Chapter 4

Noneffectively hyperbolic Cauchy problem I

4.1 C^∞ well-posedness

Let

$$P(x,D) = D_0^2 + \sum_{|\alpha|\le 2, \alpha_0<2} a_\alpha(x) D^\alpha = P_2 + P_1 + P_0$$

be a second order differential operator, defined in an open neighborhood of the origin of $\mathbb{R}^{n+1}$, hyperbolic with respect to the x_0 direction and with principal symbol $p(x,\xi)$ where $x = (x_0, x_1, ..., x_n)$, $\xi = (\xi_0, \xi_1, ..., \xi_n)$.

We now state more precisely our assumptions. We shall assume in the following that p vanishes exactly of order 2 on a C^∞ submanifold Σ on which σ has constant rank and p is noneffectively hyperbolic, that is we assume that $\Sigma = \{(x,\xi) \mid p(x,\xi) = 0, dp(x,\xi) = 0\}$ is a C^∞ manifold and

$$(4.1.1) \qquad \begin{cases} \mathrm{Sp}(F_p(\rho)) \subset i\mathbb{R}, \quad \rho \in \Sigma, \\ \dim T_\rho\Sigma = \dim \mathrm{Ker}\, F_p(\rho), \quad \rho \in \Sigma, \\ \mathrm{rank}\,\big(\sigma|_\Sigma\big) = \text{constant}, \quad \text{on } \Sigma \end{cases}$$

where $\mathrm{Sp}(F_p(\rho))$ denotes the spectrum of $F_p(\rho)$. According to the spectral structure of $F_p(\rho)$, two different possible cases may arise

$$(4.1.2) \qquad \mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) = \{0\}$$

and

$$\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) \neq \{0\}$$

about which we made detailed studies in the previous chapter.

As shown in Proposition 3.2.1 if p verifies (4.1.2) then p admits an elementary decomposition. In this chapter, assuming (4.1.2), we prove that the Cauchy problem is C^∞ well posed deriving energy estimates via elementary decomposition under the Levi or the strict Ivrii-Petkov-Hörmander condition.

Definition 4.1.1 *We say that P satisfies the Levi condition on Σ if*

$$P_{sub}(\rho) = 0, \quad \forall \rho \in \Sigma.$$

Definition 4.1.2 *We say that P satisfies the Ivrii-Petkov-Hörmander condition on Σ if*

$$\mathsf{Im}\, P_{sub}(\rho) = 0, \quad -\mathrm{Tr}^+ F_p(\rho) \le P_{sub}(\rho) \le \mathrm{Tr}^+ F_p(\rho), \quad \forall \rho \in \Sigma$$

and we say that P satisfies the strict Ivrii-Petkov-Hörmander condition on Σ if

$$\mathsf{Im}\, P_{sub}(\rho) = 0, \quad -\mathrm{Tr}^+ F_p(\rho) < P_{sub}(\rho) < \mathrm{Tr}^+ F_p(\rho), \quad \forall \rho \in \Sigma.$$

Note that if $\mathrm{Tr}^+ F_p = 0$ on Σ then the Ivrii-Petkov-Hörmander condition reduces to the Levi condition.

Theorem 4.1.1 ([24], [18]) *Assume* (4.1.1), (4.1.2) *and the subprincipal symbol P_{sub} verifies the strict Ivrii-Petkov-Hörmander condition on Σ. Then the Cauchy problem for P is C^∞ well posed.*

In the case $\mathrm{Ker}\, F_p^2 \cap \mathrm{Im}\, F_p^2 = \{0\}$ on Σ, thanks to Lemma 3.1.1 and Proposition 3.2.1, the Hamilton flow H_p never touch Σ tangentially.

4.2 Pseudodifferential operators

In this monograph we use several classes of pseudodifferential operators. We first introduce symbol classes of pseudodifferential operators.

Definition 4.2.1 *Let $g = \phi(x',\xi')^{-2}|dx'|^2 + \Phi(x',\xi')^{-2}|d\xi'|^2$ be a (splitting) metric on $\mathbb{R}^{2n}$ and let $m(x',\xi')$ be a positive function on $\mathbb{R}^{2n}$. Then $S(m,g)$ is defined as the set of all $a(x',\xi') \in C^\infty(\mathbb{R}^{2n})$ such that for all multi-indices α, $\beta \in \mathbb{N}^n$ there exists $C_{\alpha\beta}$ such that*

$$|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} a(x',\xi')| \le C_{\alpha\beta} m(x',\xi')\phi(x',\xi')^{-|\alpha|}\Phi(x',\xi')^{-|\beta|}$$

holds in $\mathbb{R}^{2n}$.

In particular if $g = \langle\xi'\rangle^{2\delta}|dx'|^2 + \langle\xi'\rangle^{-2\rho}|d\xi'|^2$ then $S(\langle\xi'\rangle^m, g) = S^m_{\rho,\delta}$ (see [19]) where $\langle\xi'\rangle^2 = 1 + |\xi'|^2$. To $a(x',\xi') \in S(m,g)$ we associate the Weyl quantized pseudodifferential operator $a(x',D')$ by

$$a(x',D')u(x') = (2\pi)^{-n}\int e^{-i(x'-y')\xi'} a(\frac{x'+y'}{2},\xi')u(y')dy'd\xi'.$$

We sometimes write $[a(x',\xi')]^w$ instead of $a(x',D')$ but we often write just a for denoting both $a(x',\xi')$ and $a(x',D')$ if there is no confusion. Let $a_i(x',\xi') \in S(m_i,g)$. Then under some suitable conditions on m_i and g (see Chapter XVIII in [19], we do not touch on these conditions in this monograph) we have with some $b(x',\xi') \in S(m_1 m_2, g)$

$$a_1(x',D')a_2(x',D') = b(x',D').$$

We denote $b(x',\xi')$ by $a_1(x',\xi')\#a_2(x',\xi')$.

Here is a brief summary of calculus of pseudodifferential operators which we use in this monograph (as before under suitable conditions on m_i and g).

Proposition 4.2.1 *Let $a_i \in S(m_i, g)$. Then*

$$\begin{aligned} &a_1\#a_2 - a_2\#a_1 - \{a_1,a_2\}/i \in S(m_1 m_2 (\phi\Phi)^{-3}, g), \\ &a_1\#a_2 + a_2\#a_1 - 2a_1 a_2 \in S(m_1 m_2 (\phi\Phi)^{-2}, g), \\ &a_1\#a_2\#a_1 - a_1^2 a_2 \in S(m_1^2 m_2 (\phi\Phi)^{-2}, g). \end{aligned}$$

Let $(\cdot,\cdot)$ denote the inner product in $L^2(\mathbb{R}^n)$ and $\|\cdot\|$ stands for the $L^2(\mathbb{R}^n)$ norm.

Proposition 4.2.2 *Let $a \in S(m,g)$. Then*

$$\mathsf{Re}(au,u) = ((\mathsf{Re}\, a)u,u), \;\; \mathsf{Im}(au,u) = ((\mathsf{Im}\, a)u,u).$$

In particular if $a(x',\xi')$ is real valued then

$$(au,u) = (u,au).$$

Proposition 4.2.3 (L^2-boundedness) *Let $a \in S(1,g)$. Then we have*

$$\|au\| \le C\|u\|, \;\; u \in \mathcal{S}(\mathbb{R}^n)$$

with some $C > 0$.

Proposition 4.2.4 (Fefferman-Phong inequality) *Assume that $a(x',\xi')$ is non negative and $a \in S((\phi\Phi)^2, g)$. Then we have*

$$(au,u) \ge -C\|u\|^2, \;\; u \in \mathcal{S}(\mathbb{R}^n)$$

with some $C > 0$.

4.3 Energy estimates

In this subsection we derive an energy identity for

$$p = -M\Lambda + B\Lambda + Q$$

with

$$M = D_0 - m(x,D'), \;\; \Lambda = D_0 - \lambda(x,D')$$

where $\lambda(x,\xi')\in S(\langle\xi'\rangle,g_0)$, $m(x,\xi')\in S(\langle\xi'\rangle,g_0)$, $B(x,\xi')\in S(\langle\xi'\rangle,g_0)$, $Q(x,\xi')\in S(\langle\xi'\rangle^2,g_0)$ and

$$g_0=|dx'|^2+\langle\xi'\rangle^{-2}|d\xi'|^2.$$

Let us put

$$P_\theta(x,D)=P(x,D_0-i\theta,D')$$

with a large positive parameter $\theta>0$ so that

$$P(e^{\theta x_0}u)=e^{\theta x_0}P_\theta u.$$

We also put

$$M_\theta=M-i\theta,\quad \Lambda_\theta=\Lambda-i\theta$$

so that $P_\theta=-M_\theta\Lambda_\theta+B\Lambda_\theta+Q$. Then we have

Proposition 4.3.1 *We have*

$$2\mathsf{Im}(P_\theta u,\Lambda_\theta u)=\frac{d}{dx_0}(\|\Lambda_\theta u\|^2+\mathsf{Re}(Qu,u))+2\theta\|\Lambda_\theta u\|^2$$
$$+2((\mathsf{Im}\,B)\Lambda_\theta u,\Lambda_\theta u)+2((\mathsf{Im}\,m)\Lambda_\theta u,\Lambda_\theta u)+2\mathsf{Re}(\Lambda_\theta u,(\mathsf{Im}\,Q)u)$$
$$+2\theta\mathsf{Re}(Qu,u)+\mathsf{Im}([D_0-\mathsf{Re}\,\lambda,\mathsf{Re}\,Q]u,u)+2\mathsf{Re}((\mathsf{Re}\,Q)u,(\mathsf{Im}\,\lambda)u).$$

Proof: Since $2\mathsf{Im}(B\Lambda_\theta u,\Lambda_\theta u)=2((\mathsf{Im}\,B)\Lambda_\theta u,\Lambda_\theta u)$ is clear we compute

$$-2\mathsf{Im}(M_\theta\Lambda_\theta u,\Lambda_\theta u)+2\mathsf{Im}(Qu,\Lambda_\theta u)=I_1+I_2.$$

Noting $\dfrac{d}{dx_0}=iM_\theta+im-\theta$ it is easy to see

$$I_1=\frac{d}{dx_0}\|\Lambda_\theta u\|^2+2\theta\|\Lambda_\theta u\|^2+2((\mathsf{Im}\,m)\Lambda_\theta u,\Lambda_\theta u).$$

We now consider $I_2=2\mathsf{Im}(Qu,\Lambda_\theta u)$

$$I_2=2\mathsf{Im}(Qu,D_0u)+2\theta\mathsf{Re}(Qu,u)+2\mathsf{Im}(Qu,-\lambda u)$$

where we see

$$2\mathsf{Im}(Qu,D_0u)=2\mathsf{Im}\big\{-D_0(Qu,u)+(QD_0u,u)+([D_0,Q]u,u)\big\}$$
$$=2\mathsf{Re}\frac{d}{dx_0}(Qu,u)+2\mathsf{Im}(D_0u,Q^*u)+2\mathsf{Im}([D_0,Q]u,u)$$
$$=2\frac{d}{dx_0}((\mathsf{Re}\,Q)u,u)+2\mathsf{Im}(D_0u,Qu)+2\mathsf{Im}(D_0u,(Q^*-Q)u)+2\mathsf{Im}([D_0,Q]u,u).$$

Therefore we get

$$2\mathsf{Im}(Qu,D_0u)=\frac{d}{dx_0}((\mathsf{Re}\,Q)u,u)+\mathsf{Im}(D_0u,(Q^*-Q)u)+\mathsf{Im}([D_0,Q]u,u).$$

Noting that

$$\mathsf{Im}([D_0, Q]u, u) = \mathsf{Im}([D_0, \mathsf{Re}\, Q]u, u),$$
$$\mathsf{Im}(D_0 u, (Q^* - Q)u) = 2\mathsf{Re}(D_0 u, (\mathsf{Im}\, Q)u)$$

because $([\frac{d}{dx_0}, \mathsf{Im}\, Q]u, u)$ is real and $D_0 = \Lambda_\theta + i\theta + \lambda$ we get

$$I_2 = \frac{d}{dx_0}((\mathsf{Re}\, Q)u, u) + \mathsf{Im}([D_0, \mathsf{Re}\, Q]u) + 2\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\, Q)u)$$
$$+2\mathsf{Re}(\lambda u, (\mathsf{Im}\, Q)u) + 2\theta\mathsf{Re}(Qu, u) + 2\mathsf{Im}(Qu, -\lambda u).$$

Since

$$2\mathsf{Re}(\lambda u, (\mathsf{Im}\, Q)u) + 2\mathsf{Im}(Qu, -\lambda u) = -2\mathsf{Im}((\mathsf{Re}\, Q)u, \lambda u),$$
$$-2\mathsf{Im}((\mathsf{Re}\, Q)u, \lambda u) = -2\mathsf{Im}((\mathsf{Re}\, Q)u, (\mathsf{Re}\, \lambda)u) + 2\mathsf{Re}((\mathsf{Re}\, Q)u, (\mathsf{Im}\, \lambda)u)$$
$$= -\mathsf{Im}([\mathsf{Re}\, \lambda, \mathsf{Re}\, Q]u, u) + 2\mathsf{Re}((\mathsf{Re}\, Q)u, (\mathsf{Im}\, \lambda)u)$$

we have

$$I_2 = \frac{d}{dx_0}((\mathsf{Re}\, Q)u, u) + \mathsf{Im}([D_0 - \mathsf{Re}\, \lambda, \mathsf{Re}\, Q]u)$$
$$+2\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\, Q)u) + 2\theta\mathsf{Re}(Qu, u) + 2\mathsf{Re}((\mathsf{Re}\, Q)u, (\mathsf{Im}\, \lambda)u).$$

and hence the result. $\square$

Note that from

$$-2\mathsf{Im}(\Lambda_\theta u, u) = 2\theta\|u\|^2 + \frac{d}{dx_0}\|u\|^2 + 2\mathsf{Im}(\lambda u, u) \tag{4.3.1}$$

one gets

$$\|\Lambda_\theta u\|^2 \geq \theta^2\|u\|^2 + \theta\frac{d}{dx_0}\|u\|^2 + 2\theta((\mathsf{Im}\, \lambda)u, u). \tag{4.3.2}$$

Replacing $\|\Lambda_\theta u\|^2$ in Proposition 4.3.1 by the estimate (4.3.2) we get

Proposition 4.3.2 *We have*

$$2\mathsf{Im}(P_\theta u, \Lambda_\theta u) \geq \frac{d}{dx_0}(\|\Lambda_\theta u\|^2 + ((\mathsf{Re}\, Q)u, u) + \theta^2\|u\|^2) + \theta\|\Lambda_\theta u\|^2$$
$$+2\theta\mathsf{Re}(Qu, u) + 2((\mathsf{Im}\, B)\Lambda_\theta u, \Lambda_\theta u) + 2((\mathsf{Im}\, m)\Lambda_\theta u, \Lambda_\theta u)$$
$$+2\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\, Q)u) + \mathsf{Im}([D_0 - \mathsf{Re}\lambda, \mathsf{Re}\, Q]u, u)$$
$$+2\mathsf{Re}((\mathsf{Re}\, Q)u, (\mathsf{Im}\, \lambda)u) + \theta^3\|u\|^2 + 2\theta^2((\mathsf{Im}\, \lambda)u, u).$$

4.4 Levi condition

Note that if $\mathrm{Tr}^+F_p = 0$ on Σ then Theorem 4.1.1 is empty and the Ivrii-Petkov-Hörmander condition reduces to the Levi condition. Taking this into account we study the Cauchy problem under the Levi condition in this section. Recall that from Proposition 3.2.1 it follows that p admits an elementary decomposition

$$p(x,\xi) = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q.$$

Here we note that

$$P(x,D) = (p + P_{sub})^w + R, \quad R \in S(1, g_0)$$

where $P(x,D)$ is our original differential operator. We also assume that the Levi condition is satisfied. Let us write

$$P_{sub} = P_s(x,\xi') + b(x,\xi')(\xi_0 - \lambda)$$

then we have

Lemma 4.4.1 *Assume that P satisfies the Levi condition on Σ. Then we have*

$$|P_s(x,\xi')| \le C\sqrt{Q(x,\xi')}, \quad |\{\xi_0 - \lambda, P_s\}| \le C\sqrt{Q(x,\xi')}$$

with some $C > 0$.

Proof: Let us recall $p = -\xi_0^2 + q$. It is clear that $P_s = 0$ on $\Sigma' = \{q = 0\}$ because $\Sigma = \{\xi_0 = 0, q = 0\}$ and $|\lambda| \le C\sqrt{q}$ by Proposition 3.2.1. Let $\{V_i\}$, $\{\chi_i\}$ be as in Proposition 3.2.1. Let $(x,\xi') \in V_i$. Since $V_i \cap \Sigma' = \{\phi_{i\alpha} = 0\}$ and hence P_s is a linear combination of $\{\phi_{i\alpha}\}$ in V_i and it is clear that

$$|P_s(x,\xi')| \le C\sum_\alpha |\phi_{i\alpha}| \le C\sqrt{q_i} \le C'\sqrt{Q}.$$

We turn to $\{\xi_0 - \lambda, P_s\}$. Arguing as before we see

$$\{\xi_0 - \lambda, P_s\} = \sum \chi_i\{\xi_0 - \lambda_i, P_s\} - \sum \lambda_i\{\chi_i, P_s\}.$$

Since P_s is a linear combination of $\{\phi_{i\alpha}\}$ on the support of χ_i and $\{\xi_0 - \lambda_i, \phi_{i\alpha}\}$ is also a linear combination of $\{\phi_{i\alpha}\}$ in V_i we see easily

$$\sum |\chi_i\{\xi_0 - \lambda_i, P_s\}| \le C\sum \chi_i|\phi_{i\alpha}| \le C'\sum \chi_i\sqrt{q_i} \le C''\sqrt{Q}.$$

Together with the estimate $|\lambda_i| \le C\sqrt{Q}$ this proves the assertion. □

We return to $P(x,D)$. From Proposition 3.2.1 we have

$$|\{\xi_0 - \lambda, Q\}| \le CQ, \quad |\lambda| \le C\sqrt{Q}$$

with some $C > 0$. Noting that

$$p^w = -(\xi_0 + \lambda)^w(\xi_0 - \lambda)^w + Q^w - \frac{i}{2}\{\xi_0 + \lambda, \xi_0 - \lambda\}^w + R$$

with $R \in S(1, g_0)$ and recalling $|\{\xi_0 + \lambda, \xi_0 - \lambda\}| \le C\sqrt{Q}$ we have

Proposition 4.4.1 *Assume that p satisfies* (4.1.1), (4.1.2) *and that P verifies the Levi condition. Then one can write*

$$P = -M\Lambda + Q + \hat{P}_1 + B_0\Lambda + \hat{P}_0, \quad M = (\xi_0 + \lambda)^w, \ \Lambda = (\xi_0 - \lambda)^w$$

where $p = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q$ is an elementary decomposition of p and $\hat{P}_1$ verifies

$$|\hat{P}_1| \le C\sqrt{Q}, \ \ |\{\xi_0 - \lambda, \mathsf{Re}\,\hat{P}_1\}| \le C\sqrt{Q}$$

with some $C > 0$ and B_0, $\hat{P}_0 \in S(1, g_0)$.

Let $\hat{P}_0 = 0$ and we apply Proposition 4.3.2. Since λ and Q are real we have

$$\begin{aligned} 2\mathsf{Im}(P_\theta u, \Lambda_\theta u) \ge{}& \frac{d}{dx_0}(\|\Lambda_\theta u\|^2 + ((Q + \mathsf{Re}\,\hat{P}_1)u, u) \\ &+\theta^2\|u\|^2) + \theta\|\Lambda_\theta u\|^2 + 2((\mathsf{Im}\,B_0)\Lambda_\theta u, \Lambda_\theta u) \\ &+2\theta((Q + \mathsf{Re}\,\hat{P}_1)u, u) + 2\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\,\hat{P}_1)u) \\ &+\mathsf{Im}([D_0 - \lambda, Q + \mathsf{Re}\,\hat{P}_1]u, u) + \theta^3\|u\|^2. \end{aligned} \tag{4.4.1}$$

We first check that

$$((Q + \mathsf{Re}\,\hat{P}_1)u, u) \ge c(Qu, u) - C\|u\|^2$$

with some $c > 0$, $C > 0$. Indeed we have

$$2((\mathsf{Re}\,\hat{P}_1)u, u) \le \epsilon\|(\mathsf{Re}\,\hat{P}_1)u\|^2 + \epsilon^{-1}\|u\|^2.$$

Note $(\mathsf{Re}\,\hat{P}_1)^*(\mathsf{Re}\,\hat{P}_1) - [(\mathsf{Re}\,\hat{P}_1)^2]^w \in S(1, g_0)$ and $Q/2 - \epsilon(\mathsf{Re}\,\hat{P}_1)^2 \ge 0$ choosing $\epsilon > 0$ small we get

$$(Qu, u)/2 - \epsilon\|(\mathsf{Re}\,\hat{P}_1)u\|^2 \ge -C\|u\|^2$$

by the Fefferman-Phong inequality.

We next consider $\mathsf{Im}([D_0 - \lambda, Q]u, u)$. Note that

$$\mathsf{Im}([D_0 - \lambda, Q]u, u) \ge -\mathsf{Re}(\{\xi_0 - \lambda, Q\}^w u, u) - C\|u\|^2.$$

Since $CQ - \{\xi_0 - \lambda, Q\} \ge 0$ with some $C > 0$ then the Fefferman-Phong inequality shows

$$C(Qu, u) + \mathsf{Im}([D_0 - \lambda, Q]u, u) \ge -C'\|u\|^2$$

with some $C' > 0$. We turn to $|\mathsf{Im}([D_0 - \lambda, \mathsf{Re}\,\hat{P}_1]u, u)|$. Note

$$2|\mathsf{Im}([D_0 - \lambda, \mathsf{Re}\,\hat{P}_1]u, u)| \le \|[D_0 - \lambda, \mathsf{Re}\,\hat{P}_1]u\|^2 + \|u\|^2.$$

Since $CQ - \{\xi_0 - \lambda, \mathsf{Re}\,\hat{P}_1\}^2 \ge 0$ with some $C > 0$, the same argument as above gives

$$C(Qu, u) \ge \|[D_0 - \lambda, \mathsf{Re}\,\hat{P}_1]u\|^2 - C'\|u\|^2$$

with some $C' > 0$. Thus we conclude that

$$2C(Qu,u) + \mathsf{Im}([D_0 - \lambda, Q + \mathsf{Re}\,\hat{P}_1]u, u) \geq -(2C'+1)\|u\|^2. \tag{4.4.2}$$

We now consider $|\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\,\hat{P}_1)u)|$. Note

$$2|\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\,\hat{P}_1)u)| \leq \|\Lambda_\theta u\|^2 + \|(\mathsf{Im}\,\hat{P}_1)u\|^2$$

and $CQ - (\mathsf{Im}\,\hat{P}_1)^2 \geq 0$ with some $C > 0$ then we obtain

$$2|\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\,\hat{P}_1)u)| \leq \|\Lambda_\theta u\|^2 + C(Qu,u) + C'\|u\|^2 \tag{4.4.3}$$

with some $C' > 0$.

Since the following estimates are clear

$$|(B\Lambda_\theta u, \Lambda_\theta u)| \leq C\|\Lambda_\theta u\|^2, \quad |(\hat{P}_0 u, \Lambda_\theta u)| \leq \|\Lambda_\theta u\|^2 + C\|u\|^2 \tag{4.4.4}$$

then with

$$E(u) = \|\Lambda_\theta u\|^2 + (Qu,u) + \theta^2\|u\|^2$$

we get

$$\theta^{-1}\|P_\theta u\|^2 \geq c\frac{d}{dx_0}E(u) + c\theta E(u)$$

with some $c > 0$ for $\theta \geq \theta_0$. Integrating this inequality we get

Lemma 4.4.2 *Assume that P satisfies the Levi condition on Σ. Then we have*

$$\begin{aligned} &\{\|\Lambda_\theta u(t,\cdot)\|^2 + (Qu(t),u(t)) + \theta^2\|u(t,\cdot)\|^2\} \\ &+\theta\int_T^t \{\|\Lambda_\theta u(s,\cdot)\|^2 + (Qu,u) + \theta^2\|u(s,\cdot)\|^2\}ds \\ &\qquad\qquad \leq C\theta^{-1}\int_T^t \|P_\theta u(s,\cdot)\|^2 ds \end{aligned} \tag{4.4.5}$$

for any $u \in C_0^\infty(\mathbb{R}^{n+1})$ vanishing in $x_0 \leq T$.

We now estimate higher order derivatives. Consider $\langle D'\rangle^\ell P_\theta = P_\theta\langle D'\rangle^\ell - [P_\theta, \langle D'\rangle^\ell]$. With $R_\ell = [P_\theta, \langle D'\rangle]\langle D'\rangle^{-\ell}$ one can write

$$\langle D'\rangle^\ell P_\theta = (P_\theta - R_\ell)\langle D'\rangle^\ell.$$

We denote $\|u\|_{(s)} = \|\langle D'\rangle^s u\|$.

Lemma 4.4.3 *Assume that $|M - \Lambda| \leq C\sqrt{Q}$ with some $C > 0$. Then we have*

$$|(R_\ell u, \Lambda_\theta u)| \leq C_\ell(\|\Lambda_\theta u\|^2 + \mathsf{Re}(Qu,u) + \|u\|^2).$$

Proof: Recall that

$$P_\theta = -M_\theta\Lambda_\theta + Q + \hat{P}_1 + B_0\Lambda_\theta + \hat{P}_0.$$

It is easy to see

$$\begin{aligned}[M_\theta \Lambda_\theta, \langle D'\rangle^\ell]\langle D'\rangle^{-\ell} &= a\Lambda_\theta + bM_\theta + R_1\\ &= (a+b)\Lambda_\theta + b(M-\Lambda) + R_2\end{aligned}$$

with some a, b, $R_i \in S(1, g_0)$. On the other hand we have

$$[Q, \langle D'\rangle^\ell]\langle D'\rangle^{-\ell} - T^w \in S(1, g_0)$$

with $T = -i\{Q, \langle \xi'\rangle^\ell\}\langle \xi'\rangle^{-\ell}$. From the non negativity of Q one has

$$|T^2| \le CQ.$$

Noting that

$$\|T^w u\|^2 = ((T\#T)^w u, u), \;\; T\#T - T^2 \in S(1, g_0)$$

the Fefferman-Phong inequality shows that

$$C\mathsf{Re}(Qu, u) - \|T^w u\|^2 \ge -C\|u\|^2.$$

Since $|M-\Lambda|^2 \le CQ$ the Fefferman-Phong inequality again shows

$$C\mathsf{Re}(Qu, u) - \|(M-\Lambda)u\|^2 \ge -C\|u\|^2.$$

Since $[\hat{P}_1 + \hat{P}_0, \langle D'\rangle^\ell]\langle D'\rangle^{-\ell} \in S(1, g_0)$, $[B_0\Lambda_\theta, \langle D'\rangle^\ell]\langle D'\rangle^{-\ell} = c_0\Lambda_\theta + c_1$ with $c_j \in S(\langle \xi'\rangle^{j-1}, g_0)$ one has

$$|(R_\ell u, \Lambda_\theta)| \le C\big\{\|\Lambda_\theta u\|^2 + \mathsf{Re}(Qu, u) + \|u\|^2\big\}$$

which is the desired assertion. □

Thanks to Lemma 4.4.2 and Lemma 4.4.3 we have

Proposition 4.4.2 *We have*

$$\begin{aligned}&(\|\Lambda_\theta u(t,\cdot)\|_{(\ell)}^2 + \theta^2\|u(t,\cdot)\|_{(\ell)}^2)\\ &+\theta\int_T^t (\|\Lambda_\theta u(s,\cdot)\|_{(\ell)}^2 + \theta^2\|u(s,\cdot)\|_{(\ell)}^2)ds\\ &\le C_\ell \theta^{-1}\int_T^t \|P_\theta u(s,\cdot)\|_{(\ell)}^2 ds\end{aligned}$$

for any $\ell \in \mathbb{R}$ and for any $u \in C_0^\infty(\mathbb{R}^{n+1})$ vanishing in $x_0 \le T$.

Assume that $\mathrm{Tr}^+ F_p = 0$ on Σ. Then from Theorem 2.3.1 the quadratic form p_ρ takes the form, in a suitable symplectic coordinates system

$$p_\rho = -\xi_0^2 + \sum_{j=1}^r \xi_j^2.$$

From (3.3.7) it follows that

$$F_p^2(\rho)v = \sum_{k=0}^{r} \epsilon_k \sigma(v, H_{\xi_k}) \Big(\sum_{j=0}^{r} \epsilon_j \sigma(H_{\xi_k}, H_{\xi_j}) H_{\xi_j} \Big) = 0$$

and a priori the condition (4.1.2) is verified. We also note that Σ is an involutive manifold in this case since

$$(T_\rho \Sigma)^\sigma = \langle H_{\xi_0}, H_{\xi_1}, ..., H_{\xi_r} \rangle \subset \langle H_{\xi_0}, H_{\xi_1}, ..., H_{\xi_r} \rangle^\sigma = T_\rho \Sigma, \quad \rho \in \Sigma.$$

Theorem 4.4.1 ([18]) *Assume* (4.1.1) *and* $\mathrm{Tr}^+ F_p = 0$ *on* Σ*. Then in order that the Cauchy problem for* P *is* C^∞ *well posed it is necessary and sufficient that* P *satisfies the Levi condition on* Σ*.*

Proof: Since $P(x,D) = (p + P_{sub})^w + R$ with $R \in S(1, g_0)$ and hence

$$P^*(x,D) = (p + \bar{P}_{sub})^w + R, \quad R \in S(1, g_0) \tag{4.4.6}$$

it follows that $P^*(x,D)$ verifies the Levi condition. Thus the energy estimates in Proposition 4.4.2 holds for P^*, that is we have

Proposition 4.4.3 *There exists* $T > 0$ *such that we have*

$$\begin{aligned} &(\|\Lambda_\theta u(t,\cdot)\|_{(\ell)}^2 + \theta^2 \|u(t,\cdot)\|_{(\ell)}^2) \\ &+\theta \int_t^T (\|\Lambda_\theta u(s,\cdot)\|_{(\ell)}^2 + \theta^2 \|u(s,\cdot)\|_{(\ell)}^2) ds \\ &\leq C_\ell \theta^{-1} \int_t^T \|P_\theta^* u(s,\cdot)\|_{(\ell)}^2 ds, \quad -T \leq t \leq T \end{aligned}$$

for any $\ell \in \mathbb{R}$ *and for any* $u \in C_0^\infty(\mathbb{R}^{n+1})$ *vanishing in* $x_0 \geq T$*.*

Corollary 4.4.1 *There exists* $T > 0$ *such that for any* $\ell \in \mathbb{R}$ *there is* $C_\ell > 0$ *such that*

$$\int \Big(\|D_0 u(t,\cdot)\|_{(\ell-1)}^2 + \|u(t,\cdot)\|_{(\ell)}^2 \Big) dt \leq C_\ell \int \|P^* u(t,\cdot)\|_{(\ell)}^2 dt$$

holds for any $u \in C_0^\infty((-T,T) \times \mathbb{R}^n)$*.*

Here we sketch the proof of the existence of solutions to the Cauchy problem for P. We recall that

$$H^s(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n) \mid u \in L^2_{loc}(\mathbb{R}^n), \ \langle D' \rangle^s u \in L^2(\mathbb{R}^n) \}.$$

Let $f \in L^2((-T,T); H^s)$. If v is in $C_0^\infty((-T,T) \times \mathbb{R}^n)$ then Corollary 4.4.1 implies

$$\Big| \int (f, v) dt \Big| \leq C \Big(\int \|f\|_{(s)}^2 dt \Big)^{1/2} \Big(\int \|P^* v\|_{(-s)}^2 dt \Big)^{1/2}.$$

Using the Hahn-Banach theorem to extend the anti-linear form in P^*v on the left-hand side we conclude that there is some $u \in L^2((-T,T);H^s)$ such that

$$\int (f,v)dt = \int (u, P^*v)dt$$

for any $v \in C_0^\infty((-T,T)\times\mathbb{R}^n)$. This proves that $Pu = f$ in $(-T,T)\times\mathbb{R}^n$. From the fact that $u \in L^2((-T,T);H^s)$ and $Pu \in \mathcal{S}(\mathbb{R}^{n+1})$ we can deduce $D_0^j u \in L^2((-T,T);H^{s-j})$ (see Appendix B in [19]). Once we have established enough regularity for u then applying Proposition 4.4.2 to $e^{-\theta x_0}u$ to conclude that $u = 0$ in $x_0 \le \tau$ if $f = 0$ in $x_0 \le \tau$.

The necessary part follows from Theorem 2.2.2 because $\mathrm{Tr}^+ F_p = 0$. □

4.5 Strict Ivrii-Petkov-Hörmander condition

In this section we assume that the strict Ivrii-Petkov-Hörmander condition is satisfied. We first note that

Lemma 4.5.1 *Assume that $p(x,\xi)$ admits an elementary decomposition $p = -M\Lambda + Q$. Then we have*

$$\mathrm{Tr}^+ F_p(\rho) = \mathrm{Tr}^+ Q_\rho, \quad \rho \in \Sigma.$$

Proof: From the assumption one can write

$$p(x,\xi) = -(\xi_0 + \lambda)(\xi_0 - \lambda) + Q(x,\xi')$$

where $|\{\xi_0 - \lambda, Q\}| \le CQ$. At ρ one has

$$p_\rho(x,\xi) = -(\xi_0 + d\lambda)(\xi_0 - d\lambda) + Q_\rho(x,\xi').$$

By a linear symplectic change of coordinates one may assume that

$$p_\rho = \xi_0(\xi_0 - \ell) + Q_\rho(x,\xi').$$

Since $|\{\xi_0, Q_\rho\}| \le CQ_\rho$ one concludes that Q_ρ is independent of x_0 and hence $Q_\rho = Q_\rho(x',\xi')$. By a linear symplectic change of coordinates again we may assume that $\ell = \xi_1$ or $\ell = 0$ according to $\ell \ne 0$ and $\ell = 0$. Now it is easy to see that

$$|\lambda - F_{p_\rho}| = \lambda^2 |\lambda - F_{Q_\rho}|$$

which proves that non zero eigenvalues of F_{p_ρ} coincides with those of F_{Q_ρ} counting the multiplicity. □

Proposition 4.5.1 *Assume that p satisfies* (4.1.1), (4.1.2) *and that P verifies the strict Ivrii-Petkov-Hörmander condition. Then one can write*

$$P = -M\Lambda + Q + \hat{P}_1 + B_0\Lambda + \hat{P}_0, \quad M = (\xi_0 + \lambda)^w,\ \Lambda = (\xi_0 - \lambda)^w$$

where $p = -(\xi_0+\lambda)(\xi_0-\lambda)+Q$ *is an elementary decomposition of* p *and* $\hat{P}_1$ *verifies*

$$|\mathsf{Im}\,\hat{P}_1| \le C\sqrt{Q}, \quad \mathrm{Tr}^+Q_\rho + \mathsf{Re}\,\hat{P}_1(\rho) > 0, \;\; \rho \in \Sigma$$

with some $C > 0$ *and* B_0*,* $\hat{P}_0 \in S(1, g_0)$*.*

Proof: We know from Proposition 3.2.1 that

$$|\{\xi_0 - \lambda, Q\}| \le CQ, \quad |\lambda| \le C\sqrt{Q}$$

with some $C > 0$. Assume that P satisfies the strict Ivrii-Petkov-Hörmander condition and in particular $\mathsf{Im}\,P_{sub} = 0$ on Σ. Here we note that

$$P(x, D) = (p + P_{sub})^w + R, \;\; R \in S(1, g_0)$$

and let us write

$$P_{sub} = P_s(x, \xi') + b(x, \xi')(\xi_0 - \lambda).$$

Repeating the same arguments as in Section 5.3 it is clear that

$$\mathsf{Im}\,P_{sub}\big|_\Sigma = \mathsf{Im}\,P_s\big|_{q=0}$$

so that $\mathsf{Im}\,P_s = 0$ if $q = 0$. Let $(x, \xi') \in V_i$ and $\{\phi_{i\alpha}\}$ be as before. Since $\mathsf{Im}\,P_s$ is a linear combination of $\{\phi_{i\alpha}\}$ in V_i and hence

$$|\mathsf{Im}\,P_s(x, \xi')| \le C \sum_\alpha |\phi_{i\alpha}| \le C\sqrt{q_i} \le C'\sqrt{Q}.$$

We return to $P(x, D)$. Recalling that

$$p^w = -(\xi_0 + \lambda)^w(\xi_0 - \lambda)^w + Q^w - \frac{i}{2}\{\xi_0 + \lambda, \xi_0 - \lambda\}^w + R$$

with $R \in S(1, g_0)$ and $|\{\xi_0 + \lambda, \xi_0 - \lambda\}| \le C\sqrt{Q}$ we get the first assertion because

$$\mathsf{Im}\hat{P}_1 = \mathsf{Im}\,P_s - \frac{1}{2}\{\xi_0 + \lambda, \xi_0 - \lambda\}.$$

To check the second assertion note that $\mathsf{Re}\,\hat{P}_1 = \mathsf{Re}\,P_s$. Since $\mathsf{Re}\,P_s = \mathsf{Re}\,P_{sub}$ on Σ, then taking a partition $\{V_i\}$ in Proposition 3.2.1 enough fine the assertion follows from Lemma 4.5.1. □

Let $\hat{P}_0 = 0$. Since λ and Q are real then the inequality (4.4.1) holds. Note that choosing $\epsilon > 0$ sufficiently small we have

$$(1 - \epsilon)\mathrm{Tr}^+Q_\rho + \mathsf{Re}\,\hat{P}_1(\rho) > 0, \quad \rho \in \Sigma$$

and hence by the Melin's inequality [35] we see that

$$(((1 - \epsilon)Q + \mathsf{Re}\,\hat{P}_1)u, u) \ge c\|u\|_{(1/2)} - C\|u\|^2 \tag{4.5.1}$$

with some $c > 0$, $C > 0$. This proves that

$$\begin{aligned}2\mathsf{Im}(P_\theta u, \Lambda_\theta u) \geq \frac{d}{dx_0}(\|\Lambda_\theta u\|^2 + ((Q + \mathsf{Re}\,\hat{P}_1)u, u)\\ +\theta^2\|u\|^2) + \theta\|\Lambda_\theta u\|^2 + 2((\mathsf{Im}\,B_0)\Lambda_\theta u, \Lambda_\theta u)\\ +2\epsilon\theta(Qu, u) + 2c\theta\|u\|^2_{(1/2)} + 2\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\,\hat{P}_1)u)\\ +\mathsf{Im}([D_0 - \lambda, Q + \mathsf{Re}\,\hat{P}_1]u, u) + c'\theta^3\|u\|^2.\end{aligned}$$

On the other hand, repeating the same arguments as in Section 5.3 we get

$$C(Qu, u) + \mathsf{Im}([D_0 - \lambda, Q]u, u) \geq -C'\|u\|^2.$$

We turn to $|\mathsf{Im}([D_0 - \lambda, \mathsf{Re}\,\hat{P}_1]u, u)|$. Noting $[D_0 - \lambda, \mathsf{Re}\,\hat{P}_1] \in S(\langle\xi'\rangle, g_0)$ it is clear that

$$|\mathsf{Im}([D_0 - \lambda, \mathsf{Re}\,\hat{P}_1]u, u)| \leq C\|u\|^2_{(1/2)}$$

Thus we conclude that

$$C(Qu, u) + C\|u\|^2_{(1/2)} + \mathsf{Im}([D_0 - \lambda, Q + \mathsf{Re}\,\hat{P}_1]u, u) \geq -C'\|u\|^2. \tag{4.5.2}$$

As for $|\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\,\hat{P}_1)u)|$ repeating the same arguments as above we get

$$2|\mathsf{Re}(\Lambda_\theta u, (\mathsf{Im}\,\hat{P}_1)u)| \leq \|\Lambda_\theta u\|^2 + C(Qu, u) + C'\|u\|^2. \tag{4.5.3}$$

Taking (4.4.4) into account and choosing θ large we get

$$\begin{aligned}2\mathsf{Im}(P_\theta u, \Lambda_\theta u) \geq \frac{d}{dx_0}(\|\Lambda_\theta u\|^2 + ((Q + \mathsf{Re}\,\hat{P}_1)u, u)\\ +\theta^2\|u\|^2) + c\theta\|\Lambda_\theta u\|^2 + 2c\theta(Qu, u)\\ +2c\theta\|u\|^2_{(1/2)} + c\theta^3\|u\|^2.\end{aligned}$$

Integrating this inequality we obtain

Lemma 4.5.2 *Assume that P satisfies the strict Ivrii-Petkov-Hörmander condition on Σ. Then we have*

$$\begin{aligned}\big\{\|\Lambda_\theta u(t,\cdot)\|^2 + \mathsf{Re}(Qu(t), u(t)) + \|u(t)\|^2_{(1/2)} + \theta^2\|u(t,\cdot)\|^2\big\}\\ +c\theta\int_T^t \big\{\|\Lambda_\theta u(s,\cdot)\|^2 + \mathsf{Re}(Qu(t), u(t))\\ +\|u(t,\cdot)\|_{(1/2)} + \theta^2\|u(s,\cdot)\|^2\big\}ds \leq C\theta^{-1}\int_T^t \|P_\theta u(s,\cdot)\|^2 ds\end{aligned} \tag{4.5.4}$$

with some $c > 0$ and $\theta \geq \theta_0$ for any $u \in C_0^\infty(\mathbb{R}^{n+1})$ vanishing in $x_0 \leq T$.

Applying Lemma 4.4.3 we get

Proposition 4.5.2 *We have*

$$\begin{aligned}
&(\|\Lambda_\theta u(t,\cdot)\|_{(\ell)}^2+\|u(t,\cdot)\|_{(\ell+1/2)}^2+\theta^2\|u(t,\cdot)\|_{(\ell)}^2)\\
&+c_\ell\theta\int_T^t(\|\Lambda_\theta u(s,\cdot)\|_{(\ell)}^2+\|u(s,\cdot)\|_{(\ell+1/2)}^2+\theta^2\|u(s,\cdot)\|_{(\ell)}^2)ds\\
&\leq C_\ell\theta^{-1}\int_T^t\|P_\theta u(s,\cdot)\|_{(\ell)}^2ds
\end{aligned}$$

with some $c_\ell>0$ and $\theta\geq\theta_\ell$ for any $\ell\in\mathbb{R}$ and for any $u\in C_0^\infty(\mathbb{R}^{n+1})$ vanishing in $x_0\leq T$.

It is clear from (4.4.6) that $P^*(x,D)$ satisfies the strict Ivrii-Petkov-Hörmander condition and hence Proposition 4.5.2 holds for P^*. Repeating the same arguments that we have used in the end of Section 4.4 we can prove Theorem 4.1.1 from Proposition 4.5.2.

In order to make more precise studies about the sufficiency of the Ivrii-Petkov-Hörmander condition for the C^∞ well-posedness we need to improve the Melin's inequality. To do so we first study the lower bound of Q.

We now change notations and study the Weyl quantized pseudodifferential operator A with classical real symbol

$$A_2(x,\xi)+A_1(x,\xi)+\cdots$$

where $a(x,\xi)=A_2(x,\xi)\geq 0$. Recall that we assume that the doubly characteristic set Σ of a is a smooth manifold verifying

$$\dim T_\rho\Sigma=\dim\operatorname{Ker}F_a(\rho),\quad \rho\in\Sigma \tag{4.5.5}$$

and

$$\operatorname{rank}\left(\sigma|_\Sigma\right)=\text{const.} \tag{4.5.6}$$

In what follows we denote by Q_ρ the polar form of the Hesse matrix of a at $\rho\in\Sigma$. Recall that all eigenvalues of $F_a(\rho)$, $\rho\in\Sigma$ are pure imaginary. Let V_λ denotes the space of generalized eigenvectors of F_a belonging to the eigenvalue λ and set

$$V_\rho^+=\bigoplus_{\mu>0}V_{i\mu}.$$

Let V_ρ^{0r} denote the real subspace of V_ρ^0, the generalized eigenspace associated to the eigenvalue 0, and N_ρ the kernel of $F_a(\rho)$.

Lemma 4.5.3 *If $0\neq v\in V_\rho^+$ then $Q_\rho(v,\bar v)>0$.*

Proof: In the proof we drop the suffix ρ. Let $F_av=i\mu v$. Since $F_a\bar v=-i\mu\bar v$ then

$$0\leq Q(v+\bar v,v+\bar v)=2Q(v,\bar v).$$

This shows that $Q(\mathsf{Re}\, v, \mathsf{Re}\, v) = 0$ if $Q(v, \bar{v}) = 0$ and hence $F_a \mathsf{Re}\, v = 0$. From $F_a \mathsf{Re}\, v = -\mu \mathsf{Im}\, v$ it follows that $\mathsf{Im}\, v = 0$ because $\mu \neq 0$. Repeating the same argument we get $\mathsf{Re}\, v = 0$. This is a contradiction. □

From the assumption $\dim V_\rho^+$ is constant when $\rho \in \Sigma$ so that V_ρ^+ is a vector bundle over Σ. Then one can choose a basis $v_1(\rho), ..., v_k(\rho)$ for V_ρ^+ which is smooth in $\rho \in \Sigma$ and verifies

$$Q_\rho(v_i(\rho), \bar{v}_j(\rho)) = 2\delta_{ij}$$

thanks to Lemma 4.5.3. Note that V_ρ^{0r}/N_ρ is a real vector bundle over Σ. We remark that $Q_\rho(v, v) = 0$ for real $v \neq 0$ implies that $v \in T_\rho \Sigma$. From this one can choose a basis $v_{k+1}(\rho)$,...,$v_{k+\ell}(\rho)$ for V_ρ^{0r}/N_ρ such that

$$Q_\rho(v_i(\rho), v_j(\rho)) = \delta_{ij}, \quad k+1 \le i, j \le k+\ell.$$

Let us set

$$L_j(\rho; v) = Q_\rho(v_j(\rho), v), \quad 1 \le j \le k+\ell$$

which is smooth in $\rho \in \Sigma$. We examine that for real v

$$\sum_{j=1}^{k+\ell} |L_j(\rho; v)|^2 = Q_\rho(v, v).$$

To see this we first note that $Q_\rho(v_i(\rho), v_j(\rho)) = 0$, $1 \le i, j \le k$ and $Q_\rho(v_i(\rho), v_j(\rho)) = 0$ for $1 \le i \le k$, $k+1 \le j \le k+\ell$ because $Q_\rho(V_\lambda, V_\mu) = 0$ if $\lambda + \mu \neq 0$. Writing

$$v = \sum_{j=1}^{k} \alpha_j v_j(\rho) + \sum_{j=1}^{k} \bar{\alpha}_j \bar{v}_j(\rho) + \sum_{j=k+1}^{k+\ell} \gamma_j v_j(\rho)$$

we see that $\sum_{j=1}^{k+\ell} |L_j(\rho; v)|^2 = 2\sum_{j=1}^{k} |\alpha_j|^2 + \sum_{j=k+1}^{k+\ell} \gamma_j^2$. On the other hand we see easily that

$$Q_\rho(v, v) = 2\sum_{j=1}^{k} |\alpha_j|^2 + \sum_{j=k+1}^{k+\ell} \gamma_j^2$$

and hence the assertion. Put

$$\begin{gathered} \Lambda(\rho; v) = (\mathsf{Re}\, L_1(\rho; v), \mathsf{Im}\, L_1(\rho; v), ..., \\ \mathsf{Re}\, L_k(\rho; v), \mathsf{Im}\, L_k(\rho; v), L_{k+1}(\rho; v), ..., L_{k+\ell}(\rho; v)) \end{gathered}$$

so that we have

$$Q_\rho(v, v) = \sum_{j=1}^{2k+\ell} \Lambda_j(\rho; v)^2.$$

Since one can write $a(\rho) = \sum_{j=1}^{2k+\ell} b_j(\rho)^2$ we have

$$Q_\rho(v) = \sum_{j=1}^{2k+\ell} db_j(\rho; v)^2 = \sum_{j=1}^{2k+\ell} \Lambda_j(\rho; v)^2.$$

Since $db_j(\rho;\cdot)$ are linearly independent one can write

$$\Lambda_j(\rho;\cdot)=\sum_{j=1}^{2k+\ell}O_{jk}(\rho)db_k(\rho;\cdot)$$

where $O(\rho)=(O_{jk}(\rho))$ is a non singular matrix which is smooth in $\rho\in\Sigma$. Since $\mathbb{R}^{2k+\ell}\ni v\mapsto(db_1(\rho;v),...,db_{2k+\ell}(\rho;v))$ is surjective we conclude that $O(\rho)$ is orthogonal. Let us define

$$c_j(\rho)=\sum_{i=1}^{2k+\ell}O_{ji}(\rho)b_i(\rho)$$

and hence $dc_j(\rho;v)=\Lambda_j(\rho;v)$ for $\rho\in\Sigma$ and $a(\rho)=\sum_{j=1}^{2k+\ell}c_j(\rho)^2$. Let $F_a(\rho)v_j(\rho)=i\mu v_j(\rho)$ then

$$\begin{aligned}\sigma(L_j(\rho;\cdot),\bar{L}_j(\rho;\cdot))=-\mu^2\sigma(\bar{v}_j(\rho),v_j(\rho)),\\ 2=Q_\rho(v_j(\rho),\bar{v}_j(\rho))=i\mu\sigma(\bar{v}_j(\rho),v_j(\rho))\end{aligned}$$

and hence

$$\sum_{j=1}^{k}\sigma(\mathsf{Im}\,L_j(\rho;\cdot),\mathsf{Re}\,L_j(\rho;\cdot))=\sum_{j=1}^{k}\{\mathsf{Im}\,L_j(\rho;\cdot),\mathsf{Re}\,L_j(\rho;\cdot)\}=2\mathrm{Tr}^+F_a(\rho)$$

for $\rho\in\Sigma$. Let us set

$$\begin{cases}X_j(x,\xi)=c_{2j-1}(x,\xi)+ic_{2j}(x,\xi), & j=1,...,k,\\ X_j(x,\xi)=c_{k+j}(x,\xi), & j=k+1,...,k+\ell.\end{cases}$$

Note that

$$\bar{X}_j\#X_j=|X_j|^2+\frac{1}{2i}\{\bar{X}_j,X_j\}+R_1,\quad R_1\in S(1,g_0)$$

and $A=(a+A_{sub})^w+R_2$, $R_2\in S(1,g_0)$. Let us set

$$B=a+A_{sub}-\sum_{j=1}^{k+\ell}\bar{X}_j\#X_j=A_{sub}+\sum_{j=1}^{k}\frac{i}{2}\{\bar{X}_j,X_j\}+R_3$$

where $R_3\in S(1,g_0)$. We assume that

$$A_{sub}+\mathrm{Tr}^+F_a(\rho)\geq 0,\quad \rho\in\Sigma. \tag{4.5.7}$$

Denoting the principal symbol of B by b_1 we see from (4.5.7) that $b_1(x,\xi)\geq 0$ on Σ. Let q be an extension of b_1 outside Σ such that $q(x,\xi)\geq 0$. Then one can write

$$b_1(x,\xi)-q(x,\xi)=\sum_{j=1}^{k+\ell}(\bar{r}_jX_j+r_j\bar{X}_j)$$

$$=\sum_{j=1}^{k+\ell}(\bar{r}_j\#X_j+\bar{X}_j\#r_j)+R_0,\quad R_0\in S(1,g_0)$$

because $b_1(x,\xi) - q(x,\xi)$ is real. Then one has

$$a(x,\xi) + A_{sub}(x,\xi) = B(x,\xi) + \sum_{j=1}^{k+\ell} \bar{X}_j \# X_j$$
$$= q(x,\xi) + \sum_{j=1}^{k+\ell} \overline{(X_j + r_j)} \# (X_j + r_j) + R_0'$$

where $R_0' \in S(1, g_0)$. Then we have

$$(Au,u) \geq (q^w u, u) + \sum_{j=1}^{k+\ell} \|(X_j + r_j)^w u\|^2 - C\|u\|^2 \geq -C\|u\|^2.$$

We summarize what we have proved in

Proposition 4.5.3 ([18]) *Let A be a pseudodifferential operator with classical symbol $A_2 + A_1 + \cdots$. Assume that $a = A_2 \geq 0$ and A_{sub} verify the assumptions* (4.5.5), (4.5.6) *and* (4.5.7) (*in particular A_{sub} is assumed to be real*). *Then we have*

$$(Au,u) \geq -C\|u\|^2.$$

In [50], we find more detailed discussions on this inequality, called the Melin-Hörmander inequality.

Let us consider

$$P = -D_0^2 + A$$

where $A = A_2 + A_1 + \cdots$ is a classical pseudodifferential operator which is *real* and satisfies all conditions in Proposition 4.5.3. From Proposition 4.3.2 it follows that

$$2\mathsf{Im}(P_\theta u, \Lambda_\theta u) \geq \frac{d}{dx_0}(\|\Lambda_\theta u\|^2 + (Au,u) + \theta^2\|u\|^2)$$
$$+\theta\|\Lambda_\theta u\|^2 + 2\theta(Au,u) + \mathsf{Im}([D_0, A]u, u) + \theta^3\|u\|^2$$

where $\Lambda_\theta = D_0 - i\theta$. Thus, for example, if

$$2\theta(Au,u) + \mathsf{Im}([D_0, A]u, u) \geq -C\|u\|^2$$

for large θ with some $C > 0$ we get an energy estimate. But in general the Ivrii-Petkov-Hörmander condition does not assure the C^∞ well-posedness. We discuss about this question in the next section. On the other hand we find discussions on the sufficiency of the (non strict) Ivrii-Petkov-Hörmander condition for C^∞ well-posedness in [50].

4.6 An example

In this section we show that the Ivrii-Petkov-Hörmander condition is not sufficient *in general* for the Cauchy problem to be C^∞ well posed for such operators verifying (4.1.1) and (4.1.2) by giving an example. This example will be obtained by reducing the problem to that which was studied by F.Colombini and S.Spagnolo [11].

Let P be

$$P(x,D)=-D_0^2+\sum_{j=1}^{\ell}\mu_j(x_j^2D_n^2+D_j^2)+b(x_0)D_n \tag{4.6.1}$$

where $\ell\leq n-1$ and μ_j are positive constants and $b(x_0)$ is a C^∞ function of x_0 defined near the origin which we will make to be precise later. We consider the Cauchy problem for P near the origin in $\mathbb{R}^{n+1}$

$$\begin{cases} P(x,D)u(x)=0,\\ u(0,x')=\phi_0(x'),\ D_0u(0,x')=\phi_1(x'). \end{cases} \tag{4.6.2}$$

For this Cauchy problem the Ivrii-Petkov-Hörmander condition asserts

$$b(x_0)\in\mathbb{R},\ \ |b(x_0)|\leq\sum_{j=1}^{k}\mu_j \quad \text{near}\ \ x_0=0. \tag{4.6.3}$$

The following assertion was proved in [46].

Theorem 4.6.1 ([46]) *There exist a C^∞ function $b(x_0)$ defined near x_0 satisfying* (4.6.3) *and $\phi_1(x')\in C_0^\infty(\mathbb{R}^n)$ such that the Cauchy problem* (4.6.2) *with $\phi_0(x')=0$ has no solution in $C^2([0,\epsilon];\mathcal{E}'(\mathbb{R}^n))$ for any $\epsilon>0$.*

Proof: In [11] they have constructed a C^∞ function $a(x_0)$ on $(-\infty,\rho]$ vanishing in $(-\infty,0]$, strictly positive on $(0,\rho]$, where ρ is a given positive constant, and a sequence of solutions $v_k(x_0)$ to the ordinary differential equations

$$\frac{d^2v_k}{dx_0^2}+h_k^2a(x_0)v_k=0$$

such that for every $\epsilon>0$ and $p\in\mathbb{N}$ there is $C(\epsilon,p)$ with

$$\begin{cases} |v_k(\epsilon)|,\ |D_0v_k(\epsilon)|\leq C(\epsilon,p)h_k^{-p},\ \ k=1,2,...,\\ |v_k(0)|h_k^{-p}\to\infty\ \ \text{as}\ \ k\to\infty. \end{cases} \tag{4.6.4}$$

Here $h_k\in\mathbb{N}$ and $h_k\to\infty$ as $k\to\infty$. Define $b(x_0)$ as

$$b(x_0)=\sum_{j=1}^{\ell}\mu_j-a(x_0)$$

then by virtue of the non-negativity of $a(x_0)$ it is clear that $b(x_0)$ verifies (4.6.3) in $(-\infty,\rho']$ with some positive ρ'. Put

$$P^*(x,D)=-D_0^2+\sum_{j=1}^{\ell}\mu_j(x_j^2D_n^2+D_j^2)-b(x_0)D_n,$$

$$u_N(x)=\sum_{k=1}^{N}v_k(x_0)E_k(x'),\ \ E_k(x')=\exp{(ix_nh_k^2)}\prod_{j=1}^{\ell}\exp{(-x_j^2h_k^2/2)}.$$

Then it is clear that $P^*(x,D)u_N(x)=0$.

Now we suppose that $w(x)$ would be a solution to (4.6.2) with $\phi_0(x')=0$ belonging to $C^2([0,\epsilon];\mathcal{E}'(\mathbb{R}^n))$ with some $0<\epsilon(\le\rho')$. We observe the integral

$$\int_0^\epsilon dx_0\int_{\mathbb{R}^n} Pw(x_0,x')u_N(x_0,x')dx'=\int_0^\epsilon\langle Pw,u_N\rangle dx_0=0.$$

By integration by parts we see

$$\langle\phi_1,u_N(0,\cdot)\rangle=\langle D_0w(\epsilon,\cdot),u_N(\epsilon,\cdot)\rangle-\langle w(\epsilon,\cdot),D_0u_N(\epsilon,\cdot)\rangle \tag{4.6.5}$$

because $w(0,\cdot)=\phi_0=0$. Since $w(\epsilon,0)$, $D_0w(\epsilon,0)\in\mathcal{E}'(\mathbb{R}^n)$ it follows that

$$|\langle w(\epsilon,\cdot),E_k(\cdot)\rangle|,\ |\langle D_0w(\epsilon,\cdot),E_k(\cdot)\rangle|\le Ch_k^M$$

with some integer M. We take p in (4.6.4) with $p\ge M+2$ so that the right-hand side of (4.6.5) converges as $N\to\infty$ and hence so does $\langle\phi_1,u_N(0,\cdot)\rangle$. We now choose $\phi_1(x')=\theta(x_n)\psi(x'')\prod_{j=1}^{\ell}\phi(x_j)$ with θ, $\phi\in C_0^\infty(\mathbb{R})$ and $\psi(x'')\in C_0^\infty(\mathbb{R}^{n-\ell-1})$ where $x''=(x_{\ell+1},...,x_{n-1})$ such that

$$\phi(0)=1,\quad \int_{\mathbb{R}^{n-\ell-1}}\psi(x'')dx''=1.$$

Then $\langle\phi_1,u_N(0,\cdot)\rangle$ turns to be

$$\sum_{k=1}^N v_k(0)\hat{\theta}(h_k^2)\prod_{j=1}^{\ell}\int_{\mathbb{R}}\psi(x_j)e^{-x_j^2h_k^2/2}dx_j$$

where $\hat{\theta}$ is the Fourier transform of θ. Remarking the fact

$$h_k\int_{\mathbb{R}}\psi(t)e^{-t^2h_k^2/2}dt\to(2\pi)^{1/2}$$

as $k\to\infty$ we would have

$$v_k(0)\hat{\theta}(h_k^2)h_k^{-\ell}\to 0\quad\text{as }\ k\to\infty. \tag{4.6.6}$$

Since $|v_k(0)|h_k^{-p}\to\infty$ as $k\to\infty$ for any $p\in\mathbb{N}$ it is clear that we can choose $\theta\in C_0^\infty(\mathbb{R})$ with arbitrarily small support which does not satisfy (4.6.6). In fact it is enough to take

$$\theta(t)=\sum_k v_k(0)^{-1}e^{ith_k^2}\alpha(t),\quad \alpha(t)=\beta(t)*\bar{\beta}(t)$$

where $\beta\in C_0^\infty(\mathbb{R})$ with small support and we note here that $\hat{\alpha}(\tau)=|\hat{\beta}(\tau)|^2\ge 0$. This contradiction proves the assertion. $\square$