^{-1}_{(f_1,\ldots,f_m)} \circ \pi_{\Lambda_1,\Lambda_2} \circ \varphi_{(f_1,\ldots,f_n)}(x_1,\ldots,x_n) \\
&= \varphi^{-1}_{(f_1,\ldots,f_m)}\Big(\pi_{\Lambda_1,\Lambda_2}\big(\varphi_{(f_1,\ldots,f_n)}(x_1,\ldots,x_n)\big)\Big) \\
&= \varphi^{-1}_{(f_1,\ldots,f_m)}\left(\Big(\big(\varphi_{(f_1,\ldots,f_n)}(x_1,\ldots,x_n)\big)_f\Big)_{f\in\Lambda_1}\right) \\
&= \Big(\big(\varphi_{(f_1,\ldots,f_n)}(x_1,\ldots,x_n)\big)_{f_1},\ldots,\big(\varphi_{(f_1,\ldots,f_n)}(x_1,\ldots,x_n)\big)_{f_m}\Big) \\
&= (x_1,\ldots,x_m) \\
&= \pi^{(n-m-1)} \circ \cdots \circ \pi^{(1)} \circ \pi^{(0)}(x_1,\ldots,x_n),
\end{aligned}
$$

we have

$$
\begin{aligned}
\mu_{\Lambda_2} \circ \pi^{-1}_{\Lambda_1,\Lambda_2} &= \mathbf{P}^{(f_1,\ldots,f_n)} \circ \varphi^{-1}_{(f_1,\ldots,f_n)} \circ \pi^{-1}_{\Lambda_1,\Lambda_2} \\
&= \mathbf{P}^{(f_1,\ldots,f_n)} \circ \big(\pi_{\Lambda_1,\Lambda_2} \circ \varphi_{(f_1,\ldots,f_n)}\big)^{-1} \\
&= \mathbf{P}^{(f_1,\ldots,f_n)} \circ \big(\varphi_{(f_1,\ldots,f_m)} \circ \pi^{(n-m-1)} \circ \cdots \circ \pi^{(1)} \circ \pi^{(0)}\big)^{-1} \\
&= \mathbf{P}^{(f_1,\ldots,f_n)} \circ \pi^{(0)^{-1}} \circ \pi^{(1)^{-1}} \circ \cdots \circ \pi^{(n-m-1)^{-1}} \circ \varphi^{-1}_{(f_1,\ldots,f_m)} \\
&= \mathbf{P}^{(f_1,\ldots,f_m)} \circ \varphi^{-1}_{(f_1,\ldots,f_m)} \\
&= \mu_{\Lambda_1}.
\end{aligned}
$$

$\underline{3^{\circ}}$ By 2°, we can apply Kolmogorov's extension theorem [cf. Claim A.5] to see the following:

$$
\begin{aligned}
&{}^{\exists 1}\mathbf{P}\text{: a probability measure on } \big(\mathbb{R}^{\mathbb{B}}, \mathcal{B}_{\mathrm{K}}(\mathbb{R}^{\mathbb{B}})\big) \\
&\text{s.t. } \mathbf{P} \circ \pi^{-1}_{\Lambda,\mathbb{B}} = \mu_\Lambda, \quad {}^{\forall}\Lambda \subset \mathbb{B} \text{ finite.}
\end{aligned}
$$

Since, for distinct $f_1,\ldots,f_n \in \mathbb{B}$ and $\Lambda = \{f_1,\ldots,f_n\} \subset \mathbb{B}$,

$$
\begin{aligned}
\varphi^{-1}_{(f_1,\ldots,f_n)} \circ \pi_{\Lambda,\mathbb{B}}\big((x_f)_{f\in\mathbb{B}}\big) &= \varphi^{-1}_{(f_1,\ldots,f_n)}\big((x_f)_{f\in\Lambda}\big) \\
&= (x_{f_1},\ldots,x_{f_n}) \\
&= \pi_{(f_1,\ldots,f_n)}\big((x_f)_{f\in\mathbb{B}}\big),
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathbf{P} \circ \pi^{-1}_{(f_1,\ldots,f_n)} &= \mathbf{P} \circ \big(\varphi^{-1}_{(f_1,\ldots,f_n)} \circ \pi_{\Lambda,\mathbb{B}}\big)^{-1} \\
&= \mathbf{P} \circ \pi^{-1}_{\Lambda,\mathbb{B}} \circ \big(\varphi^{-1}_{(f_1,\ldots,f_n)}\big)^{-1}
\end{aligned}
$$

$$\begin{aligned}
&= \mu_\Lambda \circ \left(\varphi^{-1}_{(f_1,\dots,f_n)}\right)^{-1} \\
&= \mathbf{P}^{(f_1,\dots,f_n)} \circ \varphi^{-1}_{(f_1,\dots,f_n)} \circ \left(\varphi^{-1}_{(f_1,\dots,f_n)}\right)^{-1} \\
&= \mathbf{P}^{(f_1,\dots,f_n)}. \qquad \blacksquare
\end{aligned}$$

$\left(\mathbb{R}^{\mathbb{B}}, \mathscr{B}_{\mathrm{K}}(\mathbb{R}^{\mathbb{B}}), \mathbf{P}\right)$ is our basic probability space in this monograph. Note that this probability space is of large volume, since it looks like including all real-valued almost periodic functions. Although it can be probably replaced by smaller one, we go ahead with this setting[†4].

2.2 Limit theorem on the probability space $(\mathbb{R}^{\mathbb{B}}, \mathbf{P})$

In the rest of this chapter, we view typical real random variables on this probability space, this type of which will appear in the subsequent chapters, and we introduce a limit theorem related to them.

Claim 2.1 *For $\lambda \in \mathbb{R} \setminus \{0\}$, the function $\mathbb{R} \ni t \mapsto \sqrt{2}\cos\lambda t \in \mathbb{R}$ belongs to $\mathbb{B}$. When it is denoted by $\sqrt{2}\cos\lambda\cdot$,*

$$\mathbf{P} \circ \pi^{-1}_{\sqrt{2}\cos\lambda\cdot}(dx) = \frac{1}{\pi}\mathbf{1}_{(-\sqrt{2},\sqrt{2})}(x)\frac{dx}{\sqrt{2-x^2}}.$$

Proof. Fix $\lambda \in \mathbb{R} \setminus \{0\}$. By Theorem 2.1 and Lemma 2.1,

$$\begin{aligned}
&\widehat{\mathbf{P} \circ \pi^{-1}_{\sqrt{2}\cos\lambda\cdot}}(\xi) \\
&= \widehat{\mathbf{P}^{\sqrt{2}\cos\lambda\cdot}}(\xi) \\
&= \lim_{T\to\infty} \widehat{P_T^{\sqrt{2}\cos\lambda\cdot}}(\xi) \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\xi\sqrt{2}\cos\lambda t}\,dt \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{\sqrt{-1}\xi\sqrt{2}\cos|\lambda| t}\,dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} e^{\sqrt{-1}\xi\sqrt{2}\cos|\lambda| t}\,dt \\
&\qquad \left[\because\ t \mapsto e^{\sqrt{-1}\xi\sqrt{2}\cos|\lambda| t} \text{ is even}\right] \\
&= \lim_{T\to\infty} \frac{1}{|\lambda|T}\int_{0}^{|\lambda|T} e^{\sqrt{-1}\xi\sqrt{2}\cos\tau}\,d\tau \\
&\qquad \left[\because \text{ change of variable: } \tau = |\lambda|t\right] \\
&= \lim_{n\to\infty} \frac{1}{(2n+1)\pi}\int_{0}^{(2n+1)\pi} e^{\sqrt{-1}\xi\sqrt{2}\cos\tau}\,d\tau \\
&= \lim_{n\to\infty} \left(\frac{1}{(2n+1)\pi}\int_{0}^{\pi} e^{\sqrt{-1}\xi\sqrt{2}\cos\tau}\,d\tau\right.
\end{aligned}$$

[†4] Too big is better than too small.

$$
\begin{aligned}
&\quad + \frac{1}{(2n+1)\pi} \sum_{k=1}^{n} \int_{(2k-1)\pi}^{(2k+1)\pi} e^{\sqrt{-1}\xi\sqrt{2}\cos\tau} d\tau \Bigg) \\
&= \lim_{n\to\infty} \frac{1}{(2n+1)\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} e^{\sqrt{-1}\xi\sqrt{2}\cos(\sigma+2k\pi)} d\sigma \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sqrt{-1}\xi\sqrt{2}\cos\sigma} d\sigma \\
&= \frac{1}{\pi} \int_{0}^{\pi} e^{\sqrt{-1}\xi\sqrt{2}\cos\sigma} d\sigma \\
&= \frac{1}{\pi} \int_{-\sqrt{2}}^{\sqrt{2}} e^{\sqrt{-1}\xi x} \frac{dx}{\sqrt{2-x^2}} \\
&\quad \Big[\because \text{ change of variable: } \sigma = \cos^{-1}\big(\tfrac{x}{\sqrt{2}}\big) \ \big(-\sqrt{2} \le x \le \sqrt{2}\,\big)\Big].
\end{aligned}
$$
■

Definition 2.3 For a real sequence $\{\lambda_k\}_{k=1}^{\infty}$,

$\{\lambda_k\}_{k=1}^{\infty}$ is *AI* (= *algebraically independent*)[†5]

$$\underset{\text{def}}{\Longleftrightarrow} \sum_{i=1}^{r} m_i\lambda_i \neq 0 \ \text{ for } {}^{\forall}r \in \mathbb{N} \text{ and } {}^{\forall}(m_1, \ldots, m_r) \in \mathbb{Z}^r \setminus \{(0, \ldots, 0)\},$$

$\{\lambda_k\}_{k=1}^{\infty}$ is *ASSNZ* (= *any signed sums (are) not zero*)[†6]

$$\underset{\text{def}}{\Longleftrightarrow} \sum_{i=1}^{r} \varepsilon_i\lambda_{k_i} \neq 0 \ \big({}^{\forall}(\varepsilon_1, \ldots, \varepsilon_r) \in \{-1, 1\}^r\big) \ \text{ for } {}^{\forall}r \in \mathbb{N} \text{ and } {}^{\forall}k_1 < \cdots < {}^{\forall}k_r.$$

Clearly AI $\Rightarrow$ ASSNZ.

Example 2.1 If $\{p_k\}_{k=1}^{\infty}$ is an arrangement of prime numbers in ascending order, then $\{\log p_k\}_{k=1}^{\infty}$ is AI.

Proof. For $m_1, \ldots, m_r \in \mathbb{Z} \setminus \{0\}$, put

$$
\begin{aligned}
I_+ &:= \big\{1 \le i \le r; m_i > 0\big\}, \\
I_- &:= \big\{1 \le i \le r; m_i < 0\big\}.
\end{aligned}
$$

Then $I_+ \cap I_- = \emptyset$, and $I_+ \neq \emptyset$ or $I_- \neq \emptyset$. Since, for prime numbers $q_1 < \cdots < q_r$,

$$\prod_{i\in I_+} q_i^{m_i} \neq \prod_{i\in I_-} q_i^{|m_i|}$$

by the uniqueness of prime factorization, we have

$$\prod_{i=1}^{r} q_i^{m_i} = \prod_{i\in I_+} q_i^{m_i} \prod_{i\in I_-} q_i^{m_i} = \frac{\prod_{i\in I_+} q_i^{m_i}}{\prod_{i\in I_-} q_i^{|m_i|}} \neq 1,$$

so that $\sum_{i=1}^{r} m_i \log q_i \neq 0$. This shows that $\{\log p_k\}_{k=1}^{\infty}$ is AI. ■

[†5] In other words, $\{\lambda_k\}_{k=1}^{\infty}$ is linearly independent over $\mathbb{Q}$.

[†6] In [12], $\{\lambda_k\}_{k=1}^{\infty}$ is said to satisfy the *signed sum condition* (abbr. *SS-condition*).

Lemma 2.3 *If* $\{\lambda_k\}_{k=1}^{\infty}$ *is ASSNZ, then* $\cos\lambda_k t \neq \cos\lambda_l t$ $(k \neq l)$.

Proof. Let $\cos\lambda_k t = \cos\lambda_l t$ $({}^{\forall}t \in \mathbb{R})$. Differentiating it twice, we have

$$-\lambda_k \sin\lambda_k t = -\lambda_l \sin\lambda_l t,$$
$$-\lambda_k^2 \cos\lambda_k t = -\lambda_l^2 \cos\lambda_l t.$$

Letting $t = 0$ yields that $\lambda_k^2 = \lambda_l^2$. Thus $\lambda_k + \lambda_l = 0$ or $\lambda_k - \lambda_l = 0$. When $k \neq l$, this is contrary to ASSNZ, so that $k = l$. ■

Claim 2.2 (i) *If* $\{\lambda_k\}_{k=1}^{\infty}$ *is AI, then a sequence* $\{\pi_{\sqrt{2}\cos\lambda_k\cdot}\}_{k=1}^{\infty}$ *of real random variables is independent, identically distributed (abbr. i.i.d.).*
(ii) *When* $\{\lambda_k\}_{k=1}^{\infty}$ *is ASSNZ,* $\{\pi_{\sqrt{2}\cos\lambda_k\cdot}\}_{k=1}^{\infty}$ *is an augmented multiplicative system*[†7] *(abbr. AMS). Namely, for* ${}^{\forall}r \in \mathbb{N}$ *and* ${}^{\forall}k_1 < \cdots < {}^{\forall}k_r$,

$$E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_1}\cdot}\big) \times \cdots \times \big(\pi_{\sqrt{2}\cos\lambda_{k_r}\cdot}\big)\Big] = 0,$$
$$E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_1}\cdot}\big)^2 \times \cdots \times \big(\pi_{\sqrt{2}\cos\lambda_{k_r}\cdot}\big)^2\Big] = 1.$$

In this monograph, $E^{\mathbf{P}}$ *stands for expectation w.r.t.* **P**.

Proof. First, note that $|\pi_{\sqrt{2}\cos\lambda\cdot}| \leq \sqrt{2}$ **P**-a.e. $(\lambda \neq 0)$ by Claim 2.1 and that $\lambda_k \neq 0$ $({}^{\forall}k)$, $\cos\lambda_i t \neq \cos\lambda_j t$ $(i \neq j)$ by Lemma 2.3. In the following, we divide the proof into four steps:

<u>1°</u> For $r \in \mathbb{N}$, $k_1 < \cdots < k_r$ and $n_1, \ldots, n_r \in \mathbb{N}$,

$$\begin{aligned} &E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_1}\cdot}\big)^{n_1} \times \cdots \times \big(\pi_{\sqrt{2}\cos\lambda_{k_r}\cdot}\big)^{n_r}\Big] \\ &= \Big(\frac{1}{\sqrt{2}}\Big)^{n_1+\cdots+n_r} \sum_{\substack{0\leq l_1\leq n_1,\ldots,0\leq l_r\leq n_r;\\ (2l_1-n_1)\lambda_{k_1}+\cdots+(2l_r-n_r)\lambda_{k_r}=0}} \binom{n_1}{l_1}\cdots\binom{n_r}{l_r}. \end{aligned}$$

$\because$ Since, by Theorem 2.1 and Lemma 2.1,

$$\begin{aligned} &P_T \circ \big(\sqrt{2}\cos\lambda_{k_1}\cdot, \ldots, \sqrt{2}\cos\lambda_{k_r}\cdot\big)^{-1} \\ &\to \mathbf{P} \circ \pi^{-1}_{(\sqrt{2}\cos\lambda_{k_1}\cdot,\ldots,\sqrt{2}\cos\lambda_{k_r}\cdot)} \quad \text{weakly as } T \to \infty, \end{aligned}$$

it follows that

$$\begin{aligned} &E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_1}\cdot}\big)^{n_1} \times \cdots \times \big(\pi_{\sqrt{2}\cos\lambda_{k_r}\cdot}\big)^{n_r}\Big] \\ &= \int_{\mathbb{R}^r} x_1^{n_1}\cdots x_r^{n_r}\, \mathbf{P} \circ \pi^{-1}_{(\sqrt{2}\cos\lambda_{k_1}\cdot,\ldots,\sqrt{2}\cos\lambda_{k_r}\cdot)}(dx_1\cdots dx_r) \\ &= \lim_{T\to\infty} \int_{\mathbb{R}^r} x_1^{n_1}\cdots x_r^{n_r}\, P_T \circ \big(\sqrt{2}\cos\lambda_{k_1}\cdot, \ldots, \sqrt{2}\cos\lambda_{k_r}\cdot\big)^{-1}(dx_1\cdots dx_r) \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \big(\sqrt{2}\cos\lambda_{k_1}t\big)^{n_1} \cdots \big(\sqrt{2}\cos\lambda_{k_r}t\big)^{n_r}\, dt \end{aligned}$$

[†7] In [12], this is called an *equinormed multiplicative system* (abbr. *EMS*).

$$
\begin{aligned}
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\sqrt{2}\cos\lambda_{k_1} t\big)^{n_1} \cdots \big(\sqrt{2}\cos\lambda_{k_r} t\big)^{n_r}\, dt \\
&= \Big(\frac{1}{\sqrt{2}}\Big)^{n_1+\cdots+n_r} \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum_{l_1=0}^{n_1} \binom{n_1}{l_1} e^{\sqrt{-1}(2l_1-n_1)\lambda_{k_1} t}\right) \\
&\qquad\qquad \times \cdots \times \left(\sum_{l_r=0}^{n_r} \binom{n_r}{l_r} e^{\sqrt{-1}(2l_r-n_r)\lambda_{k_r} t}\right) dt
\end{aligned}
$$

$$
\left[\because\ (\cos\theta)^n = \Big(\frac{e^{\sqrt{-1}\theta}+e^{-\sqrt{-1}\theta}}{2}\Big)^n = \frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l} e^{\sqrt{-1}l\theta} e^{-\sqrt{-1}(n-l)\theta} = \frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l} e^{\sqrt{-1}(2l-n)\theta}\right]
$$

$$
\begin{aligned}
&= \Big(\frac{1}{\sqrt{2}}\Big)^{n_1+\cdots+n_r} \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{\substack{0\le l_1\le n_1,\\ \dots,\\ 0\le l_r\le n_r}} \binom{n_1}{l_1}\cdots\binom{n_r}{l_r} \\
&\qquad\qquad \times e^{\sqrt{-1}\big((2l_1-n_1)\lambda_{k_1}+\cdots+(2l_r-n_r)\lambda_{k_r}\big)t}\, dt \\
&= \Big(\frac{1}{\sqrt{2}}\Big)^{n_1+\cdots+n_r} \sum_{\substack{0\le l_1\le n_1,\\ \dots,\\ 0\le l_r\le n_r}} \binom{n_1}{l_1}\cdots\binom{n_r}{l_r} \\
&\qquad\qquad \times \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{\sqrt{-1}\big((2l_1-n_1)\lambda_{k_1}+\cdots+(2l_r-n_r)\lambda_{k_r}\big)t}\, dt \\
&= \Big(\frac{1}{\sqrt{2}}\Big)^{n_1+\cdots+n_r} \sum_{\substack{0\le l_1\le n_1,\dots,0\le l_r\le n_r;\\ (2l_1-n_1)\lambda_{k_1}+\cdots+(2l_r-n_r)\lambda_{k_r}=0}} \binom{n_1}{l_1}\cdots\binom{n_r}{l_r}.
\end{aligned}
$$

2° For $\lambda \neq 0$,

$$
\begin{aligned}
E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda\cdot}\big)^{2n-1}\Big] &= 0, \\
E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda\cdot}\big)^{2n}\Big] &= \Big(\frac{1}{2}\Big)^n \binom{2n}{n}.
\end{aligned}
$$

$\because$ By Claim 2.1,

$$
\begin{aligned}
E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda\cdot}\big)^{2n-1}\Big] &= \int_{\mathbb{R}} x^{2n-1} \mathbf{P}\circ\pi_{\sqrt{2}\cos\lambda\cdot}^{-1}(dx) = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} \frac{x^{2n-1}}{\sqrt{2-x^2}}dx \\
&= 0 \quad \Big[\because\ x \mapsto \tfrac{x^{2n-1}}{\sqrt{2-x^2}} \text{ is odd}\Big],
\end{aligned}
$$

$$
\begin{aligned}
E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda\cdot}\big)^{2n}\Big] &= \int_{\mathbb{R}} x^{2n} \mathbf{P}\circ\pi_{\sqrt{2}\cos\lambda\cdot}^{-1}(dx) \\
&= \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} \frac{x^{2n}}{\sqrt{2-x^2}}dx \\
&= \frac{2}{\pi}\int_{0}^{\sqrt{2}} \frac{x^{2n}}{\sqrt{2-x^2}}dx \quad \Big[\because\ x \mapsto \tfrac{x^{2n}}{\sqrt{2-x^2}} \text{ is even}\Big]
\end{aligned}
$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{(\sqrt{2})^{2n}\sin^{2n}\theta}{\sqrt{2}\cos\theta}\sqrt{2}\cos\theta d\theta$$
$$\left[\because \text{ change of variable: } x = \sqrt{2}\sin\theta\right]$$
$$= \frac{2^{n+1}}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n}\theta d\theta.$$

Let $I_n := \int_0^{\frac{\pi}{2}} \sin^{2n}\theta d\theta$ for simplicity. Then

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} d\theta = \frac{\pi}{4}, \\
I_{n+1} &= \int_0^{\frac{\pi}{2}} \sin^{2n+2}\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^{2n+1}\theta(-\cos\theta)' d\theta \\
&= \Big[-\sin^{2n+1}\theta\cos\theta\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (2n+1)\sin^{2n}\theta\cos^2\theta d\theta \\
&= (2n+1)\int_0^{\frac{\pi}{2}} \sin^{2n}\theta(1-\sin^2\theta) d\theta \\
&= (2n+1)I_n - (2n+1)I_{n+1}.
\end{aligned}$$

Thus, by $(2n+2)I_{n+1} = (2n+1)I_n$ and $I_1 = \frac{\pi}{4}$,

$$\begin{aligned}
I_n = \Big(\prod_{k=1}^{n-1} \frac{I_{k+1}}{I_k}\Big) I_1 &= \frac{\pi}{4}\prod_{k=1}^{n-1}\frac{2k+1}{2k+2} \\
&= \frac{\pi}{4}\prod_{k=1}^{n-1}\frac{(2k+2)(2k+1)}{(2k+2)^2} \\
&= \frac{\pi}{4}\frac{\prod_{k=1}^{n-1}(2k+2)(2k+1)}{(\prod_{k=1}^{n-1} 2(k+1))^2} \\
&= \frac{\pi}{8}\frac{(2n)!}{(2^{n-1}n!)^2} \\
&= \frac{\pi}{2^3}\frac{1}{2^{2n-2}}\binom{2n}{n} \\
&= \frac{\pi}{2^{2n+1}}\binom{2n}{n}.
\end{aligned}$$

Therefore

$$E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda\cdot}\big)^{2n}\Big] = \frac{2^{n+1}}{\pi}\frac{\pi}{2^{2n+1}}\binom{2n}{n} = \Big(\frac{1}{2}\Big)^n\binom{2n}{n}.$$

3° Let $\{\lambda_k\}_{k=1}^{\infty}$ be AI. Note that for $k_1 < \cdots < k_r$,

$$(2l_1 - n_1)\lambda_{k_1} + \cdots + (2l_r - n_r)\lambda_{k_r} = 0 \Leftrightarrow 2l_1 - n_1 = \cdots = 2l_r - n_r = 0$$

$$\Leftrightarrow 2l_1 = n_1, \ldots, 2l_r = n_r.$$

By 1° and 2°, in case $1 \leq {}^{\exists}i \leq r$ s.t. $n_i \notin 2\mathbb{N}$,

$$\begin{aligned}
&E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_1}}\cdot\big)^{n_1} \times \cdots \times \big(\pi_{\sqrt{2}\cos\lambda_{k_r}}\cdot\big)^{n_r}\Big] \\
&= 0 \\
&= E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_1}}\cdot\big)^{n_1}\Big] \times \cdots \times E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_r}}\cdot\big)^{n_r}\Big];
\end{aligned}$$

in case $n_i \in 2\mathbb{N}$ $(1 \leq {}^{\forall}i \leq r)$,

$$\begin{aligned}
&E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_1}}\cdot\big)^{n_1} \times \cdots \times \big(\pi_{\sqrt{2}\cos\lambda_{k_r}}\cdot\big)^{n_r}\Big] \\
&= \Big(\frac{1}{\sqrt{2}}\Big)^{n_1+\cdots+n_r} \binom{n_1}{\frac{n_1}{2}} \times \cdots \times \binom{n_r}{\frac{n_r}{2}} \\
&= \Big(\frac{1}{2}\Big)^{\frac{n_1}{2}} \binom{n_1}{\frac{n_1}{2}} \times \cdots \times \Big(\frac{1}{2}\Big)^{\frac{n_r}{2}} \binom{n_r}{\frac{n_r}{2}} \\
&= E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_1}}\cdot\big)^{n_1}\Big] \times \cdots \times E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_{k_r}}\cdot\big)^{n_r}\Big].
\end{aligned}$$

Putting them together, we have that for ${}^{\forall}s \geq 1$, ${}^{\forall}m_1, \ldots, {}^{\forall}m_s \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_1}\cdot\big)^{m_1} \times \cdots \times \big(\pi_{\sqrt{2}\cos\lambda_s}\cdot\big)^{m_s}\Big] \\
&= E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_1}\cdot\big)^{m_1}\Big] \times \cdots \times E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_s}\cdot\big)^{m_s}\Big].
\end{aligned}$$

Now, for $s \geq 1$ and $\xi_1, \ldots, \xi_s \in \mathbb{R}$,

$$\begin{aligned}
&E^{\mathbf{P}}\Big[\prod_{j=1}^{s} e^{\sqrt{-1}\xi_j \pi_{\sqrt{2}\cos\lambda_j}\cdot}\Big] \\
&= E^{\mathbf{P}}\Big[\prod_{j=1}^{s}\sum_{m_j=0}^{\infty} \frac{1}{m_j!}\big(\sqrt{-1}\xi_j \pi_{\sqrt{2}\cos\lambda_j}\cdot\big)^{m_j}\Big] \\
&= E^{\mathbf{P}}\Big[\sum_{m_1,\ldots,m_s\geq 0} \frac{1}{m_1!\cdots m_s!}\big(\sqrt{-1}\xi_1\big)^{m_1}\cdots\big(\sqrt{-1}\xi_s\big)^{m_s} \\
&\qquad\qquad \times \big(\pi_{\sqrt{2}\cos\lambda_1}\cdot\big)^{m_1}\cdots\big(\pi_{\sqrt{2}\cos\lambda_s}\cdot\big)^{m_s}\Big] \\
&= \sum_{m_1,\ldots,m_s\geq 0} \frac{1}{m_1!\cdots m_s!}\big(\sqrt{-1}\xi_1\big)^{m_1}\cdots\big(\sqrt{-1}\xi_s\big)^{m_s} \\
&\qquad\qquad \times E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_1}\cdot\big)^{m_1}\cdots\big(\pi_{\sqrt{2}\cos\lambda_s}\cdot\big)^{m_s}\Big] \\
&= \sum_{m_1,\ldots,m_s\geq 0} \frac{1}{m_1!\cdots m_s!}\big(\sqrt{-1}\xi_1\big)^{m_1}\cdots\big(\sqrt{-1}\xi_s\big)^{m_s} \\
&\qquad\qquad \times E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_1}\cdot\big)^{m_1}\Big]\cdots E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_s}\cdot\big)^{m_s}\Big] \\
&= \Big(\sum_{m_1\geq 0} \frac{1}{m_1!}\big(\sqrt{-1}\xi_1\big)^{m_1} E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_1}\cdot\big)^{m_1}\Big]\Big)
\end{aligned}$$

$$\times \cdots \times \left(\sum_{m_s \geq 0} \frac{1}{m_s!} (\sqrt{-1}\xi_s)^{m_s} E^{\mathbf{P}}\left[(\pi_{\sqrt{2}\cos\lambda_s \cdot})^{m_s} \right] \right)$$
$$= E^{\mathbf{P}}\left[\sum_{m_1=0}^{\infty} \frac{1}{m_1!} (\sqrt{-1}\xi_1 \pi_{\sqrt{2}\cos\lambda_1 \cdot})^{m_1} \right]$$
$$\times \cdots \times E^{\mathbf{P}}\left[\sum_{m_s=0}^{\infty} \frac{1}{m_s!} (\sqrt{-1}\xi_s \pi_{\sqrt{2}\cos\lambda_s \cdot})^{m_s} \right]$$
$$= E^{\mathbf{P}}\left[e^{\sqrt{-1}\xi_1 \pi_{\sqrt{2}\cos\lambda_1 \cdot}} \right] \times \cdots \times E^{\mathbf{P}}\left[e^{\sqrt{-1}\xi_s \pi_{\sqrt{2}\cos\lambda_s \cdot}} \right].$$

This shows the independence of $\pi_{\sqrt{2}\cos\lambda_1 \cdot}, \ldots, \pi_{\sqrt{2}\cos\lambda_s \cdot}$. Therefore we obtain the assertion (i).

$\underline{4^{\text{o}}}$ Let $\{\lambda_k\}_{k=1}^{\infty}$ be ASSNZ. By 1^{o},

$$E^{\mathbf{P}}\left[(\pi_{\sqrt{2}\cos\lambda_{k_1} \cdot}) \times \cdots \times (\pi_{\sqrt{2}\cos\lambda_{k_r} \cdot}) \right]$$
$$= \left(\frac{1}{\sqrt{2}}\right)^r \sum_{\substack{0 \leq l_1 \leq 1, \ldots, 0 \leq l_r \leq 1; \\ (2l_1-1)\lambda_{k_1} + \cdots + (2l_r-1)\lambda_{k_r} = 0}} \binom{1}{l_1} \cdots \binom{1}{l_r},$$

$$E^{\mathbf{P}}\left[(\pi_{\sqrt{2}\cos\lambda_{k_1} \cdot})^2 \times \cdots \times (\pi_{\sqrt{2}\cos\lambda_{k_r} \cdot})^2 \right]$$
$$= \left(\frac{1}{\sqrt{2}}\right)^{2r} \sum_{\substack{0 \leq l_1 \leq 2, \ldots, 0 \leq l_r \leq 2; \\ (l_1-1)\lambda_{k_1} + \cdots + (l_r-1)\lambda_{k_r} = 0}} \binom{2}{l_1} \cdots \binom{2}{l_r}.$$

As for the former in the above, since $(2l_1 - 1)\lambda_{k_1} + \cdots + (2l_r - 1)\lambda_{k_r} \neq 0$ by $2l_i - 1 \in \{-1, 1\}$ $(1 \leq i \leq r)$,

$$E^{\mathbf{P}}\left[(\pi_{\sqrt{2}\cos\lambda_{k_1} \cdot}) \times \cdots \times (\pi_{\sqrt{2}\cos\lambda_{k_r} \cdot}) \right] = 0.$$

As for the latter in the above, since $(l_1 - 1)\lambda_{k_1} + \cdots + (l_r - 1)\lambda_{k_r} \neq 0$ provided $1 \leq {}^{\exists}i \leq r$ s.t. $l_i \neq 1$,

$$E^{\mathbf{P}}\left[(\pi_{\sqrt{2}\cos\lambda_{k_1} \cdot})^2 \times \cdots \times (\pi_{\sqrt{2}\cos\lambda_{k_r} \cdot})^2 \right] = \left(\frac{1}{2}\right)^r \underbrace{\binom{2}{1} \cdots \binom{2}{1}}_{r} = \frac{2^r}{2^r} = 1.$$

Thus we obtain the assertion (ii). ■

Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence such that $A_n := \sqrt{\sum_{k=1}^{n} a_k^2} > 0$ $({}^{\forall}n)$.

Theorem 2.2 *Let* $\{\lambda_k\}_{k=1}^{\infty}$ *be AI.*
(i) *For* ${}^{\forall}\alpha \in \mathbb{R}$ *and* ${}^{\forall}n \in \mathbb{N}$,

$$\lim_{T \to \infty} P_T\left(t; \frac{1}{A_n} \sum_{j=1}^{n} a_j \sqrt{2}\cos\lambda_j t < \alpha\right) = \mathbf{P}\left(\frac{1}{A_n} \sum_{j=1}^{n} a_j \pi_{\sqrt{2}\cos\lambda_j \cdot} < \alpha\right).$$

(ii) *If* $A_n \nearrow \infty$ *and* $a_n = o(A_n)$, *then*

$$\lim_{n\to\infty} \lim_{T\to\infty} P_T\Big(t; \frac{1}{A_n}\sum_{j=1}^n a_j\sqrt{2}\cos\lambda_j t < \alpha\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{x^2}{2}}dx, \quad {}^{\forall}\alpha \in \mathbb{R}.$$

Proof. (i) Fix $n \in \mathbb{N}$. By Claim 2.2(i) and Claim 2.1,

$$\begin{aligned}
&\mathbf{P}\circ\pi^{-1}_{(\sqrt{2}\cos\lambda_1\cdot,\ldots,\sqrt{2}\cos\lambda_n\cdot)}(dx_1\cdots dx_n)\\
&= \big(\mathbf{P}\circ\pi^{-1}_{\sqrt{2}\cos\lambda_1\cdot}\big)\times\cdots\times\big(\mathbf{P}\circ\pi^{-1}_{\sqrt{2}\cos\lambda_n\cdot}\big)(dx_1\cdots dx_n)\\
&= \Big(\frac{1}{\pi}\Big)^n \mathbf{1}_{(-\sqrt{2},\sqrt{2})\times\cdots\times(-\sqrt{2},\sqrt{2})}(x_1,\ldots,x_n)\frac{1}{\sqrt{2-x_1^2}}\cdots\frac{1}{\sqrt{2-x_n^2}}dx_1\cdots dx_n.
\end{aligned}$$

Since, for $\alpha \in \mathbb{R}$,

$$\Big\{(x_1,\ldots,x_n)\in\mathbb{R}^n; \frac{1}{A_n}\sum_{j=1}^n a_jx_j < \alpha\Big\} \text{ is open in } \mathbb{R}^n,$$
$$\Big\{(x_1,\ldots,x_n)\in\mathbb{R}^n; \frac{1}{A_n}\sum_{j=1}^n a_jx_j \le \alpha\Big\} \text{ is closed in } \mathbb{R}^n,$$

it follows that

$$\begin{aligned}
&\underline{\lim}_{T\to\infty} P_T\Big(t; \frac{1}{A_n}\sum_{j=1}^n a_j\sqrt{2}\cos\lambda_j t < \alpha\Big)\\
&= \underline{\lim}_{T\to\infty} P_T\circ\big(\sqrt{2}\cos\lambda_1\cdot,\ldots,\sqrt{2}\cos\lambda_n\cdot\big)^{-1}\Big((x_1,\ldots,x_n); \frac{1}{A_n}\sum_{j=1}^n a_jx_j < \alpha\Big)\\
&\ge \mathbf{P}\circ\pi^{-1}_{(\sqrt{2}\cos\lambda_1\cdot,\ldots,\sqrt{2}\cos\lambda_n\cdot)}\Big((x_1,\ldots,x_n); \frac{1}{A_n}\sum_{j=1}^n a_jx_j < \alpha\Big)\\
&= \Big(\frac{1}{\pi}\Big)^n \int_{-\sqrt{2}}^{\sqrt{2}}\cdots\int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-x_1^2}}\cdots\frac{1}{\sqrt{2-x_n^2}}\mathbf{1}_{\frac{1}{A_n}\sum_{j=1}^n a_jx_j<\alpha}dx_1\cdots dx_n,\\
&\overline{\lim}_{T\to\infty} P_T\Big(t; \frac{1}{A_n}\sum_{j=1}^n a_j\sqrt{2}\cos\lambda_j t \le \alpha\Big)\\
&= \overline{\lim}_{T\to\infty} P_T\circ\big(\sqrt{2}\cos\lambda_1\cdot,\ldots,\sqrt{2}\cos\lambda_n\cdot\big)^{-1}\Big((x_1,\ldots,x_n); \frac{1}{A_n}\sum_{j=1}^n a_jx_j \le \alpha\Big)\\
&\le \mathbf{P}\circ\pi^{-1}_{(\sqrt{2}\cos\lambda_1\cdot,\ldots,\sqrt{2}\cos\lambda_n\cdot)}\Big((x_1,\ldots,x_n); \frac{1}{A_n}\sum_{j=1}^n a_jx_j \le \alpha\Big)\\
&= \Big(\frac{1}{\pi}\Big)^n \int_{-\sqrt{2}}^{\sqrt{2}}\cdots\int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{\sqrt{2-x_1^2}}\cdots\frac{1}{\sqrt{2-x_n^2}}\mathbf{1}_{\frac{1}{A_n}\sum_{j=1}^n a_jx_j\le\alpha}dx_1\cdots dx_n.
\end{aligned}$$

We here note that $\left\{(x_1,\dots,x_n)\in\mathbb{R}^n;\ \frac{1}{A_n}\sum_{j=1}^n a_jx_j=\alpha\right\}$ is a hyperplane of $\mathbb{R}^n$, and so its Lebesgue measure is zero. Thus, since

$$\int_{-\sqrt{2}}^{\sqrt{2}}\cdots\int_{-\sqrt{2}}^{\sqrt{2}}\frac{1}{\sqrt{2-x_1^2}}\cdots\frac{1}{\sqrt{2-x_n^2}}\mathbf{1}_{\frac{1}{A_n}\sum_{j=1}^n a_jx_j=\alpha}\,dx_1\cdots dx_n=0,$$

we have

$$\begin{aligned}
&\lim_{T\to\infty}P_T\Big(t;\frac{1}{A_n}\sum_{j=1}^n a_j\sqrt{2}\cos\lambda_j t<\alpha\Big)\\
&=\Big(\frac{1}{\pi}\Big)^n\int_{-\sqrt{2}}^{\sqrt{2}}\cdots\int_{-\sqrt{2}}^{\sqrt{2}}\frac{1}{\sqrt{2-x_1^2}}\cdots\frac{1}{\sqrt{2-x_n^2}}\mathbf{1}_{\frac{1}{A_n}\sum_{j=1}^n a_jx_j<\alpha}\,dx_1\cdots dx_n\\
&=\mathbf{P}\Big(\frac{1}{A_n}\sum_{j=1}^n a_j\pi_{\sqrt{2}\cos\lambda_j\cdot}<\alpha\Big).
\end{aligned}$$

(ii) Suppose $A_n\nearrow\infty$ and $a_n=o(A_n)$. First

$$\frac{1}{A_n}\max_{1\le k\le n}|a_k|\to 0\quad\text{as } n\to\infty. \tag{2.1}$$

Because

$$\begin{aligned}
\frac{1}{A_n}\max_{1\le k\le n}|a_k| &= \frac{1}{A_n}\big(\max_{1\le k\le n_0}|a_k|\big)\vee\big(\max_{n_0\le k\le n}|a_k|\big)\\
&=\Big(\frac{1}{A_n}\max_{1\le k\le n_0}|a_k|\Big)\vee\Big(\max_{n_0\le k\le n}\frac{|a_k|}{A_k}\frac{A_k}{A_n}\Big)\\
&\le\Big(\frac{1}{A_n}\max_{1\le k\le n_0}|a_k|\Big)\vee\Big(\sup_{k\ge n_0}\frac{|a_k|}{A_k}\Big)\underset{\substack{\text{first } n\to\infty\\ \text{second } n_0\to\infty}}{\to} 0.
\end{aligned}$$

Let $\{X_{nj}\}_{1\le j\le n<\infty}$ be a triangular array of real random variables defined by

$$X_{nj}:=\frac{1}{A_n}a_j\pi_{\sqrt{2}\cos\lambda_j\cdot}.$$

Then, for each n,

- $\{X_{nj}\}_{1\le j\le n}$ are independent, • $|X_{nj}|\le\dfrac{|a_j|}{A_n}\sqrt{2}$ **P**-a.e.,
- $E^{\mathbf{P}}\big[X_{nj}\big]=0$.

And

$$\begin{aligned}
\sum_{j=1}^n E^{\mathbf{P}}\big[X_{nj}^2\big] &= \sum_{j=1}^n E^{\mathbf{P}}\Big[\frac{a_j^2}{A_n^2}\big(\pi_{\sqrt{2}\cos\lambda_j\cdot}\big)^2\Big]\\
&=\frac{1}{A_n^2}\sum_{j=1}^n a_j^2E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_j\cdot}\big)^2\Big]\\
&=\frac{1}{A_n^2}\sum_{j=1}^n a_j^2\quad\big[\because\ 2^{\text{o}}\text{ in the proof of Claim 2.2}\big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A_n^2}{A_n^2} = 1, \\
\sum_{j=1}^{n} E^{\mathbf{P}}\big[X_{nj}^2; |X_{nj}| \geq \varepsilon\big] &\leq \sum_{j=1}^{n} E^{\mathbf{P}}\Big[\frac{X_{nj}^4}{\varepsilon^2}\Big] \\
&= \frac{1}{\varepsilon^2}\frac{1}{A_n^4}\sum_{j=1}^{n} a_j^4 E^{\mathbf{P}}\Big[\big(\pi_{\sqrt{2}\cos\lambda_j\cdot}\big)^4\Big] \\
&= \frac{3}{2}\frac{1}{\varepsilon^2}\frac{1}{A_n^4}\sum_{j=1}^{n} a_j^4 \quad \big[\because 2^{\circ} \text{ in the proof of Claim 2.2}\big] \\
&\leq \frac{3}{2}\frac{1}{\varepsilon^2}\frac{1}{A_n^4}\Big(\sum_{j=1}^{n} a_j^2\Big)\big(\max_{1\leq k\leq n} a_k^2\big) \\
&= \frac{3}{2}\frac{1}{\varepsilon^2}\Big(\frac{1}{A_n}\max_{1\leq k\leq n}|a_k|\Big)^2 \\
&\to 0 \quad \text{as } n\to\infty \quad \big[\because (2.1)\big].
\end{aligned}
$$

Therefore, by Lindeberg's central limit theorem [cf. Claim A.7],

$$
\begin{aligned}
&\text{the distribution of } \sum_{j=1}^{n} X_{nj} \\
&\quad \to \text{the standard normal distribution weakly as } n\to\infty.
\end{aligned}
$$

Namely

$$
\lim_{n\to\infty} \mathbf{P}\Big(\frac{1}{A_n}\sum_{j=1}^{n} a_j \pi_{\sqrt{2}\cos\lambda_j\cdot} < \alpha\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{x^2}{2}}\,dx,
$$

which is the assertion (ii). ■

Remark 2.1 Historically, Theorem 2.2(ii) in the case where $a_n = 1$ $({}^{\forall}n)$ is called Kac-Steinhaus's central limit theorem. Theorem 2.2(ii) is a generalization of this central limit theorem.

Remark 2.2 When $\{\lambda_k\}_{k=1}^{\infty}$ is only ASSNZ, Theorem 2.2(ii) is valid with the following change: *If $A_n \nearrow \infty$ and $a_n = o(A_n)$, then*

$$
\begin{aligned}
&\lim_{n\to\infty} \underline{\lim}_{T\to\infty} P_T\Big(t; \frac{1}{A_n}\sum_{j=1}^{n} a_j\sqrt{2}\cos\lambda_j t < \alpha\Big) \\
&= \lim_{n\to\infty} \overline{\lim}_{T\to\infty} P_T\Big(t; \frac{1}{A_n}\sum_{j=1}^{n} a_j\sqrt{2}\cos\lambda_j t < \alpha\Big) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-\frac{x^2}{2}}\,dx, \quad {}^{\forall}\alpha \in \mathbb{R}.
\end{aligned}
$$

For the proof, cf. Fukuyama [11, 12].