CHAPTER 11

Shift operators

In this chapter we study an integer shift of spectral parameters $\lambda_{j,\nu}$ of the Fuchsian equation $P_{\mathbf{m}}(\lambda)u = 0$. Here $P_{\mathbf{m}}(\lambda)$ is the universal operator (cf. Theorem 6.14) corresponding to the spectral type $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}}$. For simplicity, we assume that $\mathbf{m}$ is rigid in this chapter unless otherwise stated.

11.1. Construction of shift operators and contiguity relations

First we construct shift operators for general shifts.

Definition 11.1. Fix a tuple of partitions $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}} \in \mathcal{P}_{p+1}^{(n)}$. Then a set of integers $\big(\epsilon_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}}$ parametrized by j and ν is called a *shift compatible to* $\mathbf{m}$ if

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\epsilon_{j,\nu}m_{j,\nu} = 0. \tag{11.1}$$

Theorem 11.2 (shift operator). *Fix a shift* $(\epsilon_{j,\nu})$ *compatible to* $\mathbf{m} \in \mathcal{P}_{p+1}^{(n)}$. *Then there is a shift operator* $R_{\mathbf{m}}(\epsilon,\lambda) \in W[x]\otimes\mathbb{C}[\lambda_{j,\nu}]$ *which gives a homomorphism of the equation* $P_{\mathbf{m}}(\lambda')v = 0$ *to* $P_{\mathbf{m}}(\lambda)u = 0$ *defined by* $v = R_{\mathbf{m}}(\epsilon,\lambda)u$. *Here the Riemann scheme of* $P_{\mathbf{m}}(\lambda)$ *is* $\{\lambda_{\mathbf{m}}\} = \big\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\big\}_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}}$ *and that of* $P_{\mathbf{m}}(\lambda')$ *is* $\{\lambda'_{\mathbf{m}}\}$ *defined by* $\lambda'_{j,\nu} = \lambda_{j,\nu} + \epsilon_{j,\nu}$. *Moreover we may assume* $\operatorname{ord} R_{\mathbf{m}}(\epsilon,\lambda) < \operatorname{ord}\mathbf{m}$ *and* $R_{\mathbf{m}}(\epsilon,\lambda)$ *never vanishes as a function of* λ *and then* $R_{\mathbf{m}}(\epsilon,\lambda)$ *is uniquely determined up to a constant multiple.*

Putting

$$\tau = \big(\tau_{j,\nu}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} \quad \textit{with} \quad \tau_{j,\nu} := \big(2 + (p-1)n\big)\delta_{j,0} - m_{j,\nu} \tag{11.2}$$

and $d = \operatorname{ord} R_{\mathbf{m}}(\epsilon,\lambda)$, *we have*

$$P_{\mathbf{m}}(\lambda+\epsilon)R_{\mathbf{m}}(\epsilon,\lambda) = (-1)^d R_{\mathbf{m}}(\epsilon,\tau-\lambda-\epsilon)^* P_{\mathbf{m}}(\lambda) \tag{11.3}$$

under the notation in Theorem 4.19 ii).

Proof. We will prove the theorem by the induction on $\operatorname{ord}\mathbf{m}$. The theorem is clear if $\operatorname{ord}\mathbf{m} = 1$.

We may assume that $\mathbf{m}$ is monotone. Then the reduction $\{\tilde\lambda_{\tilde{\mathbf{m}}}\}$ of the Riemann scheme is defined by (10.33). Hence putting

$$\begin{cases} \tilde\epsilon_1 = \epsilon_{0,1} + \cdots + \epsilon_{p,1},\\ \tilde\epsilon_{j,\nu} = \epsilon_{j,\nu} + \big((-1)^{\delta_{j,0}} - \delta_{\nu,1}\big)\tilde\epsilon_1 \quad (j = 0,\dots,p,\ \nu = 1,\dots,n_j), \end{cases} \tag{11.4}$$

there is a shift operator $R(\tilde\epsilon,\tilde\lambda)$ of the equation $P_{\tilde{\mathbf m}}(\tilde\lambda')\tilde v=0$ to $P_{\tilde{\mathbf m}}(\tilde\lambda)\tilde u=0$ defined by $\tilde v=R(\tilde\epsilon,\tilde\lambda)\tilde u$. Note that

$$\begin{aligned}
P_{\tilde{\mathbf m}}(\tilde\lambda)=\partial_{max}P_{\mathbf m}(\lambda)&=\mathrm{Ad}\Big(\prod_{j=1}^p(x-c_j)^{\lambda_{j,1}}\Big)\prod_{j=1}^p(x-c_j)^{m_{j,1}-d}\partial^{-d}\mathrm{Ad}(\partial^{-\mu})\\
&\quad\prod_{j=1}^p(x-c_j)^{-m_{j,1}}\,\mathrm{Ad}\Big(\prod_{j=1}^p(x-c_j)^{-\lambda_{j,1}}\Big)P_{\mathbf m}(\lambda),\\
P_{\tilde{\mathbf m}}(\tilde\lambda')=\partial_{max}P_{\mathbf m}(\lambda')&=\mathrm{Ad}\Big(\prod_{j=1}^p(x-c_j)^{\lambda_{j,1}}\Big)\prod_{j=1}^p(x-c_j)^{m_{j,1}-d}\partial^{-d}\mathrm{Ad}(\partial^{-\mu'})\\
&\quad\prod_{j=1}^p(x-c_j)^{-m_{j,1}}\,\mathrm{Ad}\Big(\prod_{j=1}^p(x-c_j)^{-\lambda'_{j,1}}\Big)P_{\mathbf m}(\lambda').
\end{aligned}$$

Suppose $\lambda_{j,\nu}$ are generic. Let $u(x)$ be a local solution of $P_{\mathbf m}(\lambda)u=0$ at $x=c_1$ corresponding to a characteristic exponent different from $\lambda_{1,1}$. Then

$$\tilde u(x):=\prod_{j=1}^p(x-c_j)^{\lambda_{j,1}}\partial^{-\mu}\prod_{j=1}^p(x-c_j)^{-\lambda_{j,1}}u(x)$$

satisfies $P_{\tilde{\mathbf m}}(\tilde\lambda)\tilde u(x)=0$. Putting

$$\begin{aligned}
\tilde v(x)&:=R(\tilde\epsilon,\tilde\lambda)\tilde u(x),\\
v(x)&:=\prod_{j=1}^p(x-c_j)^{\lambda'_{j,1}}\partial^{\mu'}\prod_{j=1}^p(x-c_j)^{\lambda'_{j,1}}\tilde v(x),\\
\tilde R(\tilde\epsilon,\tilde\lambda)&:=\mathrm{Ad}\Big(\prod_{j=1}^p(x-c_j)^{\lambda_{j,1}}\Big)R(\tilde\epsilon,\tilde\lambda)
\end{aligned}$$

we have $P_{\tilde{\mathbf m}}(\tilde\lambda')\tilde u(x)=0$, $P_{\mathbf m}(\lambda')v(x)=0$ and

$$\prod_{j=1}^p(x-c_j)^{\epsilon_{j,1}}\partial^{-\mu'}\prod_{j=1}^p(x-c_j)^{-\lambda'_{j,1}}v(x)=\tilde R(\tilde\epsilon,\tilde\lambda)\partial^{-\mu}\prod_{j=1}^p(x-c_j)^{-\lambda_{j,1}}u(x).$$

In general, if

$$S_2\prod_{j=1}^p(x-c_j)^{\epsilon_{j,1}}\partial^{-\mu'}\prod_{j=1}^p(x-c_j)^{-\lambda'_{j,1}}v(x)=S_1\partial^{-\mu}\prod_{j=1}^p(x-c_j)^{-\lambda_{j,1}}u(x)\tag{11.5}$$

with S_1, $S_2\in W[x]$, we have

$$R_2v(x)=R_1u(x)\tag{11.6}$$

by putting

$$\begin{aligned}
R_1&=\prod_{j=1}^p(x-c_j)^{\lambda_{j,\nu}+k_{1,j}}\partial^{\mu+\ell}\prod_{j=1}^p(x-c_j)^{k_{2,j}}S_1\prod_{j=1}^{\epsilon_{j,1}}\partial^{-\mu}\prod_{j=1}^p(x-c_j)^{-\lambda_{j,\nu}},\\
R_2&=\prod_{j=1}^p(x-c_j)^{\lambda_{j,\nu}+k_{1,j}}\partial^{\mu+\ell}\prod_{j=1}^p(x-c_j)^{k_{2,j}}S_2\prod_{j=1}^{\epsilon_{j,1}}\partial^{-\mu'}\prod_{j=1}^p(x-c_j)^{-\lambda'_{j,\nu}}
\end{aligned}\tag{11.7}$$

with suitable integers $k_{1,j}$, $k_{2,j}$ and ℓ so that $R_1,\ R_2\in W[x;\lambda]$.

We choose a non-zero polynomial $S_2\in\mathbb C[x]$ so that $S_1=S_2\tilde R(\tilde\epsilon,\tilde\lambda)\in W[x]$. Since $P_{\mathbf m}(\lambda')$ is irreducible in $W(x;\lambda)$ and $R_2v(x)$ is not zero, there exists $R_3\in W(x;\xi)$ such that $R_3R_2-1\in W(x;\lambda)P_{\mathbf m}(\lambda')$. Then $v(x)=Ru(x)$ with the operator $R=R_3R_1\in W(x;\lambda)$.

Since the equations $P_{\mathbf{m}}(\lambda)u = 0$ and $P_{\mathbf{m}}(\lambda')v = 0$ are irreducible $W(x;\lambda)$-modules, the correspondence $v = Ru$ gives an isomorphism between these two modules. Since any solutions of these equations are holomorphically continued along the path contained in $\mathbb{C}\setminus\{c_1,\dots,c_p\}$, the coefficients of the operator R are holomorphic in $\mathbb{C}\setminus\{c_1,\dots,c_p\}$. Multiplying R by a suitable element of $\mathbb{C}(\lambda)$, we may assume $R\in W(x)\otimes\mathbb{C}[\lambda]$ and R does not vanish at any $\lambda_{j,\nu}\in\mathbb{C}$.

Put $f(x) = \prod_{j=1}^{p}(x-c_j)^n$. Since $R_{\mathbf{m}}(\epsilon,\lambda)$ is a shift operator, there exists $S_{\mathbf{m}}(\epsilon,\lambda)\in W(x;\lambda)$ such that

$$f^{-1}P_{\mathbf{m}}(\lambda+\epsilon)R_{\mathbf{m}}(\epsilon,\lambda) = S_{\mathbf{m}}(\epsilon,\lambda)f^{-1}P_{\mathbf{m}}(\lambda). \tag{11.8}$$

Then Theorem 4.19 ii) shows

$$\begin{aligned}
R_{\mathbf{m}}(\epsilon,\lambda)^*\big(f^{-1}P_{\mathbf{m}}(\lambda+\epsilon)\big)^* &= \big(f^{-1}P_{\mathbf{m}}(\lambda)\big)^*S_{\mathbf{m}}(\epsilon,\lambda)^*,\\
R_{\mathbf{m}}(\epsilon,\lambda)^*\cdot f^{-1}P_{\mathbf{m}}(\lambda+\epsilon)^\vee &= f^{-1}P_{\mathbf{m}}(\lambda)^\vee\cdot S_{\mathbf{m}}(\epsilon,\lambda)^*,\\
R_{\mathbf{m}}(\epsilon,\lambda)^*f^{-1}P_{\mathbf{m}}(\rho-\lambda-\epsilon) &= f^{-1}P_{\mathbf{m}}(\rho-\lambda)S_{\mathbf{m}}(\epsilon,\lambda)^*,
\end{aligned}$$

$$R_{\mathbf{m}}(\epsilon,\rho-\mu-\epsilon)^*f^{-1}P_{\mathbf{m}}(\mu) = f^{-1}P_{\mathbf{m}}(\mu+\epsilon)S_{\mathbf{m}}(\epsilon,\rho-\mu-\epsilon)^*. \tag{11.9}$$

Here we use the notation (4.52) and put $\rho_{j,\nu} = 2(1-n)\delta_{j,0}+n-m_{j,\nu}$ and $\mu = \rho-\lambda-\epsilon$. Comparing (11.9) with (11.8), we see that $S_{\mathbf{m}}(\epsilon,\lambda)$ is a constant multiple of the operator $R_{\mathbf{m}}(\epsilon,\rho-\lambda-\epsilon)^*$ and $fR_{\mathbf{m}}(\epsilon,\rho-\lambda-\epsilon)^*f^{-1} = \big(f^{-1}R_{\mathbf{m}}(\epsilon,\rho-\lambda-\epsilon)f\big)^* = R_{\mathbf{m}}(\epsilon,\tau-\lambda-\epsilon)^*$ and we have (11.3). $\square$

Note that the operator $R_{\mathbf{m}}(\epsilon,\lambda)$ is uniquely defined up to a constant multiple.

The following theorem gives a contiguity relation among specific local solutions with a rigid spectral type and a relation between the shift operator $R_{\mathbf{m}}(\epsilon,\lambda)$ and the universal operator $P_{\mathbf{m}}(\lambda)$.

THEOREM 11.3. *Retain the notation in* Corollary 10.12 *and* Theorem 11.2 *with a rigid tuple* $\mathbf{m}$. *Assume* $m_{j,n_j}=1$ *for* $j=0,\ 1$ *and* 2. *Put* $\epsilon=(\epsilon_{j,\nu})$, $\epsilon'=(\epsilon'_{j,\nu})$,

$$\epsilon_{j,\nu} = \delta_{j,1}\delta_{\nu,n_1}-\delta_{j,2}\delta_{\nu,n_2}\quad\textit{and}\quad \epsilon'_{j,\nu} = \delta_{j,0}\delta_{\nu,n_0}-\delta_{j,2}\delta_{\nu,n_2} \tag{11.10}$$

for $j=0,\dots,p$ *and* $\nu=1,\dots,n_j$.

i) *Define* $Q_{\mathbf{m}}(\lambda)\in W(x;\lambda)$ *so that* $Q_{\mathbf{m}}(\lambda)P_{\mathbf{m}}(\lambda+\epsilon')-1\in W(x;\lambda)P_{\mathbf{m}}(\lambda+\epsilon)$. *Then*

$$R_{\mathbf{m}}(\epsilon,\lambda)-C(\lambda)Q_{\mathbf{m}}(\lambda)P_{\mathbf{m}}(\lambda+\epsilon')\in W(x;\lambda)P_{\mathbf{m}}(\lambda) \tag{11.11}$$

with a rational function $C(\lambda)$ *of* $\lambda_{j,\nu}$.

ii) *Let* $u_\lambda(x)$ *be the local solution of* $P_{\mathbf{m}}(\lambda)u=0$ *such that* $u_\lambda(x)\equiv(x-c_1)^{\lambda_{1,n_1}}$ mod $(x-c_1)^{\lambda_{1,n_1}+1}\mathcal{O}_{c_1}$ *for generic* $\lambda_{j,\nu}$. *Then we have the contiguity relation*

$$u_\lambda(x) = u_{\lambda+\epsilon'}(x)+(c_1-c_2)\prod_{\nu=0}^{K-1}\frac{\lambda(\nu+1)_{1,n_1}-\lambda(\nu)_{1,\ell(\nu)_1}+1}{\lambda(\nu)_{1,n_1}-\lambda(\nu)_{1,\ell(\nu)_1}+1}\cdot u_{\lambda+\epsilon}(x). \tag{11.12}$$

PROOF. Under the notation in Corollary 10.12, $\ell(k)_j\neq n_j$ for $j=0,1,2$ and $k=0,\dots,K-1$ and therefore the operation ∂^K_{max} on $P_{\mathbf{m}}(\lambda)$ is equals to ∂^K_{max} on $P_{\mathbf{m}}(\lambda+\epsilon)$ if they are realized by the product of the operators of the form (5.26). Hence by the induction on K, the proof of Theorem 11.2 (cf. (11.5), (11.6) and (11.7)) shows

$$P_{\mathbf{m}}(\lambda+\epsilon')u(x) = P_{\mathbf{m}}(\lambda+\epsilon')v(x) \tag{11.13}$$

for suitable functions $u(x)$ and $v(x)$ satisfying $P_{\mathbf{m}}(\lambda)u(x) = P_{\mathbf{m}}(\lambda+\epsilon)v(x) = 0$ and moreover (11.12) is calculated by (3.6). Note that the identities

$$(c_1-c_2)\prod_{j=1}^{p}(x-c_j)^{\lambda_j+\epsilon'_j} = \prod_{j=1}^{p}(x-c_j)^{\lambda_j}-\prod_{j=1}^{p}(x-c_j)^{\lambda_j+\epsilon_j},$$

$$\Big(\partial-\sum_{j=1}^p\frac{\lambda_j+\epsilon'_j}{x-c_j}\Big)\prod_{j=1}^p(x-c_j)^{\lambda_j}=\Big(\partial-\sum_{j=1}^p\frac{\lambda_j+\epsilon'_j}{x-c_j}\Big)\prod_{j=1}^p(x-c_j)^{\lambda_j+\epsilon_j}$$

correspond to (11.12) and (11.13), respectively, when $K=0$.

Note that (11.13) may be proved by (11.12). The claim i) in this theorem follows from the fact $v(x)=Q_{\mathbf{m}}(\lambda)P_{\mathbf{m}}(\lambda+\epsilon')v(x)=Q_{\mathbf{m}}(\lambda)P_{\mathbf{m}}(\lambda+\epsilon')u(x)$. □

In general, we have the following theorem for the contiguity relation.

THEOREM 11.4 (contiguity relations). *Let $\mathbf{m}\in\mathcal{P}^{(n)}$ be a rigid tuple with $m_{1,n_1}=1$ and let $u_1(\lambda,x)$ be the normalized solution of the equation $P_{\mathbf{m}}(\lambda)u=0$ with respect to the exponent λ_{1,n_1} at $x=c_1$. Let $\epsilon^{(i)}$ be shifts compatible to $\mathbf{m}$ for $i=0,\dots,n$. Then there exists polynomial functions $r_i(x,\lambda)\in\mathbb{C}[x,\lambda]$ such that $(r_0,\dots,r_n)\neq 0$ and*

$$\sum_{i=0}^n r_i(x,\lambda)u_1(\lambda+\epsilon^{(i)},x)=0. \tag{11.14}$$

PROOF. There exist $R_i\in\mathbb{C}(\lambda)R_{\mathbf{m}}(\epsilon^{(i)},\lambda)$ satisfying $u_1(\lambda+\epsilon^{(i)},x)=R_iu_1(\lambda,x)$ and $\operatorname{ord}R_i<n$. We have $r_i(x,\lambda)$ with $\sum_{i=0}^n r_i(x,\lambda)R_i=0$ and the claim. □

EXAMPLE 11.5 (Gauss hypergeometric equation). Let $P_\lambda u=0$ and $P_{\lambda'}v=0$ be Fuchsian differential equations with the Riemann Scheme

$$\begin{Bmatrix}x=\infty & 0 & 1\\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix} \text{ and } \begin{Bmatrix}x=\infty & 0 & 1\\ \lambda'_{0,1}=\lambda_{0,1} & \lambda'_{1,1}=\lambda_{1,1} & \lambda'_{2,1}=\lambda_{2,1}\\ \lambda'_{0,2}=\lambda_{0,2} & \lambda'_{1,2}=\lambda_{1,2}+1 & \lambda_{2,2}=\lambda_{2,2}-1\end{Bmatrix},$$

respectively. Here the operators $P_\lambda=P_{\lambda_{0,1},\lambda_{0,2},\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{2,1}}$ and $P_{\lambda'}$ are given in (1.51). The normalized local solution $u_\lambda(x)$ of $P_\lambda u=0$ corresponding to the exponent $\lambda_{1,2}$ at $x=0$ is

$$x^{\lambda_{1,2}}(1-x)^{\lambda_{2,1}}F(\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,1},\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1},1-\lambda_{1,1}+\lambda_{1,2};x). \tag{11.15}$$

By the reduction $\begin{Bmatrix}x=\infty & 0 & 1\\ \lambda_{0,1} & \lambda_{1,1} & \lambda_{2,1}\\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2}\end{Bmatrix}\to\begin{Bmatrix}x=\infty & 0 & 1\\ \lambda_{0,2}-\mu & \lambda_{1,2}+\mu & \lambda_{2,2}+\mu\end{Bmatrix}$ with $\mu=\lambda_{0,1}+\lambda_{1,1}+\lambda_{2,1}-1$, the contiguity relation (11.12) means

$$\begin{aligned}&x^{\lambda_{1,2}}(1-x)^{\lambda_{2,1}}F(\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,1},\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1},1-\lambda_{1,1}+\lambda_{1,2};x)\\ &=x^{\lambda_{1,2}}(1-x)^{\lambda_{2,1}}F(\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,1},\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1}+1,1-\lambda_{1,1}+\lambda_{1,2};x)\\ &\quad-\frac{\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,1}}{1-\lambda_{1,1}+\lambda_{1,2}}x^{\lambda_{1,2}+1}(1-x)^{\lambda_{2,1}}\\ &\quad\cdot F(\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,1}+1,\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1}+1,2-\lambda_{1,1}+\lambda_{1,2};x),\end{aligned}$$

which is equivalent to the contiguity relation

$$F(\alpha,\beta,\gamma,x)=F(\alpha,\beta+1,\gamma;x)-\frac{\alpha}{\gamma}xF(\alpha+1,\beta+1,\gamma+1;x). \tag{11.16}$$

Using the expression (1.51), we have

$$\begin{aligned}P_{\lambda+\epsilon'}-P_\lambda&=x^2(x-1)\partial+\lambda_{0,1}x^2-(\lambda_{0,1}+\lambda_{2,1})x,\\P_{\lambda+\epsilon'}-P_{\lambda+\epsilon}&=x(x-1)^2\partial+\lambda_{0,1}x^2-(\lambda_{0,1}+\lambda_{1,1})x-\lambda_{1,1},\\(x-1)P_{\lambda+\epsilon}&=\big(x(x-1)\partial+(\lambda_{0,2}-2)x+\lambda_{1,2}+1\big)\big(P_{\lambda+\epsilon'}-P_{\lambda+\epsilon}\big)\\&\qquad-(\lambda_{0,1}+\lambda_{1,1}+\lambda_{2,1})(\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1})x(x-1),\\x^{-1}(x-1)^{-1}\big(x(x-1)\partial&+(\lambda_{0,2}-2)x+\lambda_{1,2}+1\big)(P_{\lambda+\epsilon'}-P_\lambda)-(x-1)^{-1}P_\lambda\\&=-(\lambda_{0,1}+\lambda_{1,1}+\lambda_{2,1})\big(x\partial-\lambda_{1,2}-\frac{\lambda_{2,1}x}{x-1}\big)\end{aligned}$$

and hence (11.11) says

$$R_{\mathbf m}(\epsilon,\lambda)=x\partial-\lambda_{1,2}-\lambda_{2,1}\frac{x}{x-1}. \tag{11.17}$$

In the same way we have

$$R_{\mathbf m}(-\epsilon,\lambda+\epsilon)=(x-1)\partial-\lambda_{2,2}+1-\lambda_{1,1}\frac{x-1}{x}. \tag{11.18}$$

Then

$$\begin{aligned}&R_{\mathbf m}(-\epsilon,\lambda+\epsilon)R_{\mathbf m}(\epsilon,\lambda)-x^{-1}(x-1)^{-1}P_\lambda\\&\qquad=-(\lambda_{0,1}+\lambda_{1,2}+\lambda_{2,1})(\lambda_{0,2}+\lambda_{1,2}+\lambda_{2,1})\end{aligned} \tag{11.19}$$

and since $-R_{\mathbf m}(\epsilon,\tau-\lambda-\epsilon)^*=-\big(x\partial+(\lambda_{1,2}+2)+(\lambda_{2,1}+1)\frac{x}{x-1}\big)^*=x\partial-\lambda_{1,2}-1-(\lambda_{2,1}+1)\frac{x}{x-1}$ with τ given by (11.2), the identity (11.3) means

$$P_\lambda R_{\mathbf m}(\epsilon,\lambda)=\Big(x\partial-(\lambda_{1,2}+1)-(\lambda_{2,1}+1)\frac{x}{x-1}\Big)P_{\lambda+\epsilon}. \tag{11.20}$$

REMARK 11.6. Suppose $\mathbf m$ is irreducibly realizable but it is not rigid. If the reductions of $\{\lambda_{\mathbf m}\}$ and $\{\lambda'_{\mathbf m}\}$ to Riemann schemes with a fundamental tuple of partitions are transformed into each other by suitable additions, we can construct a shift operator as in Theorem 11.2. If they are not so, we need a shift operator for equations whose spectral type are fundamental and such an operator is called a *Schlesinger transformation*.

Now we examine the condition that a universal operator defines a shift operator.

THEOREM 11.7 (universal operator and shift operator). *Let* $\mathbf m=\big(m_{j,\nu}\big)_{\substack{0\le j\le p\\1\le\nu\le n_j}}$ *and* $\mathbf m'=\big(m'_{j,\nu}\big)_{\substack{0\le j\le p\\1\le\nu\le n_j}}\in\mathcal P_{p+1}$ *be irreducibly realizable and monotone. They may not be rigid. Suppose* $\operatorname{ord}\mathbf m>\operatorname{ord}\mathbf m'$. *Fix* j_0 *with* $0\le j_0\le p$. *Let* n'_{j_0} *be a positive integer such that* $m'_{j_0,n'_{j_0}}>m'_{j_0,n'_{j_0}+1}=0$ *and let* $P_{\mathbf m}(\lambda)$ *be the universal operator corresponding to* $\{\lambda_{\mathbf m}\}$. *Putting* $\lambda'_{j,\nu}=\lambda_{j,\nu}$ *when* $(j,\nu)\ne(j_0,n'_{j_0})$, *we define the universal operator* $P^{j_0}_{\mathbf m'}(\lambda):=P_{\mathbf m'}(\lambda')$ *with the Riemann scheme* $\{\lambda'_{\mathbf m'}\}$. *Here* $\lambda'_{j_0,n'_{j_0}}$ *is determined by the Fuchs condition. Then* $(\alpha_{\mathbf m}|\alpha_{\mathbf m'})\le m_{j_0,n'_{j_0}}m'_{j_0,n'_{j_0}}$. *Suppose*

$$(\alpha_{\mathbf m}|\alpha_{\mathbf m'})\Big(=\sum_{j=0}^p\sum_{\nu=1}^{n_j}m_{j,\nu}m'_{j,\nu}-(p-1)\operatorname{ord}\mathbf m\cdot\operatorname{ord}\mathbf m'\Big)=m_{j_0,n'_{j_0}}m'_{j_0,n'_{j_0}}. \tag{11.21}$$

Then $\mathbf m'$ *is rigid and the universal operator* $P^{j_0}_{\mathbf m'}(\lambda)$ *is the shift operator* $R_{\mathbf m}(\epsilon,\lambda)$*:*

$$\begin{aligned}&\Big\{\big[\lambda_{j,\nu}\big]_{(m_{j,\nu})}\Big\}_{\substack{0\le j\le p\\1\le\nu\le n_j}}\xrightarrow{R_{\mathbf m}(\epsilon,\lambda)=P^{j_0}_{\mathbf m'}(\lambda)}\Big\{\big[\lambda_{j,\nu}+\epsilon_{j,\nu}\big]_{(m_{j,\nu})}\Big\}_{\substack{0\le j\le p\\1\le\nu\le n_j}}\\&\qquad\text{with }\ \epsilon_{j,\nu}=\big(1-\delta_{j,j_0}\delta_{\nu,n'_{j_0}}\big)m'_{j,\nu}-\delta_{j,0}\cdot(p-1)\operatorname{ord}\mathbf m'.\end{aligned} \tag{11.22}$$

Proof. We may assume λ is generic. Let $u(x)$ be the solution of the irreducible differential equation $P_{\mathbf{m}}(\lambda)u = 0$. Then

$$\begin{aligned} P_{\mathbf{m}'}(\lambda')(x-c_j)^{\lambda_{j,\nu}}\mathcal{O}_{c_j} &\subset (x-c_j)^{\lambda_{j,\nu}+(1-\delta_{j,j_0}\delta_{\nu,n'_{j_0}})m'_{j,\nu}}\mathcal{O}_{c_j},\\ P_{\mathbf{m}'}(\lambda')x^{-\lambda_{0,\nu}}\mathcal{O}_\infty &\subset x^{-\lambda_{0,\nu}-(1-\delta_{0,j_0}\delta_{\nu,n'_{j_0}})m'_{0,\nu}+(p-1)\operatorname{ord}\mathbf{m}'}\mathcal{O}_\infty \end{aligned}$$

and $P_{\mathbf{m}'}(\lambda')u(x)$ satisfies a Fuchsian differential equation. Hence the fact $R_{\mathbf{m}}(\epsilon,\lambda) = P_{\mathbf{m}'}(\lambda')$ is clear from the characteristic exponents of the equation at each singular points. Note that the left hand side of (11.21) is never larger than the right hand side and if they are not equal, $P_{\mathbf{m}'}(\lambda')u(x)$ satisfies a Fuchsian differential equation with apparent singularities for the solutions $u(x)$ of $P_{\mathbf{m}}(\lambda)u = 0$.

It follows from Lemma 10.3 that the condition (11.21) means that at least one of the irreducibly realizable tuples $\mathbf{m}$ and $\mathbf{m}'$ is rigid and therefore if $\mathbf{m}$ is rigid, so is $\mathbf{m}'$ because $R_{\mathbf{m}}(\epsilon,\lambda)$ is unique up to constant multiple. □

If $\operatorname{ord}\mathbf{m}' = 1$, the condition (11.21) means that $\mathbf{m}$ is of Okubo type, which will be examined in the next section. It will be interesting to examine other cases. When $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ is a rigid decomposition or $\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m})$, we easily have many examples satisfying (11.21).

Here we give such examples of the pairs $(\mathbf{m}\,;\mathbf{m}')$ with $\operatorname{ord}\mathbf{m}' > 1$:

$$\begin{aligned} &(1^n,1^n,n-11\,;1^{n-1},1^{n-1},n-21) && (221,32,32,41\,;\ 110,11,11,20)\\ &(1^{2m},mm-11,m^2\,;1^2,110,1^2) && (1^{2m+1},m^21,m+1m\,;1^2,1^20,11)\\ &(221,221,221\,;110,110,110) && (2111,221,221\,;1100,110,110). \end{aligned} \tag{11.23}$$

11.2. Relation to reducibility

In this section, we will examine whether the shift operator defines a $W(x)$-isomorphism or doesn't.

Theorem 11.8. *Retain the notation in* Theorem 11.2 *and define a polynomial function* $c_{\mathbf{m}}(\epsilon;\lambda)$ *of* $\lambda_{j,\nu}$ *by*

$$R_{\mathbf{m}}(-\epsilon,\lambda+\epsilon)R_{\mathbf{m}}(\epsilon,\lambda) - c_{\mathbf{m}}(\epsilon;\lambda) \in \big(W[x]\otimes\mathbb{C}[\lambda]\big)P_{\mathbf{m}}(\lambda). \tag{11.24}$$

We call $c_{\mathbf{m}}(\epsilon;\lambda)$ *the intertwining polynomial for the differential equation* $P_{\mathbf{m}}(\lambda)u = 0$ *with respect to the shift* ϵ*.*

i) *Fix* $\lambda^o_{j,\nu} \in \mathbb{C}$*. If* $c_{\mathbf{m}}(\epsilon;\lambda^o) \neq 0$*, the equation* $P_{\mathbf{m}}(\lambda^o)u = 0$ *is isomorphic to the equation* $P_{\mathbf{m}}(\lambda^o+\epsilon)v = 0$*. If* $c_{\mathbf{m}}(\epsilon;\lambda^o) = 0$*, then the equations* $P_{\mathbf{m}}(\lambda^o)u = 0$ *and* $P_{\mathbf{m}}(\lambda^o+\epsilon)v = 0$ *are not irreducible.*

ii) *Under the notation in* Proposition 10.16, *there exists a set* Λ *whose elements* (i,k) *are in* $\{1,\ldots,N\}\times\mathbb{Z}$ *such that*

$$c_{\mathbf{m}}(\epsilon;\lambda) = C\prod_{(i,k)\in\Lambda}\big(\ell_i(\lambda)-k\big) \tag{11.25}$$

with a constant $C \in \mathbb{C}^\times$*. Here* Λ *may contain some elements* (i,k) *with multiplicities.*

Proof. Since $u \mapsto R_{\mathbf{m}}(-\epsilon,\lambda+\epsilon)R_{\mathbf{m}}(\epsilon,\lambda)u$ defined an endomorphism of the irreducible equation $P_{\mathbf{m}}(\lambda)u = 0$, the existence of $c_{\mathbf{m}}(\epsilon;\lambda)$ is clear.

If $c_{\mathbf{m}}(\epsilon;\lambda^o) = 0$, the non-zero homomorphism of $P_{\mathbf{m}}(\lambda^o)u = 0$ to $P_{\mathbf{m}}(\lambda^o+\epsilon)v = 0$ defined by $u = R_{\mathbf{m}}(\epsilon;\lambda^o)v$ is not surjective nor injective. Hence the equations are not irreducible. If $c_{\mathbf{m}}(\epsilon;\lambda^o) \neq 0$, then the homomorphism is an isomorphism and the equations are isomorphic to each other.

The claim ii) follows from Proposition 10.16. □

THEOREM 11.9. *Retain the notation in* Theorem 11.8 *with a rigid tuple* $\mathbf{m}$*. Fix a linear function* $\ell(\lambda)$ *of* λ *such that the condition* $\ell(\lambda)=0$ *implies the reducibility of the universal equation* $P_{\mathbf{m}}(\lambda)u=0$*.*

i) *If there is no irreducible realizable subtuple* $\mathbf{m}'$ *of* $\mathbf{m}$ *which is compatible to* $\ell(\lambda)$ *and* $\ell(\lambda+\epsilon)$*,* $\ell(\lambda)$ *is a factor of* $c_{\mathbf{m}}(\epsilon;\lambda)$*.*

If there is no dual decomposition of $\mathbf{m}$ *with respect to the pair* $\ell(\lambda)$ *and* $\ell(\lambda+\epsilon)$*,* $\ell(\lambda)$ *is not a factor of* $c_{\mathbf{m}}(\epsilon;\lambda)$*. Here we define that the decomposition* (10.50) *is dual with respect to the pair* $\ell(\lambda)$ *and* $\ell(\lambda+\epsilon)$ *if the following conditions are valid.*

(11.26) $\mathbf{m}'$ *is an irreducibly realizable subtuple of* $\mathbf{m}$ *compatible to* $\ell(\lambda)$*,*

(11.27) $\mathbf{m}''$ *is a subtuple of* $\mathbf{m}$ *compatible to* $\ell(\lambda+\epsilon)$*.*

ii) *Suppose there exists a decomposition* $\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''$ *with rigid tuples* $\mathbf{m}'$ *and* $\mathbf{m}''$ *such that* $\ell(\lambda)=|\{\lambda_{\mathbf{m}}\}|+k$ *with* $k\in\mathbb{Z}$ *and* $\ell(\lambda+\epsilon)=\ell(\lambda)+1$*. Then* $\ell(\lambda)$ *is a factor of* $c_{\mathbf{m}}(\epsilon;\lambda)$ *if and only if* $k=0$*.*

PROOF. Fix generic complex numbers $\lambda_{j,\nu}\in\mathbb{C}$ satisfying $\ell(\lambda)=|\{\lambda_{\mathbf{m}}\}|=0$. Then we may assume $\lambda_{j,\nu}-\lambda_{j,\nu'}\notin\mathbb{Z}$ for $1\le\nu<\nu'\le n_j$ and $j=0,\dots,p$.

i) The shift operator $R:=R_{\mathbf{m}}(-\epsilon,\lambda+\epsilon)$ gives a non-zero $W(x)$-homomorphism of the equation $P_{\mathbf{m}}(\lambda+\epsilon)v=0$ to $P_{\mathbf{m}}(\lambda)u=0$ by the correspondence $v=Ru$. Since the equation $P_{\mathbf{m}}(\lambda)u=0$ is reducible, we examine the decompositions of $\mathbf{m}$ described in Proposition 10.16. Note that the genericity of $\lambda_{j,\nu}\in\mathbb{C}$ assures that the subtuple $\mathbf{m}'$ of $\mathbf{m}$ corresponding to a decomposition $P_{\mathbf{m}}(\lambda)=P''P'$ is uniquely determined, namely, $\mathbf{m}'$ corresponds to the spectral type of the monodromy of the equation $P'u=0$.

If the shift operator R is bijective, there exists a subtuple $\mathbf{m}'$ of $\mathbf{m}$ compatible to $\ell(\lambda)$ and $\ell(\lambda+\epsilon)$ because R indices an isomorphism of monodromy.

Suppose $\ell(\lambda)$ is a factor of $c_{\mathbf{m}}(\epsilon;\lambda)$. Then R is not bijective. We assume that the image of R is the equation $P''\bar{u}=0$ and the kernel of R is the equation $P'_\epsilon\bar{v}=0$. Then $P_{\mathbf{m}}(\lambda)=P''P'$ and $P_{\mathbf{m}}(\lambda+\epsilon)=P'_\epsilon P''_\epsilon$ with suitable Fuchsian differential operators P' and P''_ϵ. Note that the spectral type of the monodromy of $P'u=0$ and $P''_\epsilon v=0$ corresponds to $\mathbf{m}'$ and $\mathbf{m}''$ with $\mathbf{m}=\mathbf{m}'+\mathbf{m}''$. Applying Proposition 10.16 to the decompositions $P_{\mathbf{m}}(\lambda)=P''P'$ and $P_{\mathbf{m}}(\lambda+\epsilon)=P'_\epsilon P''_\epsilon$, we have a dual decomposition (10.50) of $\mathbf{m}$ with respect to the pair $\ell(\lambda)$ and $\ell(\lambda+\epsilon)$.

ii) Since $P_{\mathbf{m}}(\lambda)u=0$ is reducible, we have a decomposition $P_{\mathbf{m}}(\lambda)=P''P'$ with $0<\operatorname{ord}P'<\operatorname{ord}P_{\mathbf{m}}(\lambda)$. We may assume $P'u=0$ and let $\tilde{\mathbf{m}}'$ be the spectral type of the monodromy of the equation $P'u=0$. Then $\tilde{\mathbf{m}}'=\ell_1\mathbf{m}'+\ell_2\mathbf{m}''$ with integers ℓ_1 and ℓ_2 because $|\{\lambda_{\tilde{\mathbf{m}}'}\}|\in\mathbb{Z}_{\le 0}$. Since $P'u=0$ is irreducible, $2\ge\operatorname{idx}\tilde{\mathbf{m}}'=2(\ell_1^2-\ell_1\ell_2+\ell_2^2)$ and therefore $(\ell_1,\ell_2)=(1,0)$ or $(0,1)$. Hence the claim follows from i) and the identity $|\{\lambda_{\mathbf{m}'}\}|+|\{\lambda_{\mathbf{m}''}\}|=1$ □

REMARK 11.10. i) The reducibility of $P_{\mathbf{m}}(\lambda)$ implies that of the dual of $P_{\mathbf{m}}(\lambda)$.

ii) When $\mathbf{m}$ is simply reducible (cf. Definition 6.15), each linear form of $\lambda_{j,\nu}$ describing the reducibility uniquely corresponds to a rigid decomposition of $\mathbf{m}$ and therefore Theorem 11.9 gives the necessary and sufficient condition for the bijectivity of the shift operator $R_{\mathbf{m}}(\epsilon,\lambda)$.

EXAMPLE 11.11 (EO_4). Let $P(\lambda)u=0$ and $P(\lambda')v=0$ be the Fuchsian differential equation with the Riemann schemes

$$\begin{Bmatrix}\lambda_{0,1} & [\lambda_{1,1}]_{(2)} & [\lambda_{2,1}]_{(2)}\\ \lambda_{0,2} & \lambda_{1,2} & [\lambda_{2,2}]_{(2)}\\ \lambda_{0,3} & \lambda_{1,3} & \\ \lambda_{0,4} & & \end{Bmatrix} \quad\text{and}\quad \begin{Bmatrix}\lambda_{0,1} & [\lambda_{1,1}]_{(2)} & [\lambda_{2,1}]_{(2)}\\ \lambda_{0,2} & \lambda_{1,2} & [\lambda_{2,2}]_{(2)}\\ \lambda_{0,3} & \lambda_{1,3}+1 & \\ \lambda_{0,4}-1 & & \end{Bmatrix},$$

respectively. Since the condition of the reducibility of the equation corresponds to rigid decompositions (10.62), it easily follows from Theorem 11.9 that the shift operator between $P(\lambda)u = 0$ and $P(\lambda')v = 0$ is bijective if and only if

$$\begin{cases} \lambda_{0,4} + \lambda_{1,2} + \lambda_{2,\mu} - 1 \neq 0 & (1 \le \mu \le 2), \\ \lambda_{0,\nu} + \lambda_{0,\nu'} + \lambda_{1,1} + \lambda_{1,3} + \lambda_{2,1} + \lambda_{2,2} - 1 \neq 0 & (1 \le \nu < \nu' \le 3). \end{cases}$$

In general, for a shift $\epsilon = (\epsilon_{j,\nu})$ compatible to the spectral type 1111, 211, 22, the shift operator between $P(\lambda)u = 0$ and $P(\lambda + \epsilon)v = 0$ is bijective if and only if the values of each function in the list

$$\lambda_{0,\nu} + \lambda_{1,1} + \lambda_{2,\mu} \qquad (1 \le \nu \le 4,\ 1 \le \mu \le 2), \tag{11.28}$$

$$\lambda_{0,\nu} + \lambda_{0,\nu'} + \lambda_{1,1} + \lambda_{1,3} + \lambda_{2,1} + \lambda_{2,2} - 1 \qquad (1 \le \nu < \nu' \le 4) \tag{11.29}$$

are

$$\begin{cases} \quad \text{not integers for } \lambda \text{ and } \lambda + \epsilon \\ \text{or positive integers for } \lambda \text{ and } \lambda + \epsilon \\ \text{or non-positive integers for } \lambda \text{ and } \lambda + \epsilon. \end{cases} \tag{11.30}$$

Recall (2.23) and note that the shift operator gives a homomorphism between monodromies.

The following conjecture gives $c_{\mathbf{m}}(\epsilon; \lambda)$ under certain conditions.

Conjecture 11.12. Retain the assumption that $\mathbf{m} = \big(\lambda_{j,\nu}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} \in \mathcal{P}_{p+1}^{(n)}$ is rigid.

i) If $\ell(\lambda) = \ell(\lambda + \epsilon)$ in Theorem 11.9, then $\ell(\lambda)$ is not a factor of $c_{\mathbf{m}}(\epsilon; \lambda)$,

ii) Assume $m_{1,n_1} = m_{2,n_2} = 1$ and

$$\epsilon := \big(\epsilon_{j,\nu}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}}, \quad \epsilon_{j,\nu} = \delta_{j,1}\delta_{\nu,n_1} - \delta_{j,2}\delta_{\nu,n_2}, \tag{11.31}$$

Then we have

$$c_{\mathbf{m}}(\epsilon; \lambda) = C \prod_{\substack{\mathbf{m}=\mathbf{m}'\oplus\mathbf{m}''\\ m'_{1,n_1}=m''_{2,n_2}=1}} |\{\lambda_{\mathbf{m}'}\}| \tag{11.32}$$

with $C \in \mathbb{C}^\times$.

Suppose that the spectral type $\mathbf{m}$ is of *Okubo type*, namely,

$$m_{1,1} + \cdots + m_{p,1} = (p-1) \operatorname{ord} \mathbf{m}. \tag{11.33}$$

Then some shift operators are easily obtained as follows. By a suitable addition we may assume that the Riemann scheme is

$$\begin{Bmatrix} x = \infty & x = c_1 & \cdots & x = c_p \\ [\lambda_{0,1}]_{(m_0,1)} & [0]_{(m_{1,1})} & \cdots & [0]_{(m_{p,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{1,2}]_{(m_{1,2})} & \cdots & [\lambda_{p,2}]_{(m_{p,2})} \\ \vdots & \vdots & \vdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix} \tag{11.34}$$

and the corresponding differential equation $Pu = 0$ is of the form

$$\bar{P}_{\mathbf{m}}(\lambda) = \prod_{j=1}^{p} (x - c_j)^{n - m_{j,1}} \frac{d^n}{dx^n} + \sum_{k=0}^{n-1} \prod_{j=1}^{p} (x - c_j)^{\max\{k - m_{j,1}, 0\}} a_k(x) \frac{d^k}{dx^k}. \tag{11.35}$$

In this case we say that the differential operator P is of *Okubo normal form*. Here $a_k(x)$ is a polynomial of x whose degree is not larger than $k-\sum_{j=1}^{n}\max\{k-m_{j,1},0\}$ for any $k=1,\dots,p$. Moreover we have

$$a_0(x)=\prod_{\nu=1}^{n_0}\prod_{i=0}^{m_{0,\nu}-1}(\lambda_{0,\nu}+i). \tag{11.36}$$

Define the differential operators R_1 and $R_{\mathbf{m}}(\lambda)\in W[x]\otimes\mathbb{C}[\lambda]$ by

$$R_1=\tfrac{d}{dx}\quad\text{and}\quad \bar{P}_{\mathbf{m}}(\lambda)=-R_{\mathbf{m}}(\lambda)R_1+a_0(x). \tag{11.37}$$

Let $P_{\mathbf{m}}(\lambda')v=0$ be the differential equation with the Riemann scheme

$$\begin{Bmatrix} x=\infty & x=c_1 & \cdots & x=c_p\\ [\lambda_{0,1}+1]_{(m_{0,1})} & [0]_{(m_{1,1})} & \cdots & [0]_{(m_{p,1})}\\ [\lambda_{0,2}+1]_{(m_{0,2})} & [\lambda_{1,2}-1]_{(m_{1,2})} & \cdots & [\lambda_{p,2}-1]_{(m_{p,2})}\\ \vdots & \vdots & \vdots & \vdots\\ [\lambda_{0,n_0}+1]_{(m_{0,n_0})} & [\lambda_{1,n_1}-1]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}-1]_{(m_{p,n_p})} \end{Bmatrix}. \tag{11.38}$$

Then the correspondences $u=R_{\mathbf{m}}(\lambda)v$ and $v=R_1u$ give $W(x)$-homomorphisms between the differential equations.

PROPOSITION 11.13. *Let* $\mathbf{m}=\{m_{j,\nu}\}_{\substack{0\le j\le p\\ 1\le\nu\le n_j}}$ *be a rigid tuple of partitions satisfying* (11.33). *Putting*

$$\epsilon_{j,\nu}=\begin{cases}1 & (j=0,\ 1\le\nu\le n_0),\\ \delta_{\nu,0}-1 & (1\le j\le p,\ 1\le\nu\le n_j),\end{cases} \tag{11.39}$$

we have

$$c_{\mathbf{m}}(\epsilon;\lambda)=\prod_{\nu=1}^{n_0}\prod_{i=0}^{m_{0,\nu}-1}(\lambda_{0,\nu}+\lambda_{1,1}+\cdots+\lambda_{p,1}+i). \tag{11.40}$$

PROOF. By suitable additions the proposition follows from the result assuming $\lambda_{j,1}=0$ for $j=1,\dots,p$, which has been shown. □

EXAMPLE 11.14. The generalized hypergeometric equations with the Riemann schemes

$$\begin{Bmatrix} \lambda_{0,1} & \lambda_{1,1} & [\lambda_{2,1}]_{(n-1)}\\ \vdots & \vdots & \\ \lambda_{0,\nu} & \lambda_{1,\nu_o} & \\ \vdots & \vdots & \\ \lambda_{0,n} & \lambda_{1,n} & \lambda_{2,2} \end{Bmatrix}\quad\text{and}\quad \begin{Bmatrix} \lambda_{0,1} & \lambda_{1,1} & [\lambda_{2,1}]_{(n-1)}\\ \vdots & \vdots & \\ \lambda_{0,\nu} & \lambda_{1,\nu_o}+1 & \\ \vdots & \vdots & \\ \lambda_{0,n} & \lambda_{1,n} & \lambda_{2,2}-1 \end{Bmatrix}, \tag{11.41}$$

respectively, whose spectral type is $\mathbf{m}=1^n,1^n,(n-1)1$ are isomorphic to each other by the shift operator if and only if

$$\lambda_{0,\nu}+\lambda_{1,\nu_o}+\lambda_{2,1}\neq 0\quad(\nu=1,\dots,n). \tag{11.42}$$

This statement follows from Proposition 11.13 with suitable additions.

Theorem 11.9 shows that in general $P(\lambda)u=0$ with the Riemann scheme $\{\lambda_{\mathbf{m}}\}$ is $W(x)$-isomorphic to $P(\lambda+\epsilon)v=0$ by the shift operator if and only if the values of the function $\lambda_{0,\nu}+\lambda_{1,\mu}+\lambda_{2,1}$ satisfy (11.30) for $1\le\nu\le n$ and $1\le\mu\le n$. Here ϵ is any shift compatible to $\mathbf{m}$.

The shift operator between

$$\left\{\begin{matrix}\lambda_{0,1} & \lambda_{1,1} & [\lambda_{2,1}]_{(n-1)}\\ \lambda_{0,2} & \lambda_{1,2} & \lambda_{2,2}\\ \vdots & \vdots & \\ \lambda_{0,n} & \lambda_{1,n} & \end{matrix}\right\} \text{ and } \left\{\begin{matrix}\lambda_{0,1} & \lambda_{1,1}+1 & [\lambda_{2,1}]_{(n-1)}\\ \lambda_{0,2} & \lambda_{1,2}-1 & \lambda_{2,2}\\ \vdots & \vdots & \\ \lambda_{0,n} & \lambda_{1,n} & \end{matrix}\right\} \tag{11.43}$$

is bijective if and only if

$$\lambda_{0,\nu}+\lambda_{1,1}+\lambda_{2,1}\neq 0 \text{ and } \lambda_{0,\nu}+\lambda_{1,2}+\lambda_{2,1}\neq 1 \text{ for } \nu=1,\dots,n.$$

Hence if $\lambda_{1,1}=0$ and $\lambda_{1,2}=1$ and $\lambda_{0,1}+\lambda_{2,1}=0$, the shift operator defines a non-zero endomorphism which is not bijective and therefore the monodromy of the space of the solutions are decomposed into a direct sum of the spaces of solutions of two Fuchsian differential equations. The other parameters are generic in this case, the decomposition is unique and the dimension of the smaller space equals 1. When $n=2$ and $(c_0,c_1,c_2)=(\infty,1,0)$ and $\lambda_{2,1}$ and $\lambda_{2,2}$ are generic, the space equals $\mathbb{C}x^{\lambda_{2,1}}\oplus\mathbb{C}x^{\lambda_{2,2}}$

11.3. Polynomial solutions

We characterize some polynomial solutions of a differential equation of Okubo type.

Proposition 11.15. *Retain the notation in* §11.1. *Let* $\bar P_{\mathbf m}(\lambda)u=0$ *be the differential equation with the Riemann scheme* (11.34)*. Suppose that* $\mathbf m$ *is rigid and satisfies* (11.33)*. Moreover suppose* $\lambda_{j,\nu}\notin\mathbb Z$ *for* $j=0,\dots,p$ *and* $\nu=2,\dots,n_j$*. Then the equation* $\bar P_{\mathbf m}(\lambda)u=0$ *has a non-zero polynomial solution if and only if* $-\lambda_{0,1}$ *is a non-negative integer. When* $1-\lambda_{0,1}-m_{0,1}$ *is a non-negative integer* k*, the space of polynomial solutions of the equation is spanned by the polynomials*

$$p_{\lambda,\nu}:=R_{\mathbf m}(\lambda)\circ R_{\mathbf m}(\lambda+\epsilon)\circ\cdots\circ R_{\mathbf m}(\lambda+(k-1)\epsilon)x^\nu \quad (\nu=0,\dots,m_{0,1}-1) \tag{11.44}$$

under the notation (11.37) *and* $\deg p_{\lambda,\nu}=k+\nu$*.*

Proof. Since $\mathbf m=\bigl(m_{0,1}\delta_{1,\nu}\bigr)_{\substack{0\le j\le p\\1\le\nu\le n_j}}\oplus\bigl(m_{j,\nu}-m_{0,1}\delta_{1,\nu}\bigr)_{\substack{0\le j\le p\\1\le\nu\le n_j}}$ is a rigid decomposition of $\mathbf m$, Example 5.5 and (4.56) assure a decomposition $\bar P_{\mathbf m}(\lambda)^*=\partial^{m_{0,1}}P_1$ with a suitable operator $P_1\in W(x)$ when $2-m_{0,1}-\lambda_{0,1}=1$. Moreover Proposition 11.13 assures that $R_{\mathbf m}(\lambda+\ell\epsilon)$ defines an isomorphism of the equation $P_{\mathbf m}(\lambda+(\ell+1)\epsilon)u_{k+1}=0$ to the equation $P_{\mathbf m}(\lambda+\ell\epsilon)u_k=0$ by $u_k=R_{\mathbf m}(\lambda+\ell\epsilon)u_{k+1}$ if $-\lambda_{0,1}-\ell\notin\{0,1,\dots,m_{0,1}-1\}$. Hence the polynomials (11.44) are solutions of $P_{\mathbf m}(\lambda)u=0$. The remaining part of the proposition is clear. □

Remark 11.16. i) Note that we do not assume that $m_{0,1}\ge m_{0,j}$ for $j=1,\dots,n_0$ in Proposition 11.15.

ii) We have not used the assumption that $\mathbf m$ is rigid in Proposition 11.13 and Proposition 11.15 and hence the propositions are valid without this assumption.

iii) As are give in §13.2.3, most rigid spectral types are of Okubo type, namely, satisfy (11.33).

iv) A generalization of the above proposition is given by Remark 13.1 and Theorem 11.7.

v) Suppose P is a Fuchsian differential operator with the Riemann scheme (11.34) satisfying (11.33). Suppose P is of the form (11.35). Since P defines an endomorphism of the linear space of polynomial functions of degree at most m for any non-negative integer m, there exists a monic polynomial p_m of degree m such that p_m is a generalized eigenfunction of P.