CHAPTER 7

A Kac-Moody root system

In this chapter we explain a correspondence between spectral types and roots of a Kac-Moody root system. The correspondence was first introduced by Crawley-Boevey [**CB**]. In §7.2 we study fundamental tuples through this correspondence.

7.1. Correspondence with a Kac-Moody root system

We review a Kac-Moody root system to describe the combinatorial structure of middle convolutions on the spectral types. Its relation to Deligne-Simpson problem is first clarified by [**CB**].

Let

$$I := \{0,\ (j,\nu)\,;\, j = 0,1,\ldots,\ \nu = 1,2,\ldots\}. \tag{7.1}$$

be a set of indices and let $\mathfrak{h}$ be an infinite dimensional real vector space with the set of basis Π, where

$$\Pi = \{\alpha_i\,;\, i \in I\} = \{\alpha_0,\ \alpha_{j,\nu}\,;\, j = 0,1,2,\ldots,\ \nu = 1,2,\ldots\}. \tag{7.2}$$

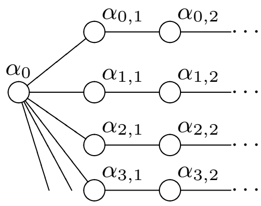
Put

$$I' := I \setminus \{0\}, \qquad \Pi' := \Pi \setminus \{\alpha_0\}, \tag{7.3}$$

$$Q := \sum_{\alpha\in\Pi} \mathbb{Z}\alpha \ \supset\ Q_+ := \sum_{\alpha\in\Pi} \mathbb{Z}_{\geq 0}\alpha. \tag{7.4}$$

We define an indefinite symmetric bilinear form on $\mathfrak{h}$ by

$$\begin{aligned}(\alpha|\alpha) &= 2 \qquad (\alpha \in \Pi),\\ (\alpha_0|\alpha_{j,\nu}) &= -\delta_{\nu,1},\\ (\alpha_{i,\mu}|\alpha_{j,\nu}) &= \begin{cases} 0 & (i \neq j \quad \text{or} \quad |\mu-\nu| > 1),\\ -1 & (i = j \quad \text{and} \quad |\mu-\nu| = 1).\end{cases}\end{aligned} \tag{7.5}$$



The element of Π is called the *simple root* of a Kac-Moody root system and the *Weyl group* W_∞ of this Kac-Moody root system is generated by the *simple reflections* s_i with $i \in I$. Here the *reflection* with respect to an element $\alpha \in \mathfrak{h}$ satisfying $(\alpha|\alpha) \neq 0$ is the linear transformation

$$s_\alpha\ :\ \mathfrak{h} \ni x \mapsto x - 2\frac{(x|\alpha)}{(\alpha|\alpha)}\alpha \in \mathfrak{h} \tag{7.6}$$

and

$$s_i = s_{\alpha_i} \quad \text{for} \quad i \in I. \tag{7.7}$$

In particular $s_i(x) = x - (\alpha_i|x)\alpha_i$ for $i \in I$ and the subgroup of W_∞ generated by s_i for $i \in I \setminus \{0\}$ is denoted by W'_∞.

The Kac-Moody root system is determined by the set of simple roots Π and its Weyl group W_∞ and it is denoted by (Π, W_∞).

Denoting $\sigma(\alpha_0) = \alpha_0$ and $\sigma(\alpha_{j,\nu}) = \alpha_{\sigma(j),\nu}$ for $\sigma \in \mathfrak{S}_\infty$, we put

$$\widetilde{W}_\infty := \mathfrak{S}_\infty \ltimes W_\infty, \tag{7.8}$$

which is an automorphism group of the root system.

REMARK 7.1 ([**Kc**]). The set Δ^{re} of *real roots* equals the W_∞-orbit of Π, which also equals $W_\infty\alpha_0$. Denoting

$$B := \{\beta \in Q_+\,;\ \operatorname{supp}\beta \text{ is connected and } (\beta,\alpha)\le 0 \quad (\forall\alpha\in\Pi)\}, \tag{7.9}$$

the set of *positive imaginary roots* Δ_+^{im} equals $W_\infty B$. Here

$$\operatorname{supp}\beta := \{\alpha\in\Pi\,;\, n_\alpha\ne 0\} \ \text{ if } \ \beta=\sum_{\alpha\in\Pi} n_\alpha\alpha. \tag{7.10}$$

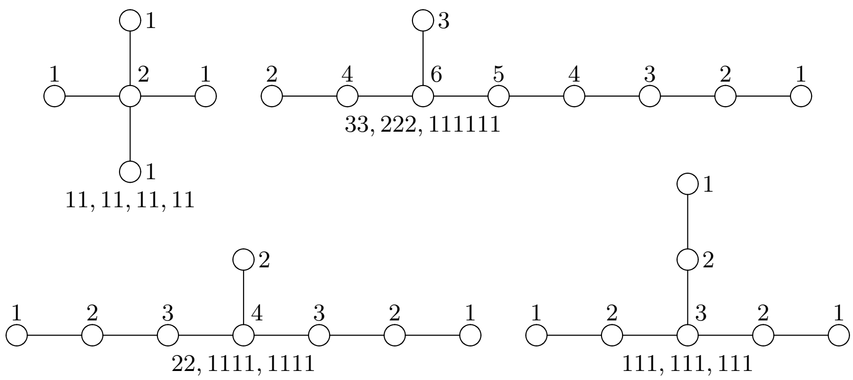
The set Δ of roots equals $\Delta^{re}\cup\Delta^{im}$ by denoting $\Delta_-^{im}=-\Delta_+^{im}$ and $\Delta^{im}=\Delta_+^{im}\cup\Delta_-^{im}$. Put $\Delta_+=\Delta\cap Q_+$, $\Delta_-=-\Delta_+$, $\Delta_+^{re}=\Delta^{re}\cap Q_+$ and $\Delta_-^{re}=-\Delta_+^{re}$. Then $\Delta=\Delta_+\cup\Delta_-$, $\Delta_+^{im}\subset\Delta_+$ and $\Delta^{re}=\Delta_+^{re}\cup\Delta_-^{re}$. The root in Δ is called *positive* if and only if $\alpha\in Q_+$.

A subset $L\subset\Pi$ is called *connected* if the decomposition $L_1\cup L_2=L$ with $L_1\ne\emptyset$ and $L_2\ne\emptyset$ always implies the existence of $v_j\in L_j$ satisfying $(v_1|v_2)\ne 0$. Note that $\operatorname{supp}\alpha\ni\alpha_0$ for $\alpha\in\Delta^{im}$.

The subset L is called classical if it corresponds to the classical Dynkin diagram, which is equivalent to the condition that the group generated by the reflections with respect to the elements in L is a finite group.

The connected subset L is called *affine* if it corresponds to affine Dynkin diagram and in our case it corresponds to $\tilde D_4$ or $\tilde E_8$ or $\tilde E_7$ or $\tilde E_6$ with the following Dynkin diagram, respectively.

(7.11)



Here the circle correspond to simple roots and the numbers attached to simple roots are the coefficients n and $n_{j,\nu}$ in the expression (7.15) of a root α.

For a tuple of partitions $\mathbf{m}=\big(m_{j,\nu}\big)_{j\ge0,\ \nu\ge1}\in\mathcal{P}^{(n)}$, we define

$$\begin{aligned} n_{j,\nu} &:= m_{j,\nu+1}+m_{j,\nu+2}+\cdots, \\ \alpha_{\mathbf{m}} &:= n\alpha_0+\sum_{j=0}^{\infty}\sum_{\nu=1}^{\infty} n_{j,\nu}\alpha_{j,\nu}\in Q_+, \\ \kappa(\alpha_{\mathbf{m}}) &:= \mathbf{m}. \end{aligned} \tag{7.12}$$

As is given in [**O6**, Proposition 2.22] we have

PROPOSITION 7.2. i) $\quad \operatorname{idx}(\mathbf{m},\mathbf{m}')=(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}'})$.

ii) *Given $i\in I$, we have $\alpha_{\mathbf{m}'}=s_i(\alpha_{\mathbf{m}})$ with*

$$\mathbf{m}'=\begin{cases}\partial\mathbf{m} & (i=0),\\ (m_{0,1}\ldots,m_{j,1}\ldots\underset{\smile}{\overset{\nu}{m_{j,\nu+1}}}\underset{\smile}{\overset{\nu+1}{m_{j,\nu}}}\ldots,\ldots) & \big(i=(j,\nu)\big).\end{cases}$$

Moreover for $\ell = (\ell_0, \ell_1, \ldots) \in \mathbb{Z}_{>0}^{\infty}$ *satisfying* $\ell_\nu = 1$ *for* $\nu \gg 1$ *we have*

$$\alpha_\ell := \alpha_{\mathbf{1}_\ell} = \alpha_0 + \sum_{j=0}^{\infty} \sum_{\nu=1}^{\ell_j - 1} \alpha_{j,\nu} = \Bigl(\prod_{j\ge 0} s_{j,\ell_j-1} \cdots s_{j,2} s_{j,1}\Bigr)(\alpha_0), \tag{7.13}$$

$$\alpha_{\partial_\ell(\mathbf{m})} = s_{\alpha_\ell}(\alpha_{\mathbf{m}}) = \alpha_{\mathbf{m}} - 2\frac{(\alpha_{\mathbf{m}}|\alpha_\ell)}{(\alpha_\ell|\alpha_\ell)}\alpha_\ell = \alpha_{\mathbf{m}} - (\alpha_{\mathbf{m}}|\alpha_\ell)\alpha_\ell. \tag{7.14}$$

Note that

$$\begin{aligned} &\alpha = n\alpha_0 + \sum_{j\ge 0}\sum_{\nu\ge 1} n_{j,\nu}\alpha_{j,\nu} \in \Delta_+ \text{ with } n > 0 \\ &\Rightarrow\ n \ge n_{j,1} \ge n_{j,2} \ge \cdots \qquad (j = 0, 1, \ldots). \end{aligned} \tag{7.15}$$

In fact, for a sufficiently large $K \in \mathbb{Z}_{>0}$, we have $n_{j,\mu} = 0$ for $\mu \ge K$ and

$$s_{\alpha_{j,\nu}+\alpha_{j,\nu+1}+\cdots+\alpha_{j,K}}\alpha = \alpha + (n_{j,\nu-1} - n_{j,\nu})(\alpha_{j,\nu} + \alpha_{j,\nu+1} + \cdots + \alpha_{j,K}) \in \Delta^+$$

for $\alpha \in \Delta_+$ in (7.15), which means $n_{j,\nu-1} \ge n_{j,\nu}$ for $\nu \ge 1$. Here we put $n_{j,0} = n$ and $\alpha_{j,0} = \alpha_0$. Hence for $\alpha \in \Delta_+$ with $\operatorname{supp}\alpha \ni \alpha_0$, there uniquely exists $\mathbf{m} \in \mathcal{P}$ satisfying $\alpha = \alpha_{\mathbf{m}}$.

It follows from (7.14) that under the identification $\mathcal{P} \subset Q_+$ with (7.12), our operation ∂_ℓ corresponds to the reflection with respect to the root α_ℓ. Moreover the rigid (resp. indivisible realizable) tuple of partitions corresponds to the positive real root (resp. indivisible positive root) whose support contains α_0, which were first established by [**CB**] in the case of Fuchsian systems of Schlesinger canonical form (cf. [**O6**]).

The corresponding objects with this identification are as follows, which will be clear in this section. Some of them are also explained in [**O6**].

$\mathcal{P}$	Kac-Moody root system
$\mathbf{m}$	$\alpha_{\mathbf{m}}$ (cf. (7.12))
$\mathbf{m}$: monotone	$\alpha \in Q_+ :\ (\alpha\|\beta) \le 0\ \ (\forall\beta \in \Pi')$
$\mathbf{m}$: realizable	$\alpha \in \overline{\Delta}_+$
$\mathbf{m}$: rigid	$\alpha \in \Delta_+^{re}$: $\operatorname{supp}\alpha \ni \alpha_0$
$\mathbf{m}$: monotone and fundamental	$\alpha \in Q_+ : \alpha = \alpha_0$ or $(\alpha\|\beta) \le 0\ \ (\forall\beta \in \Pi)$
$\mathbf{m}$: irreducibly realizable	$\alpha \in \Delta_+$, $\operatorname{supp}\alpha \ni \alpha_0$ indivisible or $(\alpha\|\alpha) < 0$
$\mathbf{m}$: basic and monotone	$\alpha \in Q_+ :\ (\alpha\|\beta) \le 0\ \ (\forall\beta \in \Pi)$ indivisible
$\mathbf{m}$: simply reducible and monotone	$\alpha \in \Delta_+ : (\alpha\|\alpha_{\mathbf{m}}) = 1\ \ \ (\forall\alpha \in \Delta(\mathbf{m}))$ $\alpha_0 \in \Delta(\mathbf{m})$, $(\alpha\|\beta) \le 0\ \ (\forall\beta \in \Pi')$
$\operatorname{ord}\mathbf{m}$	n_0 : $\alpha = n_0\alpha_0 + \sum_{i,\nu} n_{i,\nu}\alpha_{i,\nu}$
$\operatorname{idx}(\mathbf{m}, \mathbf{m}')$	$(\alpha_{\mathbf{m}}\|\alpha_{\mathbf{m}'})$
$\operatorname{idx}\mathbf{m}$	$(\alpha_{\mathbf{m}}\|\alpha_{\mathbf{m}})$
$d_\ell(\mathbf{m})$ (cf. (5.25))	$(\alpha_\ell\|\alpha_{\mathbf{m}})$ (cf. (7.13))
$\operatorname{Pidx}\mathbf{m} + \operatorname{Pidx}\mathbf{m}' = \operatorname{Pidx}(\mathbf{m} + \mathbf{m}')$	$(\alpha_{\mathbf{m}}\|\alpha_{\mathbf{m}'}) = -1$
$(\nu, \nu + 1) \in G_j \subset S'_\infty$ (cf. (4.30))	$s_{j,\nu} \in W'_\infty$ (cf. (7.7))
$H \simeq \mathfrak{S}_\infty$ (cf. (4.30))	$\mathfrak{S}_\infty$ in (7.8)
$\partial_{\mathbf{1}}$	s_0

∂_ℓ	s_{α_ℓ} (cf. (7.13))
$\langle \partial_{\mathbf{1}},\, S_\infty \rangle$	$\widetilde{W}_\infty$ (cf. (7.8))
$\{\lambda_{\mathbf{m}}\}$	$(\Lambda(\lambda), \alpha_{\mathbf{m}})$ (cf. (7.18))
$\lvert\{\lambda_{\mathbf{m}}\}\rvert$	$(\Lambda(\lambda) + \frac{1}{2}\alpha_{\mathbf{m}} \vert \alpha_{\mathbf{m}})$
$\mathrm{Ad}\big((x-c_j)^\tau\big)$	$+\tau\Lambda^0_{0,j}$ (cf. (7.18))

Here

$$\overline{\Delta}_+ := \{k\alpha\,;\, \alpha \in \Delta_+,\ k \in \mathbb{Z}_{>0},\ \operatorname{supp}\alpha \ni \alpha_0\}, \tag{7.16}$$

$\Delta(\mathbf{m}) \subset \Delta_+^{re}$ is given in (7.30) and $\Lambda(\lambda) \in \tilde{\mathfrak{h}}_p$ for $\lambda = (\lambda_{j,\nu})_{\substack{j=0,\dots,p\\ \nu=1,2,\dots}}$ with $\lambda_{j,\nu} \in \mathbb{C}$ is defined as follows.

Definition 7.3. Fix a positive integer p which may be ∞. Put

$$I_p := \{0,\ (j,\nu)\,;\, j = 0,1,\dots,p,\ \nu = 1,2,\dots\} \subset I \tag{7.17}$$

for a positive integer p and $I_\infty = I$.

Let $\mathfrak{h}_p$ be the $\mathbb{R}$-vector space of finite linear combinations the elements of $\Pi_p := \{\alpha_i\,;\, i \in \Pi_p\}$ and let $\mathfrak{h}_p^\vee$ be the $\mathbb{C}$-vector space whose elements are linear combinations of infinite or finite elements of Π_p, which is identified with $\Pi_{i\in I_p}\mathbb{C}\alpha_i$ and contains $\mathfrak{h}_p$.

The element $\Lambda \in \mathfrak{h}_p^\vee$ naturally defines a linear form of $\mathfrak{h}_p$ by $(\Lambda|\cdot)$ and the group $\widetilde{W}_\infty$ acts on $\mathfrak{h}_p^\vee$. If $p = \infty$, we assume that the element $\Lambda = \xi_0\alpha_0 + \sum \xi_{j,\nu}\alpha_{j,\nu} \in \mathfrak{h}_\infty^\vee$ always satisfies $\xi_{j,1} = 0$ for sufficiently large $j \in \mathbb{Z}_{\ge 0}$. Hence we have naturally $\mathfrak{h}_p^\vee \subset \mathfrak{h}_{p+1}^\vee$ and $\mathfrak{h}_\infty^\vee = \bigcup_{j\ge 0}\mathfrak{h}_j^\vee$.

Define the elements of $\mathfrak{h}_p^\vee$:

$$\begin{aligned}
\Lambda_0 &:= \frac{1}{2}\alpha_0 + \frac{1}{2}\sum_{j=0}^{p}\sum_{\nu=1}^{\infty}(1-\nu)\alpha_{j,\nu},\\
\Lambda_{j,\nu} &:= \sum_{i=\nu+1}^{\infty}(\nu-i)\alpha_{j,i} \quad (j=0,\dots,p,\ \nu=0,1,2,\dots),\\
\Lambda^0 &:= 2\Lambda_0 - 2\Lambda_{0,0} = \alpha_0 + \sum_{\nu=1}^{\infty}(1+\nu)\alpha_{0,\nu} + \sum_{j=1}^{p}\sum_{\nu=1}^{\infty}(1-\nu)\alpha_{j,\nu},\\
\Lambda^0_{j,k} &:= \Lambda_{j,0} - \Lambda_{k,0} = \sum_{\nu=1}^{\infty}\nu(\alpha_{k,\nu} - \alpha_{j,\nu}) \quad (0 \le j < k \le p),\\
\Lambda(\lambda) &:= -\Lambda_0 - \sum_{j=0}^{p}\sum_{\nu=1}^{\infty}\Big(\sum_{i=1}^{\nu}\lambda_{j,i}\Big)\alpha_{j,\nu}\\
&= -\Lambda_0 + \sum_{j=0}^{p}\sum_{\nu=1}^{\infty}\lambda_{j,\nu}(\Lambda_{j,\nu-1} - \Lambda_{j,\nu}).
\end{aligned} \tag{7.18}$$

Under the above definition we have

$$(\Lambda^0|\alpha) = (\Lambda^0_{j,k}|\alpha) = 0 \quad (\forall \alpha \in \Pi_p), \tag{7.19}$$

$$(\Lambda_{j,\nu}|\alpha_{j',\nu'}) = \delta_{j,j'}\delta_{\nu,\nu'} \quad (j,\, j' = 0,1,\dots,\ \nu,\, \nu' = 1,2,\dots), \tag{7.20}$$

$$(\Lambda_0|\alpha_i) = (\Lambda_{j,0}|\alpha_i) = \delta_{i,0} \quad (\forall i \in \Pi_p), \tag{7.21}$$

$$\lvert\{\lambda_{\mathbf{m}}\}\rvert = (\Lambda(\lambda) + \tfrac{1}{2}\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}), \tag{7.22}$$

$$
\begin{aligned}
s_0(\Lambda(\lambda)) &= -\Bigl(\sum_{j=0}^{p}\lambda_{j,1}-1\Bigr)\alpha_0+\Lambda(\lambda)\\
&= -\mu\Lambda^0-\Lambda_0-\sum_{\nu=1}^{\infty}\Bigl(\sum_{i=1}^{\nu}\bigl(\lambda_{0,i}-(1+\delta_{i,0})\mu\bigr)\Bigr)\alpha_{0,\nu}\\
&\quad -\sum_{j=1}^{p}\sum_{\nu=1}^{\infty}\Bigl(\sum_{i=1}^{\nu}\bigl(\lambda_{j,i}+(1-\delta_{i,0})\mu\bigr)\Bigr)\alpha_{j,\nu}
\end{aligned}
\tag{7.23}
$$

with $\mu=\sum_{j=0}^{p}\lambda_{j,1}-1$.

We identify the elements of $\mathfrak{h}_p^\vee$ if their difference are in $\mathbb{C}\Lambda^0$, namely, consider them in $\tilde{\mathfrak{h}}_p:=\mathfrak{h}_p^\vee/\mathbb{C}\Lambda^0$. Then the elements have the unique representatives in $\mathfrak{h}_p^\vee$ whose coefficients of α_0 equal $-\frac{1}{2}$.

Remark 7.4. i) If $p<\infty$, we have

$$
\{\Lambda\in\mathfrak{h}_p^\vee\,;\,(\Lambda|\alpha)=0\quad(\forall\alpha\in\Pi_p)\}=\mathbb{C}\Lambda^0+\sum_{j=1}^{p}\mathbb{C}\Lambda_{0,j}^0. \tag{7.24}
$$

ii) The invariance of the bilinear form $(\ |\ )$ under the Weyl group W_∞ proves (5.15).

iii) The addition given in Theorem 5.2 i) corresponds to the map $\Lambda(\lambda)\mapsto\Lambda(\lambda)+\tau\Lambda_{0,j}^0$ with $\tau\in\mathbb{C}$ and $1\le j\le p$.

iv) Combining the action of $s_{j,\nu}$ on $\mathfrak{h}_p^\vee$ with that of s_0, we have

$$
\Lambda(\lambda')-s_{\alpha_\ell}\Lambda(\lambda)\in\mathbb{C}\Lambda^0\quad\text{and}\quad\alpha_{\mathbf{m}'}=s_{\alpha_\ell}\alpha_{\mathbf{m}}\quad\text{when}\quad\{\lambda'_{\mathbf{m}'}\}=\partial_\ell\{\lambda_{\mathbf{m}}\} \tag{7.25}
$$

because of (5.30) and (7.23).

Thus we have the following theorem.

Theorem 7.5. *Under the above notation we have the commutative diagram*

$$
\begin{array}{ccc}
\bigl\{P_{\mathbf{m}}:\text{ Fuchsian differential operators with }\{\lambda_{\mathbf{m}}\}\bigr\} & \to & \bigl\{(\Lambda(\lambda),\alpha_{\mathbf{m}})\,;\,\alpha_{\mathbf{m}}\in\overline{\Delta}_+\bigr\}\\
\downarrow\text{fractional operations} & \circlearrowright & \downarrow W_\infty\text{-action},\ +\tau\Lambda_{0,j}^0\\
\bigl\{P_{\mathbf{m}}:\text{ Fuchsian differential operators with }\{\lambda_{\mathbf{m}}\}\bigr\} & \to & \bigl\{(\Lambda(\lambda),\alpha_{\mathbf{m}})\,;\,\alpha_{\mathbf{m}}\in\overline{\Delta}_+\bigr\}.
\end{array}
$$

Here $\Lambda(\lambda)\in\tilde{\mathfrak{h}}$, *the Riemann schemes* $\{\lambda_{\mathbf{m}}\}=\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\}_{\substack{j=0,\dots,p\\ \nu=1,2,\dots}}$ *satisfy* $|\{\lambda_{\mathbf{m}}\}|=0$ *and the defining domain of* $w\in W_\infty$ *is* $\{\alpha\in\overline{\Delta}_+\,;\,w\alpha\in\overline{\Delta}_+\}$.

Proof. Let T_i denote the corresponding operation on $\{(P_{\mathbf{m}},\{\lambda_{\mathbf{m}}\})\}$ for $s_i\in W_\infty$ with $i\in I$. Then T_0 corresponds to $\partial_{\mathbf{1}}$ and when $i\in I'$, T_i is naturally defined and it doesn't change $P_{\mathbf{m}}$. The fractional transformation of the Fuchsian operators and their Riemann schemes corresponding to an element $w\in W_\infty$ is defined through the expression of w by the product of simple reflections. It is clear that the transformation of their Riemann schemes do not depend on the expression.

Let $i\in I$ and $j\in I$. We want to prove that $(T_iT_j)^k=id$ if $(s_is_j)^k=id$ for a non-negative integer k. Note that $T_i^2=id$ and the addition commutes with T_i. Since $T_i=id$ if $i\in I'$, we have only to prove that $(T_{j,1}T_0)^3=id$. Moreover Proposition 5.8 assures that we may assume $j=0$.

Let P be a Fuchsian differential operator with the Riemann scheme (4.15). Applying suitable additions to P, we may assume $\lambda_{j,1}=0$ for $j\ge1$ to prove $(T_{0,1}T_0)^3P=P$ and then this easily follows from the definition of $\partial_{\mathbf{1}}$ (cf. (5.26)) and the relation

$$\begin{Bmatrix} \infty & c_j \ (1\le j\le p) \\ [\lambda_{0,1}]_{(m_{0,1})} & [0]_{(m_{j,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{j,2}]_{(m_{j,2})} \\ [\lambda_{0,\nu}]_{(m_{0,\nu})} & [\lambda_{j,\nu}]_{(m_{j,\nu})} \end{Bmatrix} \quad (d = m_{0,1}+\cdots+m_{p,1}-\operatorname{ord}\mathbf{m})$$

$$\xrightarrow[\partial^{1-\lambda_{0,1}}]{T_{0,1}T_0} \begin{Bmatrix} \infty & c_j \ (1\le j\le p) \\ [\lambda_{0,2}-\lambda_{0,1}+1]_{(m_{0,1})} & [0]_{(m_{j,1}-d)} \\ [-\lambda_{0,1}+2]_{(m_{0,2}-d)} & [\lambda_{j,2}+\lambda_{0,1}-1]_{(m_{j,2})} \\ [\lambda_{0,\nu}-\lambda_{0,1}+1]_{(m_{0,\nu})} & [\lambda_{j,\nu}+\lambda_{0,1}-1]_{(m_{j,\nu})} \end{Bmatrix}$$

$$\xrightarrow[\partial^{\lambda_{0,1}-\lambda_{0,2}}]{T_{0,1}T_0} \begin{Bmatrix} \infty & c_j \ (1\le j\le p) \\ [-\lambda_{0,2}+2]_{(m_{0,1}-d)} & [0]_{(m_{j,1}+m_{0,1}-m_{0,2}-d)} \\ [\lambda_{0,1}-\lambda_{0,2}+1]_{(m_{0,1})} & [\lambda_{j,2}+\lambda_{0,2}-1]_{(m_{j,2})} \\ [\lambda_{0,\nu}-\lambda_{0,2}+1]_{(m_{0,\nu})} & [\lambda_{j,\nu}+\lambda_{0,2}-1]_{(m_{j,\nu})} \end{Bmatrix}$$

$$\xrightarrow[\partial^{\lambda_{0,2}-1}]{T_{0,1}T_0} \begin{Bmatrix} \infty & c_j \ (1\le j\le p) \\ [\lambda_{0,1}]_{(m_{0,1})} & [0]_{(m_{j,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{j,2}]_{(m_{j,2})} \\ [\lambda_{0,\nu}]_{(m_{0,\nu})} & [\lambda_{j,\nu}]_{(m_{j,\nu})} \end{Bmatrix}$$

and $(T_{0,1}T_0)^3P \in \mathbb{C}[x]\operatorname{Ad}(\partial^{\lambda_{0,2}-1})\circ\operatorname{Ad}(\partial^{\lambda_{0,2}-\lambda_{0,1}})\circ\operatorname{Ad}(\partial^{1-\ \lambda_{0,1}})\operatorname{R}P = \mathbb{C}[x]\operatorname{R}P$. □

DEFINITION 7.6. For an element w of the Weyl group W_∞ we put

$$\Delta(w) := \Delta_+^{re}\cap w^{-1}\Delta_-^{re}. \tag{7.26}$$

If $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ with $i_\nu \in I$ is the *minimal expression* of w as the products of simple reflections which means k is minimal by definition, we have

$$\Delta(w) = \bigl\{\alpha_{i_k}, s_{i_k}(\alpha_{i_{k-1}}), s_{i_k}s_{i_{k-1}}(\alpha_{i_{k-2}}),\ldots, s_{i_k}\cdots s_{i_2}(\alpha_{i_1})\bigr\}. \tag{7.27}$$

The number of the elements of $\Delta(w)$ equals the number of the simple reflections in the minimal expression of w, which is called the *length* of w and denoted by $L(w)$. The equality (7.27) follows from the following lemma.

LEMMA 7.7. *Fix $w \in W_\infty$ and $i \in I$. If $\alpha_i \in \Delta(w)$, there exists a minimal expression $w = s_{i'_1}s_{i'_2}\cdots s_{i'_k}$ with $s_{i'_k} = s_i$ and $L(ws_i) = L(w)-1$ and $\Delta(ws_i) = s_i\bigl(\Delta(w)\setminus\{\alpha_i\}\bigr)$. If $\alpha_i \notin \Delta(w)$, $L(ws_i) = L(w)+1$ and $\Delta(ws_i) = s_i\Delta(w)\cup\{\alpha_i\}$. Moreover if $v \in W_\infty$ satisfies $\Delta(v) = \Delta(w)$, then $v = w$.*

PROOF. The proof is standard as in the case of classical root system, which follows from the fact that the condition $\alpha_i = s_{i_k}\cdots s_{i_{\ell+1}}(\alpha_{i_\ell})$ implies

$$s_i = s_{i_k}\cdots s_{i_{\ell+1}}s_{i_\ell}s_{i_{\ell+1}}\cdots s_{i_k} \tag{7.28}$$

and then $w = ws_is_i = s_{i_1}\cdots s_{i_{\ell-1}}s_{i_{\ell+1}}\cdots s_{i_k}s_i$. □

DEFINITION 7.8. For $\alpha \in Q$, put

$$h(\alpha) := n_0 + \sum_{j\ge0}\sum_{\nu\ge1} n_{j,\nu} \ \text{ if } \ \alpha = n_0\alpha_0 + \sum_{j\ge0}\sum_{\nu\ge1} n_{j,\nu}\alpha_{j,\nu} \in Q. \tag{7.29}$$

Suppose $\mathbf{m} \in \mathcal{P}_{p+1}$ is irreducibly realizable. Note that $sf\mathbf{m}$ is the monotone fundamental element determined by $\mathbf{m}$, namely, $\alpha_{sf\mathbf{m}}$ is the unique element of $W\alpha_{\mathbf{m}} \cap \bigl(B\cup\{\alpha_0\}\bigr)$. We inductively define $w_{\mathbf{m}} \in W_\infty$ satisfying $w_{\mathbf{m}}\alpha_{\mathbf{m}} = \alpha_{sf\mathbf{m}}$. We may assume $w_{\mathbf{m}'}$ has already defined if $h(\alpha_{\mathbf{m}'}) < h(\alpha_{\mathbf{m}})$. If $\mathbf{m}$ is not monotone, there exists $i \in I\setminus\{0\}$ such that $(\alpha_{\mathbf{m}}|\alpha_i) > 0$ and then $w_{\mathbf{m}} = w_{\mathbf{m}'}s_i$ with $\alpha_{\mathbf{m}'} = s_i\alpha_{\mathbf{m}}$. If $\mathbf{m}$ is monotone and $\mathbf{m} \ne f\mathbf{m}$, $w_{\mathbf{m}} = w_{\partial\mathbf{m}}s_0$.

We moreover define

$$\Delta(\mathbf{m}) := \Delta(w_{\mathbf{m}}). \tag{7.30}$$

Suppose $\mathbf{m}$ is monotone, irreducibly realizable and $\mathbf{m} \neq sf\mathbf{m}$. We define $w_{\mathbf{m}}$ so that there exists $K \in \mathbb{Z}_{>0}$ and $v_1, \ldots, v_K \in W'_\infty$ satisfying

$$\begin{aligned} &w_{\mathbf{m}} = v_K s_0 \cdots v_2 s_0 v_1 s_0, \\ &(v_k s_0 \cdots v_1 s_0 \alpha_{\mathbf{m}} | \alpha) \leq 0 \quad (\forall \alpha \in \Pi \setminus \{0\},\ k = 1, \ldots, K), \end{aligned} \tag{7.31}$$

which uniquely characterizes $w_{\mathbf{m}}$. Note that

$$v_k s_0 \cdots v_1 s_0 \alpha_{\mathbf{m}} = \alpha_{(s\partial)^k \mathbf{m}} \quad (k = 1, \ldots, K). \tag{7.32}$$

The following proposition gives the correspondence between the reduction of realizable tuples of partitions and the minimal expressions of the elements of the Weyl group.

Proposition 7.9. Definition 7.8 *naturally gives the product expression* $w_{\mathbf{m}} = s_{i_1} \cdots s_{i_k}$ *with* $i_\nu \in I \ \ (1 \leq \nu \leq k)$.

i) *We have*

$$L(w_{\mathbf{m}}) = k, \tag{7.33}$$

$$(\alpha | \alpha_{\mathbf{m}}) > 0 \quad (\forall \alpha \in \Delta(\mathbf{m})), \tag{7.34}$$

$$h(\alpha_{\mathbf{m}}) = h(\alpha_{sf\mathbf{m}}) + \sum_{\alpha \in \Delta(\mathbf{m})} (\alpha | \alpha_{\mathbf{m}}). \tag{7.35}$$

Moreover $\alpha_0 \in \operatorname{supp} \alpha$ *for* $\alpha \in \Delta(\mathbf{m})$ *if* $\mathbf{m}$ *is monotone.*

ii) *Suppose* $\mathbf{m}$ *is monotone and* $f\mathbf{m} \neq \mathbf{m}$. *Fix maximal integers* ν_j *such that* $m_{j,1} - d_{max}(\mathbf{m}) < m_{j,\nu_j+1}$ *for* $j = 0, 1, \ldots$ *Then*

$$\begin{aligned} \Delta(\mathbf{m}) = s_0 \Bigl(\prod_{\substack{j \geq 0 \\ \nu_j > 0}} s_{j,1} \cdots s_{j,\nu_j} \Bigr) \Delta(s\partial \mathbf{m}) \cup \{\alpha_0\} \\ \cup \{\alpha_0 + \alpha_{j,1} + \cdots + \alpha_{j,\nu}\,;\, 1 \leq \nu \leq \nu_j \text{ and } j = 0, 1, \ldots\}, \end{aligned} \tag{7.36}$$

$$(\alpha_0 + \alpha_{j,1} + \cdots + \alpha_{j,\nu} | \alpha_{\mathbf{m}}) = d_{max}(\mathbf{m}) + m_{j,\nu+1} - m_{j,1} \quad (\nu \geq 0). \tag{7.37}$$

iii) *Suppose* $\mathbf{m}$ *is not rigid. Then* $\Delta(\mathbf{m}) = \{\alpha \in \Delta_+^{re}\,;\, (\alpha|\alpha_{\mathbf{m}}) > 0\}$.

iv) *Suppose* $\mathbf{m}$ *is rigid. Let* $\alpha \in \Delta_+^{re}$ *satisfying* $(\alpha|\alpha_{\mathbf{m}}) > 0$ *and* $s_\alpha(\alpha_{\mathbf{m}}) \in \Delta_+$. *Then*

$$\begin{cases} \alpha \in \Delta(\mathbf{m}) & \text{if } (\alpha|\alpha_{\mathbf{m}}) > 1, \\ \#\Bigl(\{\alpha,\, \alpha_{\mathbf{m}} - \alpha\} \cap \Delta(\mathbf{m})\Bigr) = 1 & \text{if } (\alpha|\alpha_{\mathbf{m}}) = 1. \end{cases} \tag{7.38}$$

Moreover if a root $\gamma \in \Delta(\mathbf{m})$ *satisfies* $(\gamma|\alpha_{\mathbf{m}}) = 1$, *then* $\alpha_{\mathbf{m}} - \gamma \in \Delta_+^{re}$ *and* $\alpha_0 \in \operatorname{supp}(\alpha_{\mathbf{m}} - \gamma)$.

v) $w_{\mathbf{m}}$ *is the unique element with the minimal length satisfying* $w_{\mathbf{m}}\alpha_{\mathbf{m}} = \alpha_{sf\mathbf{m}}$.

Proof. Since $h(s_{i'}\alpha) - h(\alpha) = -(\alpha_{i'}|\alpha) = (s_{i'}\alpha_{i'}|\alpha)$, we have

$$\begin{aligned} h(s_{i'_\ell} \cdots s_{i'_1}\alpha) - h(\alpha) &= \sum_{\nu=1}^{\ell} \Bigl(h(s_{i'_\nu} \cdots s_{i'_1}\alpha) - h(s_{i'_{\nu-1}} \cdots s_{i'_1}\alpha) \Bigr) \\ &= \sum_{\nu=1}^{\ell} (\alpha_{i'_\nu} | s_{i'_\nu} \cdots s_{i'_1}\alpha) = \sum_{\nu=1}^{\ell} (s_{i'_\ell} \cdots s_{i'_{\nu+1}} \alpha_{i'_\nu} | s_{i'_\ell} \cdots s_{i'_1}\alpha) \end{aligned}$$

for i', $i'_\nu \in I$ and $\alpha \in \Delta$.

i) We show by the induction on k. We may assume $k \geq 1$. Put $w' = s_{i_1} \cdots s_{i_{k-1}}$ and $\alpha_{\mathbf{m}'} = s_{i_k}\alpha_{\mathbf{m}}$ and $\alpha(\nu) = s_{i_{k-1}} \cdots s_{i_{\nu+1}} \alpha_{i_\nu}$ for $\nu = 1, \ldots, k-1$. The hypothesis of the induction assures $L(w') = k - 1$, $\Delta(\mathbf{m}') = \{\alpha(1), \ldots, \alpha(k-1)\}$ and

$(\alpha(\nu)|\alpha_{\mathbf{m}'}) > 0$ for $\nu = 1,\ldots,k-1$. If $L(w_{\mathbf{m}}) \neq k$, there exists ℓ such that $\alpha_{i_k} = \alpha(\ell)$ and $w_{\mathbf{m}} = s_{i_1}\cdots s_{i_{\ell-1}}s_{i_{\ell+1}}\cdots s_{i_{k-1}}$ is a minimal expression. Then $h(\alpha_{\mathbf{m}}) - h(\alpha_{\mathbf{m}'}) = -(\alpha_{i_k}|\alpha_{\mathbf{m}'}) = -(\alpha(\ell)|\alpha_{\mathbf{m}'}) < 0$, which contradicts to the definition of $w_{\mathbf{m}}$. Hence we have i). Note that (7.34) implies $\operatorname{supp}\alpha \ni \alpha_0$ if $\alpha \in \Delta(\mathbf{m})$ and $\mathbf{m}$ is monotone.

ii) The equality (7.36) follows from

$$\Delta(\partial\mathbf{m}) \cap \sum_{\alpha\in\Pi\setminus\{0\}} \mathbb{Z}\alpha = \{\alpha_{j,1} + \cdots + \alpha_{j,\nu_j}\,;\, \nu = 1,\ldots,\nu_j,\ \nu_j > 0 \text{ and } j = 0,1,\ldots\}$$

because $\Delta(\mathbf{m}) = s_0\Delta(\partial\mathbf{m}) \cup \{\alpha_0\}$ and $\Bigl(\prod_{\substack{j\ge 0\\ \nu_j>0}} s_{j,\nu_j}\cdots s_{j,1}\Bigr)\alpha_{\partial\mathbf{m}} = \alpha_{s\partial\mathbf{m}}$.

The equality (7.37) follows from $(\alpha_0|\alpha_{\mathbf{m}}) = d_{\mathbf{1}}(\mathbf{m}) = d_{max}(\mathbf{m})$ and $(\alpha_{j,\nu}|\alpha_{\mathbf{m}}) = m_{j,\nu+1} - m_{j,\nu}$.

iii) Note that $\gamma \in \Delta(\mathbf{m})$ satisfies $(\gamma|\alpha_{\mathbf{m}}) > 0$.

Put $w_\nu = s_{i_{\nu+1}}\cdots s_{i_{k-1}}s_{i_k}$ for $\nu = 0,\ldots,k$. Then $w_{\mathbf{m}} = w_0$ and $\Delta(\mathbf{m}) = \{w_\nu^{-1}\alpha_{i_\nu}\,;\, \nu = 1,\ldots,k\}$. Moreover $w_{\nu'}w_\nu^{-1}\alpha_{i_\nu} \in \Delta_-^{re}$ if and only if $0 \le \nu' < \nu$.

Suppose $\mathbf{m}$ is not rigid. Let $\alpha \in \Delta_+^{re}$ with $(\alpha|\alpha_{\mathbf{m}}) > 0$. Since $(w_{\mathbf{m}}\alpha|\alpha_{\overline{\mathbf{m}}}) > 0$, $w_{\mathbf{m}}\alpha \in \Delta_-^{re}$. Hence there exists ν such that $w_\nu\alpha \in \Delta_+$ and $w_{\nu-1}\alpha \in \Delta_-$, which implies $w_\nu\alpha = \alpha_{i_\nu}$ and the claim.

iv) Suppose $\mathbf{m}$ is rigid. Let $\alpha \in \Delta_+^{re}$. Put $\ell = (\alpha|\alpha_{\mathbf{m}})$. Suppose $\ell > 0$ and $\beta := s_\alpha\alpha_{\mathbf{m}} \in \Delta_+$. Then $\alpha_{\mathbf{m}} = \ell\alpha + \beta$, $\alpha_0 = \ell w_{\mathbf{m}}\alpha + w_{\mathbf{m}}\beta$ and $(\beta|\alpha_{\mathbf{m}}) = (\alpha_{\mathbf{m}} - \ell\alpha|\alpha_{\mathbf{m}}) = 2 - \ell^2$. Hence if $\ell \ge 2$, $\mathbb{R}\beta \cap \Delta(\mathbf{m}) = \emptyset$ and the same argument as in the proof of iii) assures $\alpha \in \Delta(\mathbf{m})$.

Suppose $\ell = 1$. There exists ν such that $w_\nu\alpha$ or $w_\nu\beta$ equals α_{i_ν}. We may assume $w_\nu^{-1}\alpha = \alpha_{i_\nu}$. Then $\alpha \in \Delta(\mathbf{m})$.

Suppose there exists $w_{\nu'}\beta = \alpha_{i_{\nu'}}$. We may assume $\nu' < \nu$. Then $w_{\nu'}\alpha_{\mathbf{m}} = w_{\nu'-1}\alpha + w_{\nu'-1}\beta \in \Delta_-^{re}$, which contradicts to the definition of w_ν. Hence $w_{\nu'}\beta = \alpha_{i_{\nu'}}$ for $\nu' = 1,\ldots,k$ and therefore $\beta \notin \Delta(\mathbf{m})$.

Let $\gamma = w_\nu^{-1}\alpha_{i_\nu} \in \Delta(\mathbf{m})$ and $(\gamma|\alpha_{\mathbf{m}}) = 1$. Put $\beta = \alpha_{\mathbf{m}} - \alpha = s_\alpha\alpha_{\mathbf{m}}$. Then $w_{\nu-1}\alpha_{\mathbf{m}} = w_\nu\beta \in \Delta_+^{re}$. Since $\beta \notin \Delta(\mathbf{m})$, we have $\beta \in \Delta_+^{re}$.

Replacing $\mathbf{m}$ by $s\mathbf{m}$, we may assume $\mathbf{m}$ is monotone to prove $\alpha_0 \in \operatorname{supp}\beta$. Since $(\beta|\alpha_{\mathbf{m}}) = 1$ and $(\alpha_i|\alpha_{\mathbf{m}}) \le 0$ for $i \in I \setminus \{0\}$, we have $\alpha_0 \in \operatorname{supp}\beta$.

v) The uniqueness of $w_{\mathbf{m}}$ follows from iii) when $\mathbf{m}$ is not rigid. It follows from (7.34), Theorem 15.1 and Corollary 15.3 when $\mathbf{m}$ is rigid. □

COROLLARY 7.10. *Let* $\mathbf{m}$, $\mathbf{m}'$, $\mathbf{m}'' \in \mathcal{P}$ *and* $k \in \mathbb{Z}_{>0}$ *such that*

$$\mathbf{m} = k\mathbf{m}' + \mathbf{m}'',\ \operatorname{idx}\mathbf{m} = \operatorname{idx}\mathbf{m}'' \text{ and } \mathbf{m}' \text{ is rigid.} \tag{7.39}$$

Then $\mathbf{m}$ *is irreducibly realizable if and only if so is* $\mathbf{m}''$.

Suppose $\mathbf{m}$ *is irreducibly realizable. If* $\operatorname{idx}\mathbf{m} \le 0$ *or* $k > 1$, *then* $\mathbf{m}' \in \Delta(\mathbf{m})$. *If* $\operatorname{idx}\mathbf{m} = 2$, *then* $\{\alpha_{\mathbf{m}'}, \alpha_{\mathbf{m}''}\} \cap \Delta(\mathbf{m}) = \{\alpha_{\mathbf{m}'}\}$ *or* $\{\alpha_{\mathbf{m}''}\}$.

PROOF. The assumption implies $(\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}}) = 2k^2 + 2k(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) + (\alpha_{\mathbf{m}''}|\alpha_{\mathbf{m}''})$ and hence $(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -k$ and $s_{\alpha_{\mathbf{m}'}}\alpha_{\mathbf{m}''} = \alpha_{\mathbf{m}}$. Thus we have the first claim (cf. Theorem 7.5). The remaining claims follow from Proposition 7.9. □

REMARK 7.11. i) In general, $\gamma \in \Delta(\mathbf{m})$ does not always imply $s_\gamma\alpha_{\mathbf{m}} \in \Delta_+$.

Put $\mathbf{m} = 32,32,32,32$, $\mathbf{m}' = 10,10,10,10$ and $\mathbf{m}'' = 01,01,01,01$. Putting $v = s_{0,1}s_{1,1}s_{2,1}s_{3,1}$, we have $\alpha_{\mathbf{m}'} = \alpha_0$, $\alpha_{\mathbf{m}''} = v\alpha_0$, $(\alpha_{\mathbf{m}'}|\alpha_{\mathbf{m}''}) = -2$, $s_0\alpha_{\mathbf{m}''} = 2\alpha_{\mathbf{m}'}+\alpha_{\mathbf{m}''}$, $vs_0\alpha_{\mathbf{m}''} = \alpha_0+2\alpha_{\mathbf{m}''}$ and $s_0vs_0v\alpha_0 = s_0vs_0\alpha_{\mathbf{m}''} = 3\alpha_{\mathbf{m}'}+2\alpha_{\mathbf{m}''} = \alpha_{\mathbf{m}}$.

Then $\gamma := s_0v\alpha_0 = 2\alpha_{\mathbf{m}'} + \alpha_{\mathbf{m}''} \in \Delta(\mathbf{m})$, $(\gamma|\alpha_{\mathbf{m}}) = (s_0v\alpha_{\mathbf{m}'}|s_0vs_0v\alpha_{\mathbf{m}'}) = (\alpha_{\mathbf{m}'}|s_0v\alpha_{\mathbf{m}'}) = (\alpha_{\mathbf{m}'}|2\alpha_{\mathbf{m}'} + \alpha_{\mathbf{m}''}) = 2$ and $s_\gamma(\alpha_{\mathbf{m}}) = (3\alpha_{\mathbf{m}'} + 2\alpha_{\mathbf{m}''}) - 2(2\alpha_{\mathbf{m}'} + \alpha_{\mathbf{m}''}) = -\alpha_{\mathbf{m}'} \in \Delta_-$.

ii) Define

$$[\Delta(\mathbf{m})] := \big\{(\alpha|\alpha_{\mathbf{m}})\,;\,\alpha\in\Delta(\mathbf{m})\big\}. \tag{7.40}$$

Then $[\Delta(\mathbf{m})]$ gives a partition of the non-negative integer $h(\alpha_{\mathbf{m}})-h(sf\mathbf{m})$, which we call *the type of* $\Delta(\mathbf{m})$. It follows from (7.35) that

$$\#\Delta(\mathbf{m})\le h(\alpha_{\mathbf{m}})-h(sf\mathbf{m}) \tag{7.41}$$

for a realizable tuple $\mathbf{m}$ and the equality holds in the above if $\mathbf{m}$ is monotone and simply reducible. Moreover we have

$$[\Delta(\mathbf{m})]=[\Delta(s\partial\mathbf{m})]\cup\{d(\mathbf{m})\}\cup\bigcup_{j=0}^{p}\{m_{j,\nu}-m_{j,1}-d(\mathbf{m})\in\mathbb{Z}_{>0}\,;\,\nu>1\}, \tag{7.42}$$

$$\#\Delta(\mathbf{m})=\#\Delta(s\partial\mathbf{m})+\sum_{j=0}^{p}\Big(\min\{\nu\,;\,m_{j,\nu}>m_{j,1}-d(\mathbf{m})\}-1\Big)+1, \tag{7.43}$$

$$h(\mathbf{m})=h(sf\mathbf{m})+\sum_{i\in[\Delta(\mathbf{m})]}i \tag{7.44}$$

if $\mathbf{m}\in\mathcal{P}_{p+1}$ is monotone, irreducibly realizable and not fundamental. Here we use the notation in Definitions 4.11, 5.7 and 6.15. For example,

type	$\mathbf{m}$	$h(\alpha_{\mathbf{m}})$	$\#\Delta(\mathbf{m})$
H_n	$1^n,1^n,n-11$	n^2+1	n^2
EO_{2m}	$1^{2m},mm,mm-11$	$2m^2+3m+1$	$\binom{2m}{2}+4m$
EO_{2m+1}	$1^{2m+1},m+1m,mm1$	$2m^2+5m+3$	$\binom{2m+1}{2}+4m+2$
X_6	$111111,222,42$	29	28
	$21111,222,33$	25	24
P_n	$n-11,n-11,\ldots\in\mathcal{P}_{n+1}^{(n)}$	$2n+1$	$[\Delta(\mathbf{m})]:1^{n+1}\cdot(n-1)$
$P_{4,2m+1}$	$m+1m,m+1m,m+1m,m+1m$	$6m+1$	$[\Delta(\mathbf{m})]:1^{4m}\cdot 2^m$

Suppose $\mathbf{m}\in\mathcal{P}_{p+1}$ is basic. We may assume (6.3). Suppose $(\alpha_{\mathbf{m}}|\alpha_0)=0$, which is equivalent to $\sum_{j=0}^{p}m_{j,1}=(p-1)\operatorname{ord}\mathbf{m}$. Let k_j be positive integers such that

$$(\alpha_{\mathbf{m}}|\alpha_{j,\nu})=0\ \text{ for }\ 1\le\nu<k_j\ \text{ and }\ (\alpha_{\mathbf{m}}|\alpha_{j,k_j})<0, \tag{7.45}$$

which is equivalent to $m_{j,1}=m_{j,2}=\cdots=m_{j,k_j}>m_{j,k_j+1}$ for $j=0,\ldots,p$. Then

$$\sum_{j=0}^{p}\frac{1}{k_j}\ge\sum_{j=0}^{p}\frac{m_{j,1}}{\operatorname{ord}\mathbf{m}}=p-1. \tag{7.46}$$

If the equality holds in the above, we have $k_j\ge 2$ and $m_{j,k_j+1}=0$ and therefore $\mathbf{m}$ is of one of the types $\tilde{D}_4$ or $\tilde{E}_6$ or $\tilde{E}_7$ or $\tilde{E}_8$. Hence if $\operatorname{idx}\mathbf{m}<0$, the set $\{k_j\,;\,0\le j\le p,\ k_j>1\}$ equals one of the set $\emptyset$, $\{2\}$, $\{2,\nu\}$ with $2\le\nu\le5$, $\{3,\nu\}$ with $3\le\nu\le5$, $\{2,2,\nu\}$ with $2\le\nu\le5$ and $\{2,3,\nu\}$ with $3\le\nu\le5$. In this case the corresponding Dynkin diagram of $\{\alpha_0,\alpha_{j,\nu}\,;\,1\le\nu<k_j,\ j=0,\ldots,p\}$ is one of the types A_ν with $1\le\nu\le6$, D_ν with $4\le\nu\le7$ and E_ν with $6\le\nu\le8$. Thus we have the following remark.

Remark 7.12. Suppose a tuple $\mathbf{m}\in\mathcal{P}_{p+1}^{(n)}$ is basic and monotone. The subgroup of W_∞ generated by reflections with respect to α_ℓ (cf. (7.13)) which satisfy $(\alpha_{\mathbf{m}}|\alpha_\ell)=0$ is infinite if and only if $\operatorname{idx}\mathbf{m}=0$.

For a realizable monotone tuple $\mathbf{m}\in\mathcal{P}$, we define

$$\Pi(\mathbf{m}):=\{\alpha_{j,\nu}\in\operatorname{supp}\alpha_{\mathbf{m}}\,;\,m_{j,\nu}=m_{j,\nu+1}\}\cup\begin{cases}\{\alpha_0\} & (d_{\mathbf{1}}(\mathbf{m})=0),\\ \emptyset & (d_{\mathbf{1}}(\mathbf{m})\ne0).\end{cases} \tag{7.47}$$

Note that the condition $(\alpha_{\mathbf{m}}|\alpha_\ell)=0$, which is equivalent to say that α_ℓ is a root of the root space with the fundamental system $\Pi(\mathbf{m})$, means that the corresponding middle convolution ∂_ℓ keeps the spectral type invariant.

7.2. Fundamental tuples

We will prove some inequalities (7.48) and (7.49) for fundamental tuples which are announced in [**O6**].

PROPOSITION 7.13. *Let $\mathbf{m}\in\mathcal{P}_{p+1}\setminus\mathcal{P}_p$ be a fundamental tuple. Then*

$$\operatorname{ord}\mathbf{m}\le 3|\operatorname{idx}\mathbf{m}|+6, \tag{7.48}$$

$$\operatorname{ord}\mathbf{m}\le |\operatorname{idx}\mathbf{m}|+2 \quad \textit{if}\ \ p\ge 3, \tag{7.49}$$

$$p\le \tfrac{1}{2}|\operatorname{idx}\mathbf{m}|+3. \tag{7.50}$$

EXAMPLE 7.14. For a positive integer m we have special 4 elements

$$\begin{aligned} &D_4^{(m)}: m^2,m^2,m^2,m(m-1)1 && E_6^{(m)}: m^3,m^3,m^2(m-1)1\\ &E_7^{(m)}: (2m)^2,m^4,m^3(m-1)1 && E_8^{(m)}: (3m)^2,(2m)^3,m^5(m-1)1 \end{aligned} \tag{7.51}$$

with orders $2m$, $3m$, $4m$ and $6m$, respectively, and index of rigidity $2-2m$.

Note that $E_8^{(m)}$, $D_4^{(m)}$ and $11,11,11,\dots\in\mathcal{P}_{p+1}^{(2)}$ attain the equalities (7.48), (7.49) and (7.50), respectively.

REMARK 7.15. It follows from the Proposition 7.13 that there exist only finite basic tuples $\mathbf{m}\in\mathcal{P}$ with a fixed index of rigidity under the normalization (6.3). This result is given in [**O6**, Proposition 8.1] and a generalization is given in [**HiO**].

Hence Proposition 7.13 assures that there exist only finite *fundamental universal Fuchsian differential operators* with a fixed number of accessory parameters. Here a fundamental universal Fuchsian differential operator means a universal operator given in Theorem 6.14 whose spectral type is fundamental (cf. Definition 6.15).

Now we prepare a lemma.

LEMMA 7.16. *Let $a\ge 0$, $b>0$ and $c>0$ be integers such that $a+c-b>0$. Then*

$$\frac{b+kc-6}{(a+c-b)b}\begin{cases}<k+1 & (0\le k\le 5),\\ \le 7 & (0\le k\le 6).\end{cases}$$

PROOF. Suppose $b\ge c$. Then

$$\frac{b+kc-6}{(a+c-b)b}\le\frac{b+kb-6}{b}<k+1.$$

Next suppose $b<c$. Then

$$\begin{aligned}(k+1)(a+c-b)b-(b+kc-6)&\ge(k+1)(c-b)b-b-kc+6\\ &\ge(k+1)b-b-k(b+1)+6=6-k.\end{aligned}$$

Thus we have the lemma. □

PROOF OF PROPOSITION 7.13. Since $\operatorname{idx}k\mathbf{m}=k^2\operatorname{idx}\mathbf{m}$ for a basic tuple $\mathbf{m}$ and $k\in\mathbb{Z}_{>0}$, we may assume that $\mathbf{m}$ is basic and $\operatorname{idx}\mathbf{m}\le -2$ to prove the proposition.

Fix a basic monotone tuple $\mathbf{m}$. Put $\alpha=\alpha_{\mathbf{m}}$ under the notation (7.12) and $n=\operatorname{ord}\mathbf{m}$. Note that

$$(\alpha|\alpha)=n(\alpha|\alpha_0)+\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}n_{j,\nu}(\alpha|\alpha_{j,\nu}),\quad (\alpha|\alpha_0)\le 0,\quad (\alpha|\alpha_{j,\nu})\le 0. \tag{7.52}$$

We first assume that (7.48) is not valid, namely,

$$3|(\alpha|\alpha)| + 6 < n. \tag{7.53}$$

In view of (6.18), we have $(\alpha|\alpha) < 0$ and the assumption implies $|(\alpha|\alpha_0)| = 0$ because $|(\alpha|\alpha)| \ge n|(\alpha|\alpha_0)|$.

Let Π_0 be the connected component of $\{\alpha_i \in \Pi\,;\,(\alpha|\alpha_i) = 0 \text{ and } \alpha_i \in \operatorname{supp}\alpha\}$ containing α_0. Note that $\operatorname{supp}\alpha$ generates a root system which is neither classical nor affine but Π_0 generates a root system of finite type.

Put $J = \{j\,;\,\exists\alpha_{j,\nu} \in \operatorname{supp}\alpha_{\mathbf{m}} \text{ such that } (\alpha|\alpha_{j,\nu}) < 0\} \ne \emptyset$ and for each $j \in J$ define k_j with the condition (7.45). Then we note that

$$(\alpha|\alpha_{j,\nu}) = \begin{cases} 0 & (1 \le \nu < k_j), \\ 2n_{j,k_j} - n_{j,k_j-1} - n_{j,k_j+1} \le -1 & (\nu = k_j). \end{cases}$$

Applying the above lemma to $\mathbf{m}$ by putting $n = b + k_j c$ and $n_{j,\nu} = b + (k_j - \nu)c$ $(1 \le \nu \le k_j)$ and $n_{j,k_j+1} = a$, we have

$$\frac{n-6}{(n_{j,k_j-1} + n_{j,k_j+1} - 2n_{j,k_j})n_{j,k_j}} \begin{cases} < k_j + 1 & (1 \le k_j \le 5), \\ \le 7 & (1 \le k_j \le 6). \end{cases} \tag{7.54}$$

Here $(\alpha|\alpha_{j,k_j}) = b - c - a \le -1$ and we have $|(\alpha|\alpha)| \ge |(\alpha|\alpha_{j,\nu})| > \frac{n-6}{k_j+1}$ if $k_j < 6$ and therefore $k_j \ge 3$.

It follows from the condition $k_j \ge 3$ that $\mathbf{m} \in \mathcal{P}_3$ because Π_0 is of finite type and moreover that Π_0 is of exceptional type, namely, of type E_6 or E_7 or E_8 because $\operatorname{supp}\alpha$ is not of finite type.

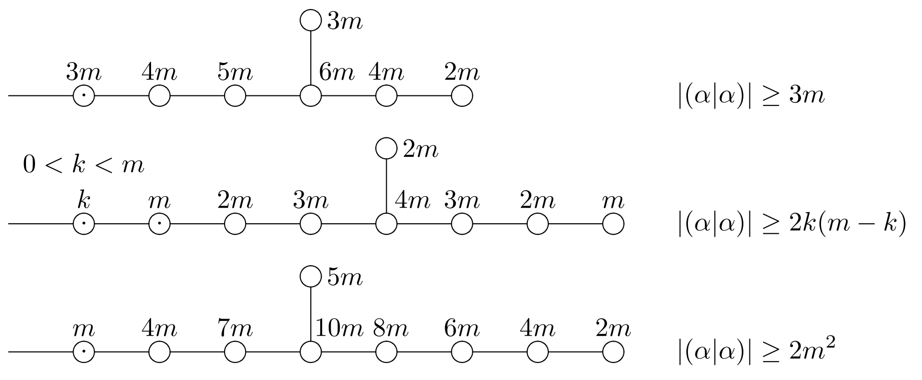
Suppose $\#J \ge 2$. We may assume $\{0,1\} \subset J$ and $k_0 \le k_1$. Since Π_0 is of exceptional type and $\operatorname{supp}\alpha$ is not of finite type, we may assume $k_0 = 3$ and $k_1 \le 5$. Owing to (7.52) and (7.54), we have

$$\begin{aligned} |(\alpha|\alpha)| &\ge n_{0,3}(n_{0,2} + n_{0,4} - 2n_{0,3}) + n_{1,k_1}(n_{1,k_1-1} + n_{1,k_1+1} - 2n_{1,k_1}) \\ &> \tfrac{n-6}{3+1} + \tfrac{n-6}{5+1} > \tfrac{n-6}{3}, \end{aligned}$$

which contradicts to the assumption.

Thus we may assume $J = \{0\}$. For $j = 1$ and 2 let n_j be the positive integer such that $\alpha_{j,n_j} \in \operatorname{supp}\alpha$ and $\alpha_{j,n_j+1} \notin \operatorname{supp}\alpha$. We may assume $n_1 \ge n_2$.

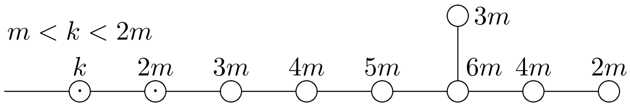
Fist suppose $k_0 = 3$. Then $(n_1, n_2) = (2,1)$, $(3,1)$ or $(4,1)$ and the Dynkin diagram of $\operatorname{supp}\alpha$ with the numbers $m_{j,\nu}$ is one of the diagrams:



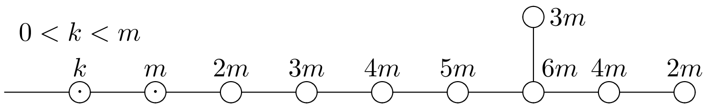
For example, when $(n_1, n_2) = (3,1)$, then $k := m_{0,4} \ge 1$ because $(\alpha|\alpha_{0,3}) \ne 0$ and therefore $0 < k < m$ and $|(\alpha|\alpha)| \ge k(m-2k)+m(2m+k-2m) = 2k(m-k) \ge 2m-2$ and $3|(\alpha|\alpha)| + 6 - 4m \ge 3(2m-2) + 6 - 4m > 0$. Hence (7.53) does not hold.

Other cases don't happen because of the inequalities $3 \cdot 3m + 6 - 6m > 0$ and $3 \cdot 2m^2 + 6 - 10m > 0$.

Lastly suppose $k_0 > 3$. Then $(k_0, n_1, n_2) = (4, 2, 1)$ or $(5, 2, 1)$.



$|(\alpha|\alpha)| \geq 2m$



$|(\alpha|\alpha)| \geq 2(m-1)$

In the above first case we have $(\alpha|\alpha)| \geq 2m$, which contradicts to (7.53). Note that $(|\alpha|\alpha)| \geq k\cdot(m-2k) + m\cdot k = 2k(m-k) \geq 2(m-1)$ in the above last case, which also contradicts to (7.53) because $3\cdot 2(m-1)+6 = 6m$.

Thus we have proved (7.48).

Assume $\mathbf{m} \notin \mathcal{P}_3$ to prove a different inequality (7.49). In this case, we may assume $(\alpha|\alpha_0) = 0$, $|(\alpha|\alpha)| \geq 2$ and $n > 4$. Note that

$$(7.55)\qquad 2n = n_{0,1} + n_{1,1} + \cdots + n_{p,1} \quad \text{with} \quad p \geq 3 \text{ and } n_{j,1} \geq 1 \text{ for } j = 0, \ldots, p.$$

If there exists j with $1 \leq n_{j,1} \leq \frac{n}{2} - 1$, (7.49) follows from (7.52) and $|(\alpha|\alpha_{j,1})| = n_{j,1}(n + n_{j,2} - 2n_{j,1}) \geq 2n_{j,1}(\frac{n}{2} - n_{j,1}) \geq n-2$.

Hence we may assume $n_{j,1} \geq \frac{n-1}{2}$ for $j = 0, \ldots, p$. Suppose there exists j with $n_{j,1} = \frac{n-1}{2}$. Then n is odd and (7.55) means that there also exists j' with $j \neq j'$ and $n_{j',1} = \frac{n-1}{2}$. In this case we have (7.49) since

$$|(\alpha|\alpha_{j,1})| + |(\alpha|\alpha_{j',1})| = n_{j,1}(n + n_{j,2} - 2n_{j,1}) + n_{j',1}(n + n_{j',2} - 2n_{j,1}) \geq \tfrac{n-1}{2} + \tfrac{n-1}{2}.$$

Now we may assume $n_{j,1} \geq \frac{n}{2}$ for $j = 0, \ldots, p$. Then (7.55) implies that $p = 3$ and $n_{j,1} = \frac{n}{2}$ for $j = 0, \ldots, 3$. Since $(\alpha|\alpha) < 0$, there exists j with $n_{j,2} \geq 1$ and

$$\begin{aligned}|(\alpha|\alpha_{j,1})| + |(\alpha|\alpha_{j,2})| &= n_{j,1}(n + n_{j,2} - 2n_{j,1}) + n_{j,2}(n_{j,1} + n_{j,3} - 2n_{j,2})\\ &= \tfrac{n}{2}n_{j,2} + n_{j,2}(\tfrac{n}{2} + n_{j,3} - 2n_{j,2})\\ &\begin{cases} \geq n & (n_{j,2} \geq 1),\\ = n-2 & (n_{j,2} = 1 \text{ and } n_{j,3} = 0).\end{cases}\end{aligned}$$

Thus we have completed the proof of (7.49).

There are 4 basic tuples with the index of the rigidity 0 and 13 basic tuples with the index of the rigidity -2, which are given in (6.18) and Proposition 6.10. They satisfy (7.50).

Suppose that (7.50) is not valid. We may assume that p is minimal under this assumption. Then $\operatorname{idx}\mathbf{m} < -2$, $p \geq 5$ and $n = \operatorname{ord}\mathbf{m} > 2$. We may assume $n > n_{0,1} \geq n_{1,1} \geq \cdots \geq n_{p,1} > 0$. Since $(\alpha|\alpha_0) \leq 0$, we have

$$(7.56)\qquad n_{0,1} + n_{1,1} + \cdots + n_{p,1} \geq 2n > n_{0,1} + \cdots + n_{p-1,1}.$$

In fact, if $n_{0,1} + \cdots + n_{p-1,1} \geq 2n$, the tuple $\mathbf{m}' = (\mathbf{m}_0, \ldots, \mathbf{m}_{p-1})$ is also basic and $|(\alpha|\alpha)| - |(\alpha_{\mathbf{m}'}, \alpha_{\mathbf{m}'})| = n^2 - \sum_{\nu\geq 1} n_{p,\nu}^2 \geq 2$, which contradicts to the minimality.

Thus we have $2n_{j,1} < n$ for $j = 3, \ldots, p$. If n is even, we have $|\operatorname{idx}\mathbf{m}| \geq \sum_{j=3}^{p} |(\alpha|\alpha_{j,1})| = \sum_{j=3}^{p}(n + n_{j,2} - 2n_{j,1}) \geq 2(p-2)$, which contradicts to the assumption. If $n = 3$, (7.56) assures $p = 5$ and $n_{0,1} = \cdots = n_{5,0} = 1$ and therefore $\operatorname{idx}\mathbf{m} = -4$, which also contradicts to the assumption. Thus $n = 2m+1$ with $m \geq 2$. Choose k so that $n_{k-1,1} \geq m > n_{k,1}$. Then $|\operatorname{idx}\mathbf{m}| \geq \sum_{j=k}^{p}(\alpha|\alpha_{j,1})| = \sum_{j=k}^{p}(n+n_{j,2}-2n_{j,1}) \geq 3(p-k+1)$. Owing to (7.56), we have $2(2m+1) > km+(p-k)$ and $k < \frac{4m+2-p}{m-1} \leq \frac{4m-3}{m-1} \leq 5$, which means $k \leq 4$, $|\operatorname{idx}\mathbf{m}| \geq 3(p-3) \geq 2p-4$ and a contradiction to the assumption. □