Chapter 10

Appendix

In this chapter we gather some basic facts on symplectic basis and symplectic coordinates, see for example [36], [19].

10.1 Symplectic vector space

Definition 10.1.1 *Let S be a finite dimensional vector space over $\mathbb{R}$ $(\mathbb{C})$ and let σ be a non degenerate anti-symmetric bilinear form on S. Then we call S a (finite dimensional) real (complex) symplectic vector space. Let S_i $(i = 1, 2)$ be two symplectic vector spaces with symplectic forms σ_i. If a linear bijection*

$$T : S_1 \to S_2$$

verifies $T^*\sigma_2 = \sigma_1$ then T is called a symplectic isomorphism.

Remark: σ is said to be non degenerate if

$$\sigma(\gamma, \gamma') = 0, \ \forall \gamma' \in S \Longrightarrow \gamma = 0.$$

$T^*\mathbb{R}^n = \{(x, \xi) \mid x, \xi \in \mathbb{R}^n\}$ is a symplectic vector space with

$$\sigma((x, \xi), (y, \eta)) = \langle \xi, y\rangle - \langle x, \eta\rangle.$$

Proposition 10.1.1 *Let S be a finite dimensional real symplectic vector space. Then the dimension of S is even and there is a symplectic isomorphism*

$$T : S \to T^*\mathbb{R}^n$$

with some n.

Proof: Let e_j, f_j be the unit vector along x_j, ξ_j axis in $T^*\mathbb{R}^n$ respectively. It is clear that

$$\sigma(e_j, e_k) = \sigma(f_j, f_k) = 0, \ \ \sigma(f_j, e_k) = \delta_{jk} \tag{10.1.1}$$

where δ_{jk} is the Kronecker's delta. To prove this proposition it is enough to show that there exists a basis of S verifying (10.1.1). Take $f_1 \in S$, $f_1 \neq 0$. Since σ is non degenerate one can take $e_1 \in S$ so that $\sigma(f_1, e_1) = 1$. Note that f_1 and e_1 are linearly independent. Let $S_0 = \mathrm{span}\{f_1, e_1\}$ and

$$S_1 = S_0^\sigma = \{\gamma \in S \mid \sigma(\gamma, S_0) = 0\}.$$

Then we have $S = S_1 \oplus S_0$ for if $\gamma \in S_1 \cap S_0$ then writing $\gamma = af_1 + be_1$ one gets

$$\sigma(\gamma, f_1) = -b = 0, \ \sigma(\gamma, e_1) = a = 0$$

and hence $\gamma = 0$. We now show that S_1 is a symplectic vector space with the symplectic form σ. It is enough to check that σ is non degenerate on S_1. Suppose $\sigma(\gamma, S_1) = 0$, $\gamma \in S_1$. By definition we see $\sigma(\gamma, S_0) = 0$ hence $\sigma(\gamma, S) = 0$ which gives $\gamma = 0$. The rest of the proof is carried out by induction. □

Definition 10.1.2 *Let S be a symplectic vector space of dimension $2n$ with the symplectic form σ. A basis $\{f_j, e_j\}_{j=1}^n$ verifying* (10.1.1) *is called a symplectic basis.*

Proposition 10.1.2 *Let S be a symplectic vector space of dimension $2n$ with the symplectic form σ. Let A, B be subsets of $J = \{1, 2, ..., n\}$. Assume that $\{e_j\}_{j\in A}$, $\{f_k\}_{k\in B}$ are linearly independent and verify* (10.1.1). *Then one can choose $\{e_j\}_{j\in J\setminus A}$, $\{f_k\}_{k\in J\setminus B}$ so that $\{e_j\}_{j\in J}$ and $\{f_k\}_{k\in J}$ become a full symplectic basis.*

Proof: Assume $B \setminus A \neq \emptyset$. Take $l \in B \setminus A$. Then there exists $g \in S$ such that $\sigma(g, f_l) = -1$. With $V = \mathrm{span}\{e_j, f_k \mid j \in A, k \in B\}$ we have $g \notin V$ because $\sigma(V, f_l) = 0$ by assumption. Choosing α_i, β_i, $i \in A \cap B$ suitably one can assume that

$$e_l = g - \sum_{i\in A\cap B} \alpha_i e_i - \sum_{i\in A\cap B} \beta_i f_i$$

verifies

$$\sigma(e_l, e_j) = 0, \ j \in A, \ \ \sigma(e_l, f_k) = -\delta_{lk}, \ k \in B.$$

Repeating this argument we may assume that $B \subset A$. Applying the same arguments to $A \setminus B$ we may assume $A = B$. If $A = B \neq J$ then with

$$S_0 = \mathrm{span}\{e_j, f_k \mid j \in A, k \in B\}$$

we consider $S_1 = S_0^\sigma$. Since S_1 is a symplectic vector space, then by Proposition 10.1.1 there is a symplectic basis for S_1 and hence it is enough to add this basis to $\{e_j, f_j\}_{j\in A=B}$. □

10.2 Darboux theorem

Let us start with

Definition 10.2.1 *Let S be a C^∞ manifold with a C^∞ closed non degenerate 2 form. We call such a manifold as a symplectic manifold. Let S_i be two symplectic manifolds with symplectic forms σ_i. Let χ be a diffeomorphism*

$$\chi : S_1 \to S_2$$

such that $\chi^*\sigma_2 = \sigma_1$. Then χ is called symplectomorphism or canonical transformation.

Note that the tangent space $T_\gamma S$ becomes a symplectic vector space by the symplectic form and hence even dimensional. Let $f \in C^k(S)$ $(k \geq 1)$. Then df is a linear form on $T_\gamma(S)$ and then

$$\langle t, df \rangle = \sigma(t, H_f)$$

defines $H_f(\gamma) \in T_\gamma(S)$. It is clear that H_f is a C^{k-1} vector field on S. Let f, $g \in C^k(S)$. Then we define the Poisson bracket $\{f, g\}$ by the formula

$$\{df(\gamma), dg(\gamma)\} = \sigma(H_f, H_g) = H_f \cdot g = \{f, g\}.$$

Here we recall the Jacobi's identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \quad f, g, h \in C^2(S). \tag{10.2.1}$$

Then we have

$$H_{\{f,g\}} = [H_f, H_g] = H_f H_g - H_g H_f.$$

Note that χ is a canonical transformation if and only if

$$\chi^*\{f, g\} = \{\chi^* f, \chi^* g\}, \quad f, g \in C^2(S). \tag{10.2.2}$$

Therefore to define a local canonical transformation $\chi : S \to T^*(\mathbb{R}^n)$ it is enough to choose local coordinates (x, ξ) verifying

$$\{x_j, x_k\} = 0, \;\; \{\xi_j, \xi_k\} = 0, \;\; \{\xi_j, x_k\} = -\{x_k, \xi_j\} = \delta_{jk}. \tag{10.2.3}$$

The next theorem is called Darboux theorem (not homogeneous).

Theorem 10.2.1 *Let S be a symplectic manifold of dimension $2n$ and let A, B be two subsets of $\{1, 2, ..., n\}$. Let U be a neighborhood of γ_0 and assume that f_α, $g_\beta \in C^\infty(U)$, $\alpha \in A$, $\beta \in B$ verify the followings*

$$\begin{aligned} df_\alpha(\gamma_0), dg_\beta(\gamma_0) \;\; (\alpha \in A, \beta \in B) \quad & \textit{are linearly independent,} \\ \{f_\alpha, f_{\alpha'}\} = \{g_\beta, g_{\beta'}\} = 0, \;\; & \{f_\alpha, g_\beta\} = \delta_{\alpha\beta} \\ \textit{in some neighborhood of } \gamma_0 \; & (\alpha, \alpha' \in A, \beta, \beta' \in B). \end{aligned}$$

Then there exists C^∞ functions f_α, $\alpha \notin A$, g_β, $\beta \notin B$ defined near γ_0 such that $\{f_\alpha\}$, $\{g_\beta\}$ satisfy (10.2.3).

To prove this we first show a theorem of Frobenius.

Theorem 10.2.2 *Let $v_1,\ldots,v_r$ be C^∞ vector fields defined near the origin of $\mathbb{R}^n$ which verify*

$$\begin{aligned} &v_1(0),\ldots,v_r(0) \quad \text{are linearly independent,} \\ &[v_i,v_j] = \sum_{k=1}^{r} c_{ijk}v_k \quad \text{(near the origin).} \end{aligned} \tag{10.2.4}$$

Let S be a C^∞ manifold with $0 \in S$ such that $T_0S + \mathrm{span}\{v_1(0),\ldots,v_r(0)\} = \mathbb{R}^n$ and let $f_1,\ldots,f_r \in C^\infty$ near the origin. Then the system of equations

$$\begin{cases} v_j u = f_j, \quad j = 1,\ldots,r, \\ u = u_0 \quad \text{on } S \end{cases}$$

has a C^∞ solution near the origin if and only if

$$v_i f_j - v_j f_i = \sum_{k=1}^{r} C_{ijk} f_k \quad i,j = 1,\ldots,r. \tag{10.2.5}$$

The solution u is unique.

Proof: Since $v_i u = f_i$ gives

$$v_i v_j u - v_j v_i u = v_i f_j - v_j f_i = \sum_{k=1}^{r} c_{ijk} v_k u = \sum_{k=1}^{r} c_{ijk} f_k$$

and hence the necessity of (10.2.5) is clear. Denoting the equation as $vu = f$ it is clear that for a non singular matrix $A = (a_{ij})$ the equation $vu = f$ has a solution u if and only if the equation $Avu = Af$ has a solution u. We note that $V_i = \sum_{j=1}^{r} a_{ij} v_j$, $i = 1,\ldots,r$ satisfy (10.2.4) because

$$\begin{aligned} [V_i,V_j] = [\sum a_{ik}v_k, \sum a_{jl}v_l] &= \sum a_{ik}a_{jl}[v_k,v_l] \\ &+ \sum a_{ik}(v_k a_{jl})v_l - \sum a_{jl}(v_l a_{ik})v_k \end{aligned}$$

and $v_i = \sum \tilde{a}_{ij} V_j$. Thus far the statement is invariant under such transformations and change of coordinates since the condition (10.2.4) is coordinates free. We show that our problem is reduced to

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{r} b_{ij} v_j, \quad i = 1,\ldots,r$$

where (b_{ij}) is a non singular matrix.

We now proceed by induction on n. We may assume that $v_1 = \partial/\partial x_1$ taking a suitable coordinates. Subtracting a smooth function times v_1 we may assume that v_i, $i = 2,\ldots,r$ contains no $\partial/\partial x_1$

$$v_i = \sum_{j=2}^{n} b_{ij} \frac{\partial}{\partial x_j}, \quad i = 2,\ldots,r.$$

Renumbering the coordinates $x_2, ..., x_n$ and considering $\sum_{j=2}^r a_{ij}v_j$ with a suitable non singular matrix $(a_{ij})_{2\le i,j\le r}$ we may assume that

$$v_i = \frac{\partial}{\partial x_i} + \sum_{j=r+1}^{n} b_{ij}\frac{\partial}{\partial x_j}, \quad i = 2, ..., r$$

leaving v_1 unchanged. Since

$$v_1v_i - v_iv_1 = \sum_{j=r+1}^{n} \frac{\partial b_{ij}}{\partial x_1}\frac{\partial}{\partial x_j} = \sum_{k=1}^{r} c_{1ik}v_k$$

it is clear that $c_{1ik} = 0$ and hence $\partial b_{ij}/\partial x_1 = 0$. Then b_{ij} are independent of x_1 and hence from the hypothesis of the induction we may assume that $v_i = \partial/\partial x_i$, $i = 2, ..., r$. Hence the result. We now study

$$\frac{\partial}{\partial x_j}u = f_j, \quad j = 1, ..., r.$$

With $x' = (x_1, ..., x_r)$, $x'' = (x_{r+1}, ..., x_n)$ we see that S is given by $x' = h(x'')$. Now the solution is given by

$$u(x) = \int_{h(x'')}^{x'} \sum_{j=1}^{r} f_j dx_j + u_0(h(x''), x'').$$

Since

$$d(\sum_{j=1}^{r} f_j dx_j) = 0$$

then the integral is well defined. $\square$

Proof of Theorem 10.2.1: Let $j \in A \setminus B$. We look for g_j which verifies

$$\{f_\alpha, g_j\} = H_{f_\alpha}g_j = \delta_{j\alpha}, \quad \alpha \in A, \tag{10.2.6}$$

$$\{g_\beta, g_j\} = H_{g_\beta}g_j = 0, \quad \beta \in B. \tag{10.2.7}$$

By assumption H_{f_α} $(\alpha \in A)$ and H_{g_β} $(\beta \in B)$ are linearly independent and in view of

$$[H_f, H_g] = H_{\{f,g\}}$$

then $\{H_{f_\alpha} \mid \alpha \in A\}$ and $\{H_{g_\beta} \mid \beta \in B\}$ verify the assumption (10.2.4) in Theorem 10.2.2. Then giving g_j on a manifold $C \ni \gamma_0$ of dimension $2n - |A| - |B|$ such that

$$T_{\gamma_0}C + \mathrm{span}\{H_{f_\alpha}(\gamma_0), H_{g_\beta}(\gamma_0) \mid \alpha \in A, \beta \in B\} = \mathbb{R}^{2n} = T_{\gamma_0}S$$

we obtain g_j satisfying (10.2.7) which is determined uniquely by Theorem 10.2.2. We examine that $\{dg_\beta\}_{\beta\in B\cup\{j\}}$ and $\{df_\alpha\}_{\alpha\in A}$ are linearly independent. In fact if we have

$$bdg_j(\gamma_0) + \sum_{\beta\in B} b_\beta dg_\beta(\gamma_0) + \sum_{\alpha\in A} a_\alpha df_\alpha(\gamma_0) = 0$$

then applying H_{f_j} we have $b(\gamma_0) = 0$ and hence $b_\beta = 0$ and $a_\alpha = 0$ which proves the assertion. Therefore we can repeat the same arguments until we arrive at $A \subset B$. For $j \in B \setminus A$, the same argument gives f_j and finally we may assume that $A = B$. Assume that $A = B \neq \{1, 2, ..., n\}$. Take $j \in \{1, ..., n\} \setminus B$. We want to construct g_j satisfying (10.2.7). Take the manifold C of dimension $2n - 2|B|$ given by

$$C = \{f_\alpha = g_\beta = 0,\ \forall\alpha \in A,\ \forall\beta \in B\}.$$

Note that H_{f_α}, H_{g_β} do not belong to $T_{\gamma_0}C$ and this shows that

$$T_{\gamma_0}C + \mathrm{span}\{H_{f_\alpha}, H_{g_\beta} \mid \alpha \in A, \beta \in B\} = \mathbb{R}^{2n}.$$

Then by the Frobenius theorem one can construct g_j verifying (10.2.7) giving g_j on C. If we choose g_j on C so that

$$d(g_j|_C)(\gamma_0) \neq 0$$

then it is clear that $\{dg_\beta\}_{\beta\in B\cup\{j\}}$ and $\{df_\beta\}_{\alpha\in A}$ are linearly independent. The rest of the proof is clear. □

10.3 Homogeneous Darboux theorem

Let us consider T^*X where X is a smooth manifold. With $M_t(x, \xi) = (x, t\xi)$ we define the radial vector field as

$$\rho f = \frac{d}{dt} M_t^* f|_{t=1},\quad \rho = \sum \xi_j \frac{\partial}{\partial \xi_j}.$$

We can define ρ in a coordinates free way

$$\sigma(\rho, t) = \omega(t), \quad t \in T(T^*X)$$

where $\omega = \xi dx$ is the canonical 1–form. Note that

$$\sum \xi_j \frac{\partial}{\partial \xi_j} = r\frac{\partial}{\partial r}$$

where $r^2 = \sum \xi_j^2$. Indeed with $\xi_j = \omega_j r$ we have

$$\frac{\partial}{\partial \xi_j} = \frac{\partial r}{\partial \xi_j}\frac{\partial}{\partial r} + \sum \frac{\partial \omega_k}{\partial \xi_j}\frac{\partial}{\partial \omega_k} = \omega_j \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \omega_j}.$$

Here $\sum \omega_j^2 = 1$ implies $\sum \omega_j \frac{\partial}{\partial \omega_j} = 0$.

Theorem 10.3.1 *Let X be a smooth N dimensional manifold and let*

$$A \subset \{1, ..., N-1\}, \quad B \subset \{1, ..., N\}$$

be two subsets. Let f_α, $\alpha \in A$ and g_β, $\beta \in B$ be C^∞ functions defined in a conic neighborhood of $\gamma_0 \in T^*X \setminus 0$ satisfying

$$f_\alpha,\ g_\beta \ \text{ are homogeneous of degree } 0 \text{ and } 1 \text{ respectively}, \tag{10.3.1}$$

$$f_\alpha(\gamma_0) = g_\beta(\gamma_0) = 0, \quad \forall \alpha \in A,\ \forall \beta \in B \setminus \{N\},\ g_N(\gamma_0) \neq 0, \tag{10.3.2}$$

$$df_\alpha(\gamma_0),\ dg_\beta(\gamma_0),\ \xi dx \quad \text{are linearly independent at } \gamma_0 \tag{10.3.3}$$

and the commutation relations

$$\{f_\alpha, f_{\alpha'}\} = \{g_\beta, g_{\beta'}\} = 0,\ \ \{f_\alpha, g_\beta\} = \delta_{\alpha\beta},\ \ \alpha, \alpha' \in A,\ \ \beta, \beta' \in B.$$

Then we can find f_α, $\alpha \notin A$ with $f_\alpha(\gamma_0) = 0$ and g_β, $\beta \notin B$ with $g_\beta(\gamma_0) = 0$ if $\beta \neq N$ and $g_N(\gamma_0) \neq 0$ so that $\{f_\alpha\}$ and $\{g_\beta\}$ will be a full homogeneous canonical coordinates.

Proof: We first make some comments on the necessity. Assume that we have a full homogeneous coordinates $\{f_\alpha, g_\beta\}$. Then one can express

$$\rho = \sum_\alpha r_\alpha H_{f_\alpha} + \sum_\beta s_\beta H_{g_\beta}.$$

Since $\rho f_\alpha = 0$ because of the homogeneity and we conclude that $\rho = \sum_\alpha r_\alpha H_{f_\alpha}$ and then

$$r_\alpha(\gamma_0) = \rho g_\alpha(\gamma_0) = g_\alpha(\gamma_0).$$

This implies $g_\alpha(\gamma_0) \neq 0$ with some $1 \leq \alpha \leq N$. If $g_\alpha(\gamma_0) = 0$, $\alpha = 1, ..., N-1$ and $\{f_\alpha, g_\beta\}$ is a full homogeneous coordinates then necessarily ρ, $\{H_{g_\alpha}\}_{\alpha\in\{1,\ldots,N\}}$ and $\{H_{f_\alpha}\}_{\alpha\in\{1,\ldots,N-1\}}$ are linearly independent.

Let $j \in A \setminus B$ ($1 \leq j \leq N-1$). We construct g_j which is homogeneous of degree 1 and verifies

$$H_{f_\alpha} g_j = \delta_{j\alpha}, \quad H_{g_\beta} g_j = 0.$$

Recall that H_{f_α}, H_{g_β} and ρ are linearly independent. Take a submanifold C of dimension $2N - |A| - |B|$ such that

$$T_{\gamma_0} C \ni \rho \tag{10.3.4}$$

and

$$T_{\gamma_0} C + \operatorname{span}\{H_{f_\alpha}, H_{g_\beta} \mid \alpha \in A, \beta \in B\} = \mathbb{R}^{2N}. \tag{10.3.5}$$

By the Frobenius theorem g_j is uniquely determined. We now examine that $\rho g_j = g_j$. Since

$$\rho H_{f_\alpha} g_j = 0 = H_{f_\alpha} \rho g_j + [\rho, H_{f_\alpha}] g_j = H_{f_\alpha} \rho g_j - H_{f_\alpha} g_j$$

it follows that

$$H_{f_\alpha}(\rho g_j) = \delta_{j\alpha}.$$

Similarly from

$$\rho H_{g_\beta} g_j = 0 = H_{g_\beta}(\rho g_j) + [\rho, H_{g_\beta}] g_j = H_{g_\beta}(\rho g_j)$$

we have $H_{g_\beta}(\rho g_j) = 0$. On the other hand since $\rho g_j|_C = \rho(g_j|_C) = g_j|_C$ the uniqueness part of the Frobenius theorem gives that $\rho g_j = g_j$. It is easy to see that $\{H_{g_\beta}\}_{\beta \in B \cup \{j\}}$ and $\{H_{f_\alpha}\}_{\alpha \in A}$ are linearly independent. In fact assume that

$$\rho = \sum_{\alpha \in A} r_\alpha H_{f_\alpha} + \sum_{\beta \in B \cup \{j\}} s_\beta H_{g_\beta}.$$

Applying this to f_j we get $\rho f_j = -s_j(\gamma_0) = f_j(\gamma_0) = 0$. Thus the assertion follows from the assumption. We continue this arguments to arrive at the case $A \subset (B \setminus \{N\})$.

Let $j \in B \setminus A$. We want to construct f_j which is homogeneous of degree 0 and satisfies

$$H_{f_\alpha} f_j = 0, \quad H_{g_\beta} f_j = -\delta_{j\beta}, \;\; \alpha \in A, \;\; \beta \in B.$$

Take a submanifold C of dimension $2N - |A| - |B|$ such that (10.3.4) and (10.3.5) are verified. We apply the Frobenius theorem imposing $f_j|_C = 0$. Then it is easy to check that

$$H_{f_\alpha}(\rho f_j) = 0, \quad H_{g_\beta}(\rho f_j) = 0$$

and from the uniqueness one has $\rho f_j = 0$, that is f_j is homogeneous of degree 0. It is clear that $\{H_{f_\alpha}\}_{\alpha \in A \cup \{j\}}$ and $\{H_{g_\beta}\}_{\beta \in B}$ are linearly independent. Repeating these arguments we may assume that

$$A = B \quad \text{or} \quad B = A \cup \{N\}.$$

Assume $A = B \neq \{1, ..., N-1\}$. Then taking $j \notin B$ we construct g_j and f_j. Let C be defined by

$$C = \{f_\alpha = 0, \; g_\beta = 0, \;\; \alpha \in A, \;\; \beta \in B\}.$$

It is clear that $\rho \in T_{\gamma_0} C$ because f_α and g_β are homogeneous. We see that

$$T_{\gamma_0}(T^*X \setminus 0) = T_{\gamma_0} C + \mathrm{span}\{H_{f_\alpha}, H_{g_\beta} \mid \alpha \in A, \beta \in B\}.$$

From the Frobenius theorem one can solve g_j

$$H_{f_\alpha} g_j = 0, \;\; H_{g_\beta} g_j = 0, \;\; \alpha \in A, \;\; \beta \in B$$

with $g_j(\gamma_0) = 0$ where g_j is chosen so that $d(g_j|_C)(\gamma_0)$ is not proportional to ρ, this is clearly possible because $\dim T_{\gamma_0} C \geq 2$. Thus we may assume that

$$A = \{1, ..., N-1\}, \quad B = \{1, ..., N-1\} \;\text{ or }\; B = \{1, ..., N\}.$$

Assume $B = \{1, ..., N-1\}$ and solve g_N

$$H_{f_\alpha} g_N = 0, \quad H_{g_\beta} g_N = 0, \quad \alpha = 1, ..., N-1, \; \beta = 1, ..., N-1$$

with $g_N(\gamma_0) \neq 0$. By the Frobenius theorem one can construct g_N. Suppose that

$$\rho = \sum_{\alpha=1}^{N-1} r_\alpha H_{f_\alpha} + \sum_{\beta=1}^{N} s_\beta H_{g_\beta}.$$

Then we would have $\rho = s_N H_{g_N}$ at γ_0 and hence $\rho g_N = g_N(\gamma_0) = 0$. This is a contradiction. Thus ρ, $\{H_{f_\alpha}\}$ and $\{H_{g_\beta}\}$ are linearly independent. Finally we construct f_N. Solve f_N as a solution to

$$\begin{cases} H_{f_k} f_N = 0, \quad H_{g_k} f_N = 0, \quad k = 1, ..., N-1, \\ H_{g_N} f_N = -1, \quad \rho f_N = 0. \end{cases}$$

Since $[H_{f_k}, \rho] = H_{f_k}$, $[H_{g_k}, \rho] = 0$ then the hypothesis of Theorem 10.2.2 is verified and hence f_N with $f_N(\gamma_0) = 0$ exists. $\square$