Chapter 8

Optimality of the Gevrey index

8.1 Non solvability in C^∞ and the Gevrey class

In this chapter we study the following model operator

$$P_{mod}(x,D) = -D_0^2 + 2x_1 D_0 D_n + D_1^2 + x_1^3 D_n^2. \tag{8.1.1}$$

It is worthwhile to note that if we make the change of coordinates

$$y_j = x_j\ (0 \le j \le n-1),\ \ y_n = x_n + x_0 x_1$$

which preserves the initial plane $x_0 = const.$, the operator P_{mod} is written in these coordinates as

$$P_{mod} = -D_0^2 + (D_1 + x_0 D_n)^2 + (x_1\sqrt{1+x_1}D_n)^2 = -D_0^2 + A^2 + B^2.$$

Here we have $A^* = A$ and $B^* = B$ while $[D_0, A] \neq 0$ and $[A,B] \neq 0$.

Let us denote by $p(x,\xi)$ the symbol of $P_{mod}(x,D)$ then it is clear that the double characteristic manifold near the double characteristic point $\bar{\rho} = (0,(0,...,0,1)) \in \mathbb{R}^{2(n+1)}$ is given by

$$\Sigma = \{(x,\xi) \in \mathbb{R}^{2(n+1)} \mid \xi_0 = 0, x_1 = 0, \xi_1 = 0\}$$

and the localization of p at $\rho \in \Sigma$ is given by $p_\rho(x,\xi) = -\xi_0^2 + 2x_1\xi_0 + \xi_1^2$. This is just (2) in Theorem 2.3.1 with $k = l = 1$ where ξ_1 and x_1 is exchanged. Since $(x_1, \xi_1) \mapsto (\xi_1, -x_1)$ is a symplectic change of the coordinates system then we see

$$\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) \neq \{0\},\ \ \rho \in \Sigma.$$

The main feature of p is that the Hamilton flow H_p lands tangentially on Σ. Indeed the integral curve of H_p

$$\xi_1 = -\frac{x_0^2}{4},\ x_n = \frac{x_0^5}{8},\ \xi_0 = 0,\ \xi_1 = \frac{x_0^3}{8},\ x_j, \xi_j = \text{constants},\ |x_0| > 0$$

parametrized by x_0 lands on Σ tangentially as $\pm x_0 \downarrow 0$.

We are now concerned with the Cauchy problem for P_{mod}.

Definition 8.1.1 *We say that the Cauchy problem for P_{mod} is locally solvable in $\gamma^{(s)}$ at the origin if for any $\Phi = (u_0, u_1) \in (\gamma^{(s)}(\mathbb{R}^n))^2$, there exists a neighborhood U_Φ of the origin such that the Cauchy problem*

$$\begin{cases} Pu = 0 \quad \text{in } U_\Phi, \\ D_0^j u(0, x') = u_j(x'), \quad j = 0, 1, \quad x \in U_\Phi \cap \{x_0 = 0\} \end{cases}$$

has a solution $u(x) \in C^\infty(U_\Phi)$.

We prove the next result following [7], modifying the argument there about the existence of zeros with "negative imaginary part" of some Stokes multiplier.

Theorem 8.1.1 *If $s > 5$ then the Cauchy problem for P_{mod} is not locally solvable in $\gamma^{(s)}$. In particular the Cauchy problem for P_{mod} is not C^∞ solvable.*

Our strategy to prove Theorem 8.1.1 is to find a family of exact solutions U to $P_{mod}U = 0$ and apply some duality arguments.

8.2 Construction of solutions

We look for a solution to $P_{mod}U = 0$ of the form

$$U(x) = \exp(i\rho^5 x_n + \frac{i}{2}\zeta\rho x_0) w(x_1\rho^2), \ \ \zeta \in \mathbb{C}, \ \rho > 0.$$

It is clear that if w verifies

$$w''(x) = (x^3 + \zeta x - \zeta^2\rho^{-2}/4) w(x)$$

then $P_{mod}U = 0$. Taking this into account we study the following ordinary differential equation

$$w''(x) = (x^3 + \zeta x + \epsilon) w(x) \tag{8.2.1}$$

where ζ, ϵ are complex numbers and ϵ will be thought of as small in the final arguments. We briefly recap, for this special situation, the general theory of subdominant solutions of the equation (8.2.1), according to the exposition for instance in the book of Sibuya [51]. Theorem 6.1 in [51] states that the differential equation (8.2.1) has a solution

$$w(x; \zeta, \epsilon) = \mathcal{Y}(x; \zeta, \epsilon)$$

such that

(i) $\mathcal{Y}(x; \zeta, \epsilon)$ is an entire function of (x, ζ, ϵ),

(ii) $\mathcal{Y}(x;\zeta,\epsilon)$ admits an asymptotic representation

$$\mathcal{Y}(x;\zeta,\epsilon) \sim x^{-3/4}\Big(1+\sum_{N=1}^{\infty} B_N x^{-N/2}\Big)\exp\{-E(x;\zeta)\}$$

uniformly on each compact set in the (ζ,ϵ) space as x goes to infinity in any closed subsector of the open sector

$$|\arg x| < \frac{3\pi}{5}$$

moreover

$$E(x;\zeta) = \frac{2}{5}x^{5/2} + \zeta x^{1/2}$$

and B_N are polynomials in (ζ,ϵ).

We note that if we set $\omega = \exp[i\frac{2\pi}{5}]$ and

$$\mathcal{Y}_k(x;\zeta,\epsilon) = \mathcal{Y}(\omega^{-k}x;\omega^{-2k}\zeta,\omega^{-3k}\epsilon)$$

where $k = 0,1,2,3,4$ then all the five functions $\mathcal{Y}_k(x;\zeta,\epsilon)$ solve (8.2.1). In particular $\mathcal{Y}_0(x;\zeta,\epsilon) = \mathcal{Y}(x;\zeta,\epsilon)$. Let us denote

$$Y = x^{-3/4}\Big(1+\sum_{N=1}^{\infty} B_N x^{-N/2}\Big)\exp\{-E(x;\zeta)\}$$

then we have immediately

(i) $\mathcal{Y}_k(x;\zeta,\epsilon)$ is an entire function of (x,ζ,ϵ),

(ii) $\mathcal{Y}(x;\zeta,\epsilon) \sim Y(\omega^{-k}x;\omega^{-2k}\zeta,\omega^{-3k}\epsilon)$ uniformly on each compact set in the (ζ,ϵ) space as x goes to infinity in any closed subsector of the open sector

$$|\arg x - \frac{2k}{5}\pi| < \frac{3\pi}{5}.$$

Let S_k denote the open sector defined by $|\arg x - 2k\pi/5| < \pi/5$. We say that a solution of (8.2.1) is subdominant in the sector S_k if it tends to 0 as x tends to infinity along any direction in the sector S_k. Analogously a solution is called dominant in the sector S_k if this solution tends to ∞ as x tends to infinity along any direction in the sector S_k. Since

$$\mathrm{Re}\, x^{5/2} > 0 \quad \text{for } x \in S_0 \tag{8.2.2}$$

and $\mathrm{Re}\, x^{5/2} < 0$ for $x \in S_{-1} = S_4$ and for $x \in S_1$ the solution $\mathcal{Y}_0(x;\zeta,\epsilon)$ is subdominant in S_0 and dominant in S_4 and S_1. Similarly $\mathcal{Y}_k(x;\zeta,\epsilon)$ is subdominant in S_k and dominant in S_{k-1} and S_{k+1}. From (8.2.2) we conclude that $\mathcal{Y}_{k+1}$ and $\mathcal{Y}_{k+2}$ are linearly independent. Therefore $\mathcal{Y}_k$ is a linear combination of those two $\mathcal{Y}_{k+1}$ and $\mathcal{Y}_{k+2}$

$$\mathcal{Y}_k(x;\zeta,\epsilon) = C_k(\zeta,\epsilon)\mathcal{Y}_{k+1}(x;\zeta,\epsilon) + \tilde{C}_k(\zeta,\epsilon)\mathcal{Y}_{k+2}(x;\zeta,\epsilon).$$

In the above relation the coefficients C_k, $\tilde{C}_k$ are called the Stokes multipliers for $\mathcal{Y}_k(x;\zeta,\epsilon)$.

Proposition 8.2.1 *There exists a zero of* $C_0(\zeta, 0)$ *with negative imaginary part.*

We first summarize in the following statement some of the known and useful facts about the Stokes multipliers for our particular equation (8.2.1). Proofs can be found in Chapter 5 of [51].

Proposition 8.2.2 *The following results hold.*

(i) $C_0(0,0) = 1 + \omega$,

(ii) $\tilde{C}_k(\zeta, \epsilon) = -\omega$, *for all* k, ϵ *and* ζ,

(iii) $C_k(\zeta, \epsilon) = C_0(\omega^{-2k}\zeta, \omega^{-3k}\epsilon)$, *for all* k, ϵ, ζ *and* $C_0(\zeta, \epsilon)$ *is an entire function of* (ζ, ϵ),

(vi) $\partial_\zeta C_0(\zeta, \epsilon)\big|_{(\zeta,\epsilon)=(0,0)} \neq 0$.

We now prove key lemmas to prove Proposition 8.2.1.

Lemma 8.2.1 *Let us denote* $c_k(\zeta) = C_k(\zeta, 0)$. *Then we have*

$$c_k(\zeta) + \omega^2 c_{k+2}(\zeta) c_{k+3}(\zeta) - \omega^3 = 0 \mod 5.$$

Or otherwise stated

$$c(\zeta) + \omega^2 c(\omega\zeta) c(\omega^4\zeta) - \omega^3 = 0, \quad \forall \zeta \in \mathbb{C}$$

where $c(\zeta) = c_0(\zeta) = C_0(\zeta, 0)$.

Proof: For the proof, see Section 5, (27.5) in [51]. □

The next lemma is found in [52].

Lemma 8.2.2 *We have*

$$\overline{C_0(\zeta, \epsilon)} = \bar{\omega} C_0(\bar{\omega}\bar{\zeta}, \omega\bar{\epsilon}).$$

In particular we have $\overline{c(\zeta)} = \bar{\omega} c(\bar{\omega}\bar{\zeta})$.

Proof: Let us write $a = (\zeta, \epsilon)$ and $\bar{a} = (\bar{\zeta}, \bar{\epsilon})$. Since $w(x) = \overline{\mathcal{Y}_0(\bar{x}; \bar{a})}$ verifies the equation

$$w''(x) = (x^3 + \zeta x + \epsilon) w(x)$$

and hence $w(x) = C\mathcal{Y}_0(x; a)$ with some constant C. Checking the asymptotic behavior of both sides as $x \to +\infty$, $|\arg x| < \pi/5$ we conclude $C = 1$ so that $w(x) = \mathcal{Y}_0(x; a)$ that is

$$\mathcal{Y}_0(\bar{x}; \bar{a}) = \overline{\mathcal{Y}_0(x; a)}.$$

From this we see

$$\overline{\mathcal{Y}_4(x; a)} = \mathcal{Y}_0(\bar{\omega}\bar{x}; \bar{\omega}^2\bar{\zeta}, \omega^2\bar{\epsilon}) = \mathcal{Y}_0(\omega^{-1}\bar{x}; \omega^{-2}\bar{\zeta}, \omega^{-3}\bar{\epsilon}) = \mathcal{Y}_1(\bar{x}; \bar{\zeta}, \bar{\epsilon}) = \mathcal{Y}_1(\bar{x}; \bar{a}).$$

Similarly we have $\overline{\mathcal{Y}_1(x;a)} = \mathcal{Y}_4(\bar{x};\bar{a})$. Thus from $\mathcal{Y}_4(x;a) = C_4(a)\mathcal{Y}_0(x;a) - \omega\mathcal{Y}_1(x;a)$ it follows that

$$\mathcal{Y}_1(\bar{x};\bar{a}) = \overline{C_4(a)}\mathcal{Y}_0(\bar{x};\bar{a}) - \bar{\omega}\mathcal{Y}_4(\bar{x};\bar{a}),$$
$$\mathcal{Y}_4(\bar{x};\bar{a}) = C_4(\bar{a})\mathcal{Y}_0(\bar{x};\bar{a}) - \omega\mathcal{Y}_1(\bar{x};\bar{a}).$$

Multiply the first identity by ω we get

$$\mathcal{Y}_4(\bar{x};\bar{a}) = \omega\overline{C_4(a)}\mathcal{Y}_0(\bar{x};\bar{a}) - \omega\mathcal{Y}_1(\bar{x};\bar{a})$$

which proves

$$C_4(\bar{\zeta},\bar{\epsilon}) = \omega\overline{C_4(\zeta,\epsilon)}.$$

This proves the assertion. □

Lemma 8.2.3 *The Stokes multiplier $C_0(\zeta,0)$ vanishes in at least one $\zeta_0(\neq 0)$.*

Proof: Suppose that $c(\zeta) \neq 0$ for all $\zeta \in \mathbb{C}$. Then from Lemma 8.2.1 it follows that $c(\zeta) \neq \omega^3$ for all $\zeta \in \mathbb{C}$. Since $c(\zeta)$ is an entire function by Picard's Little Theorem implies that $c(\zeta)$ would be constant because $c(\zeta)$ avoids two distinct values 0 and ω^3. But this contradicts (vi) of Proposition 8.2.2. Since $C_0(0,0) = 1+\omega$ from Proposition 8.2.2 we see that $\zeta_0 \neq 0$. □

Lemma 8.2.4 *For real ζ and ϵ we have $C_0(\zeta,\epsilon) \neq 0$. In particular $c(\zeta) \neq 0$ for real ζ.*

Proof: Suppose that $C_0(\zeta,\epsilon) = 0$ for some real ζ and ϵ. From Lemma 8.2.2 it follows that $C_0(\bar{\omega}\bar{\zeta},\omega\bar{\epsilon}) = C_0(\bar{\omega}\zeta,\omega\epsilon) = 0$ which contradicts Lemma 8.2.1. □

Lemma 8.2.5 *The closed sector $3\pi/5 \leq \arg\zeta \leq \pi$ is zero free set of $c(\zeta)$.*

Proof: Let us recall that $\mathcal{Y}_0(x;\zeta,0)$ verifies

$$\mathcal{Y}_0''(x;\zeta,0) = (x^3+\zeta x)\mathcal{Y}_0(x;\zeta,0)$$

which is subdominant in $|\arg x| < \pi/5$. Let us put

$$u(x) = \mathcal{Y}_0(\alpha(x+1); -3\alpha^2, 0)$$

where $-\pi/5 < \arg\alpha < 0$ then we have

$$u''(x) = (\alpha^5x^3 + 3\alpha^5x^2 - 2\alpha^5)u(x) = \alpha^5(x^3+3x^2-2)u(x). \tag{8.2.3}$$

Note that

$$\mathcal{Y}_0(\alpha(x+1); -3\alpha^2,0) = c(-3\alpha^2)\mathcal{Y}_1(\alpha(x+1); -3\alpha^2,0) \\ -\omega\mathcal{Y}_2(\alpha(x+1); -3\alpha^2,0).$$

Suppose that $c(-3\alpha^2) = 0$ so that

$$\begin{aligned}\mathcal{Y}_0(\alpha(x+1); -3\alpha^2, 0) &= -\omega\mathcal{Y}_2(\alpha(x+1); -3\alpha^2, 0)\\ &= -\omega\mathcal{Y}_0(\omega^{-2}\alpha(x+1); -3\omega^{-4}\alpha^2, 0).\end{aligned}$$

Since $\mathsf{Re}\,(\omega^{-2}\alpha x)^{5/2} = \mathsf{Re}\,(e^{i\pi/5}|x|\alpha)^{5/2} > 0$ for $x < 0$ it is clear from the asymptotic behavior that $\mathcal{Y}_0(\alpha(x+1); -3\alpha^2, 0)$ is exponentially decaying in $\mathbb{R}$ as $|x| \to \infty$ and in particular $u(x) \in \mathcal{S}(\mathbb{R})$. We multiply $\bar{u}(x)$ on (8.2.3) then integration by parts gives

$$-\int_{\mathbb{R}} |u'(x)|^2 dx = \alpha^5 \int_{\mathbb{R}} (x^3 + 3x^2 - 2)|u(x)|^2 dx.$$

Since $\mathsf{Im}\,\alpha^5 \neq 0$, taking the imaginary part we get

$$\int_{\mathbb{R}} (x^3 + 3x^2 - 2)|u(x)|^2 dx = 0$$

hence $u'(x) = 0$ so that $u(x) = 0$. This is a contradiction. So we conclude that

$$c(-3\alpha^2) \neq 0 \quad \text{if} \quad -\frac{\pi}{5} < \arg\alpha < 0$$

which proves that $c(\zeta) \neq 0$ for $3\pi/5 < \arg\zeta < \pi$. From Lemma 8.2.4 we see $c(\zeta) \neq 0$ if $\arg\zeta = \pi$. We finally check that $c(\zeta) \neq 0$ with $\arg\zeta = 3\pi/5$. Indeed if $c(\zeta) = 0$ with $\arg\zeta = 3\pi/5$ then $c(\bar{\omega}\bar{\zeta}) = 0$ by Lemma 8.2.2 but since $\bar{\omega}\bar{\zeta} \in \mathbb{R}$ which contradicts Lemma 8.2.4 again and hence the assertion. □

Proof of Proposition 8.2.1: From Lemma 8.2.3 there exists $\zeta \neq 0$ with $c(\zeta) = 0$. From Lemma 8.2.5 we see $-\pi < \arg\zeta < 3\pi/5$. If $0 \leq \arg\zeta < 3\pi/5$ then $-\pi < \arg\bar{\omega}\bar{\zeta} < -2\pi/5$ which proves the assertion because $c(\bar{\omega}\bar{\zeta}) = 0$. □

Let us now consider the equation

$$C_0(\zeta, -\frac{1}{4}\zeta^2\epsilon) = 0.$$

Let ζ_0 be a zero of $c(\zeta) = C_0(\zeta, 0)$ with negative imaginary part. Let μ be the multiplicity of the root ζ_0. Since $C_0(\zeta, 0)$ is holomorphic μ is finite and by the Weierstrass preparation theorem we can write

$$C_0(\zeta, -\zeta^2\epsilon/4) = \gamma(\zeta, \epsilon)((\zeta - \zeta_0)^\mu + \sum_{j=1}^{\mu} a_j(\epsilon)(\zeta - \zeta_0)^{\mu-j})$$

where $\gamma(\zeta_0, 0) \neq 0$, $a_j(0) = 0$ and $a_j(\epsilon)$ is holomorphic at $\epsilon = 0$. Then each root $\zeta(\epsilon)$ of $C_0(\zeta, -\zeta^2\epsilon/4) = 0$ admits the Puiseux expansion

$$\zeta(\epsilon) = \zeta_0 + \sum_{j=0}^{\infty} \zeta_j (\epsilon^{1/p})^j = \tilde{\zeta}(\epsilon^{1/p})$$

with some positive integer p and a holomorphic $\tilde{\zeta}(z)$ near $z = 0$. In what follows we consider the equation

$$w''(x) = \Big(x^3 + \zeta x - \frac{1}{4}\zeta^2\epsilon^p\Big)w(x) \tag{8.2.4}$$

so that the equation $C_0(\zeta, -\zeta^2\epsilon^p/4) = 0$ has a solution $\zeta(\epsilon^p) = \tilde{\zeta}(\epsilon)$ where $\tilde{\zeta}(\epsilon)$ is holomorphic in a neighborhood of $\epsilon = 0$ and

$$\tilde{\zeta}(0) = \zeta_0, \quad \mathsf{Im}\,\zeta_0 < 0.$$

With $\eta(\epsilon) = -\tilde{\zeta}(\epsilon)^2\epsilon^p/4$ we have

$$\mathcal{Y}_0(x; \tilde{\zeta}(\epsilon), \eta(\epsilon)) = -\omega\mathcal{Y}_2(x; \tilde{\zeta}(\epsilon), \eta(\epsilon)), \quad \forall x \in \mathbb{C} \tag{8.2.5}$$

where $|\epsilon| \ll 1$. We now examine the behavior of $\mathcal{Y}_0(x; \tilde{\zeta}(\epsilon), \eta(\epsilon))$ as $\mathbb{R} \ni x \to \pm\infty$. Recall

$$\mathcal{Y}_0(x; \tilde{\zeta}, \eta) = x^{-3/4}(1 + R(x, \tilde{\zeta}, \eta))e^{-(\frac{2}{5}x^{5/2} + \tilde{\zeta}x^{1/2})} \quad \text{in} \quad |\arg x| < 3\pi/5$$

and hence as $\mathbb{R} \ni x \to +\infty$ the function $\mathcal{Y}_0(x; \tilde{\zeta}, \eta)$ decays as $\exp(-2x^{5/2}/5)$. On the other hand from (8.2.5) we have

$$\mathcal{Y}_0(x; \tilde{\zeta}(\epsilon), \eta(\epsilon)) = -\omega\mathcal{Y}_0(\omega^{-2}x; \omega^{-4}\tilde{\zeta}(\epsilon), \omega^{-6}\eta(\epsilon))$$

and for negative $x < 0$ since $\omega^{-2}x = e^{\pi i/5}|x|$ and

$$(\omega^{-2}x)^{5/2} = i|x|^{5/2}, \quad \omega^{-4}\tilde{\zeta}(\omega^{-2}x)^{1/2} = i\tilde{\zeta}|x|^{1/2}$$

it follows that $\mathcal{Y}_0(x; \tilde{\zeta}, \eta)$ decays or grows as $\exp(\mathsf{Im}\,\tilde{\zeta}|x|^{1/2})$ as $\mathbb{R} \ni x \to -\infty$. This is one of the main reasons that we need to find a zero with *negative* imaginary part (a non real root is not enough). We conclude that $\mathcal{Y}_0(x; \tilde{\zeta}(\epsilon), \eta(\epsilon)) \in \mathcal{S}(\mathbb{R})$ and in particular is bounded uniformly in $x \in \mathbb{R}$ and $|\epsilon| \ll 1$;

$$|\mathcal{Y}_0(x; \tilde{\zeta}(\epsilon), \eta(\epsilon))| \le B, \quad x \in \mathbb{R}, \quad |\epsilon| \ll 1.$$

8.3 Proof of non solvability

Take $T > 0$ small and let us set

$$\begin{aligned} U_\rho(x) = \exp\big[-i\rho^5 x_n + \frac{i}{2}\tilde{\zeta}(\rho^{-2/p})\rho(T - x_0)\big] \\ \times\mathcal{Y}(x_1\rho^2; \tilde{\zeta}(\rho^{-2/p}), \eta(\rho^{-2/p})) \end{aligned} \tag{8.3.1}$$

where $\rho > 0$. It is clear that $P_{mod}U_\rho = 0$. Let us consider the following Cauchy problem

$$\begin{cases} P_{mod}u = 0, \\ u(0, x') = 0, \\ D_0u(0, x') = \bar{\phi}(x_1)\bar{\psi}(x'')\bar{\theta}(x_n) \end{cases} \tag{8.3.2}$$

where $x'' = (x_2, ..., x_{n-1})$ and $\phi \in C_0^\infty(\mathbb{R})$, $\psi \in C_0^\infty(\mathbb{R}^{n-2})$ and $\theta \in C_0^\infty(\mathbb{R})$. We now prove

Proposition 8.3.1 *Assume that* $\theta \in C_0^\infty(\mathbb{R})$ *is an even function such that* $\theta \notin \gamma_0^{(5)}(\mathbb{R})$ *and* $\int \psi(x'')dx'' \neq 0$. *Then unless* $\phi^{(k)}(0)$, $k = 0, 1, 2$ *satisfy some special relations the Cauchy problem* (8.3.2) *has no* C^∞ *solution in any neighborhood of the origin.*

Before going into the details of the proof we remark that we can assume that solutions u to (8.3.2) have compact supports with respect to x'. To examine this we recall the Holmgren uniqueness theorem (see, for example [39], Theorem 4.2). Let us set

$$D_\delta = \{x \in \mathbb{R}^{n+1} \mid |x'|^2 + |x_0| < \delta\}$$

then we have

Proposition 8.3.2 *There exists* $\delta > 0$ *such that if* $u(x) \in C^2(D_\delta)$ *verifies*

$$\begin{cases} P_{mod}u = 0 \quad \text{in } D_\delta, \\ D_0^j u(0, x') = 0, \quad j = 0, 1, \quad x' \in D_\delta \cap \{x_0 = 0\} \end{cases}$$

then $u(x)$ *vanishes identically in* D_δ.

Proof of Proposition 8.3.1: Assume that (8.3.2) has a C^∞ solution in a neighborhood of the origin. Applying Proposition 8.3.2 we conclude that we can assume

$$u(x) = 0 \quad \text{for } |x_0| \leq T, \; |x'| \geq r$$

with some small $T > 0$ and $r > 0$. Note that

$$\int_0^T (P_{mod}U_\rho, u)dx_0 = \int_0^T (U_\rho, P_{mod}u)dx_0 + i(D_0U_\rho(T), u(T))$$
$$+ i(U_\rho(T), D_0u(T)) - i(U_\rho(0), D_0u(0)) - i(2x_1D_nU_\rho(T), u(T))$$

because $u(0) = 0$. From this we have

$$(D_0U_\rho(T), u(T)) + (U_\rho(T), D_0u(T))$$
$$-(2x_1D_nU_\rho(T), u(T)) = (U_\rho(0), D_0u(0)). \tag{8.3.3}$$

Recalling that $\mathcal{Y}(\rho^2x_1; \tilde{\zeta}, \eta)$ is bounded uniformly in ρ and x_1 we see that the left-hand side of (8.3.3) is $O(\rho^5)$. On the other hand the right-hand side is

$$\int_{\mathbb{R}^n} e^{-i\rho^5 x_n + i\tilde{\zeta}\rho T/2}\mathcal{Y}(\rho^2x_1; \tilde{\zeta}, \eta)\phi(x_1)\psi(x'')\theta(x_n)dx'$$
$$= e^{i\tilde{\zeta}\rho T/2}\hat{\theta}(\rho^5)\Big(\int_{\mathbb{R}^{n-2}} \psi(x'')dx''\Big)\rho^{-2}\int \mathcal{Y}(x_1; \tilde{\zeta}, \eta)\phi(\rho^{-2}x_1)dx_1$$

where $\hat{\theta}$ is the Fourier transform of θ. We note that for large ρ one has

$$\big|e^{i\tilde{\zeta}\rho T/2}\big| \geq e^{c\rho T}$$

with some $c > 0$ because $\tilde{\zeta}(\rho^{-2/p}) \to \zeta_0$ as $\rho \to \infty$ and $\mathsf{Im}\,\zeta_0 < 0$. Thus we conclude that

$$\rho^{-7} e^{c\rho T} |\hat{\theta}(\rho^5)| \left| \int \mathcal{Y}(x_1; \tilde{\zeta}, \eta) \phi(\rho^{-2} x_1) dx_1 \right| = O(1). \tag{8.3.4}$$

Since $\theta \notin \gamma_0^{(5)}(\mathbb{R})$ and even it follows that for any $N \in \mathbb{N}$ and $c > 0$

$$e^{c\rho} \rho^{-N} |\hat{\theta}(\rho^5)|$$

is not bounded as $\rho \to \infty$. Indeed if $e^{c\rho}\rho^{-N}|\hat{\theta}(\rho^5)|$ is bounded then we have

$$|\hat{\theta}(\rho)| \le C \rho^{N/5} e^{-c\rho^{1/5}} \le C' e^{-c' \rho^{1/5}}$$

with some $c' > 0$ and hence $\theta \in \gamma_0^{(5)}(\mathbb{R})$. Let us write

$$\int \mathcal{Y}(x_1; \tilde{\zeta}, \eta) \phi(\rho^{-2} x_1) dx_1 = \sum_{k=0}^{2} \frac{\rho^{-2k}}{k!} \phi^{(k)}(0) \int \mathcal{Y}(x_1; \tilde{\zeta}, \eta) x_1^k dx_1 + O(\rho^{-6}).$$

Then to complete the proof, noting that

$$\int \mathcal{Y}(x_1; \tilde{\zeta}, \eta) x_1^k dx_1 \to \int \mathcal{Y}(x_1; \zeta_0, 0) x_1^k dx_1$$

as $\rho \to \infty$, it suffices to show

Lemma 8.3.1 *At least one of*

$$\int \mathcal{Y}(x_1; \zeta_0, 0) x_1^k dx_1, \quad k = 0, 1, 2$$

is different from 0.

Proof: Let us denote by $w(\xi)$ the Fourier transform of $\mathcal{Y}(x; \zeta_0, 0)$

$$w(\xi) = \int e^{-ix\xi} \mathcal{Y}(x; \zeta_0, 0) dx.$$

Then since $\mathcal{Y}(x; \zeta_0, 0)$ verifies $\mathcal{Y}'' = (x^3 + \zeta_0 x)\mathcal{Y}$ and $\mathcal{Y}(x; \zeta_0, 0) \in \mathcal{S}(\mathbb{R})$ then $w(\xi)$ satisfies

$$w'''(\xi) - \zeta_0 w'(\xi) + i\xi^2 w(\xi) = 0. \tag{8.3.5}$$

Noting that

$$w^{(k)}(0) = \int \mathcal{Y}(x; \zeta_0, 0) x^k dx$$

the proof follows from the uniqueness of solution to the initial value problem for the ordinary differential equation (8.3.5). □

It is now clear that, choosing $\phi^{(k)}(0)$, $k = 0, 1, 2$ suitably, (8.3.4) does not hold. Thus the proof is completed. □