CHAPTER 6

Deligne-Simpson problem

In this chapter we give an answer for the existence and the construction of Fuchsian differential equations with given Riemann schemes and examine the irreducibility for generic spectral parameters.

6.1. Fundamental lemmas

First we prepare two lemmas to construct Fuchsian differential operators with a given spectral type.

DEFINITION 6.1. For $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ 1\le\nu\le n_j}} \in \mathcal{P}_{p+1}^{(n)}$, we put

$$\begin{aligned} N_\nu(\mathbf{m}) := {}&(p-1)(\nu+1)+1 \\ &- \#\{(j,i)\in\mathbb{Z}^2\,;\, i\ge 0,\ 0\le j\le p,\ \widetilde{m}_{j,i}\ge n-\nu\}, \end{aligned} \tag{6.1}$$

$$\widetilde{m}_{j,i} := \sum_{\nu=1}^{n_j} \max\big\{m_{j,\nu}-i,0\big\}. \tag{6.2}$$

See the Young diagram in (6.32) and its explanation for an interpretation of the number $\widetilde{m}_{j,i}$.

LEMMA 6.2. *We assume that* $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ 1\le\nu\le n_j}} \in \mathcal{P}_{p+1}^{(n)}$ *satisfies*

$$m_{j,1}\ge m_{j,2}\ge\cdots\ge m_{j,n_j}>0 \quad\textit{and}\quad n>m_{0,1}\ge m_{1,1}\ge\cdots\ge m_{p,1} \tag{6.3}$$

and

$$m_{0,1}+\cdots+m_{p,1}\le (p-1)n. \tag{6.4}$$

Then

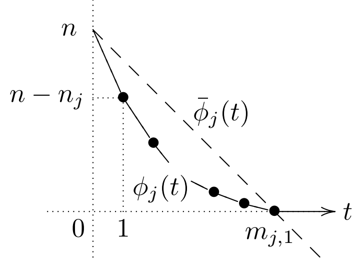
$$N_\nu(\mathbf{m})\ge 0 \qquad (\nu=2,3,\dots,n-1) \tag{6.5}$$

if and only if $\mathbf{m}$ *is not any one of*

$$\begin{gathered} (k,k;k,k;k,k;k,k),\quad (k,k,k;k,k,k;k,k,k),\\ (2k,2k;k,k,k,k;k,k,k,k)\\ \textit{and}\ \ (3k,3k;2k,2k,2k;k,k,k,k,k,k)\ \ \textit{with}\ \ k\ge 2. \end{gathered} \tag{6.6}$$

PROOF. Put

$$\begin{aligned} \phi_j(t) &:= \sum_{\nu=1}^{n_j}\max\{m_{j,\nu}-t,0\},\\ \bar\phi_j(t) &:= n\Big(1-\frac{t}{m_{j,1}}\Big)\quad\text{for}\quad j=0,\dots,p. \end{aligned}$$



Then $\phi_j(t)$ and $\bar\phi_j(t)$ are strictly decreasing continuous functions of $t \in [0, m_{j,1}]$ and

$$\begin{aligned}
\phi_j(0) &= \bar\phi_j(0) = n, \\
\phi_j(m_{j,1}) &= \bar\phi_j(m_{j,1}) = 0, \\
2\phi_j(\tfrac{t_1+t_2}{2}) &\le \phi_j(t_1) + \phi_j(t_2) && (0 \le t_1 \le t_2 \le m_{j,1}), \\
\phi_j'(t) &= -n_j \le -\tfrac{n}{m_{j,1}} = \bar\phi_j'(t) && (0 < t < 1).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\phi_j(t) = \bar\phi_j(t) && (0 < t < m_{j,1},\ n = m_{j,1}n_j), \\
&\phi_j(t) < \bar\phi_j(t) && (0 < t < m_{j,1},\ n < m_{j,1}n_j)
\end{aligned}$$

and for $\nu = 2, \dots, n-1$

$$\begin{aligned}
\sum_{j=0}^{p} \#\{i \in \mathbb{Z}_{\ge 0}\,;\, \phi_j(i) \ge n-\nu\} &= \sum_{j=0}^{p} \bigl[\phi_j^{-1}(n-\nu) + 1\bigr] \\
&\le \sum_{j=0}^{p} \bigl(\phi_j^{-1}(n-\nu) + 1\bigr) \\
&\le \sum_{j=0}^{p} \bigl(\bar\phi_j^{-1}(n-\nu) + 1\bigr) = \sum_{j=0}^{p} \Bigl(\frac{\nu m_{j,1}}{n} + 1\Bigr) \\
&\le (p-1)\nu + (p+1) = (p-1)(\nu+1) + 2.
\end{aligned}$$

Here $[r]$ means the largest integer which is not larger than a real number r.

Suppose there exists ν with $2 \le \nu \le n-1$ such that (6.5) doesn't hold. Then the equality holds in the above each line, which means

$$\begin{aligned}
\phi_j^{-1}(n-\nu) &\in \mathbb{Z} && (j = 0, \dots, p), \\
n &= m_{j,1}n_j && (j = 0, \dots, p), \\
(p-1)n &= m_{0,1} + \cdots + m_{p,1}.
\end{aligned} \tag{6.7}$$

Note that $n = m_{j,1}n_j$ implies $m_{j,1} = \cdots = m_{j,n_j} = \frac{n}{n_j}$ and $p - 1 = \frac{1}{n_0} + \cdots + \frac{1}{n_p} \le \frac{p+1}{2}$. Hence $p = 3$ with $n_0 = n_1 = n_2 = n_3 = 2$ or $p = 2$ with $1 = \frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2}$. If $p = 2$, $\{n_0, n_1, n_2\}$ equals $\{3,3,3\}$ or $\{2,4,4\}$ or $\{2,3,6\}$. Thus we have (6.6) with $k = 1, 2, \dots$. Moreover since

$$\phi_j^{-1}(n-\nu) = \bar\phi_j^{-1}(n-\nu) = \frac{\nu m_{j,1}}{n} = \frac{\nu}{n_j} \in \mathbb{Z} \quad (j = 0, \dots, p),$$

ν is a common multiple of $n_0, \dots, n_p$ and thus $k \ge 2$. If ν is the least common multiple of $n_0, \dots, n_p$ and $k \ge 2$, then (6.7) is valid and the equality holds in the above each line and hence (6.5) is not valid. □

Corollary 6.3 (Kostov [**Ko**]). *Let* $\mathbf{m} \in \mathcal{P}$ *satisfying* $d_{max}(\mathbf{m}) \le 0$. *When* $\operatorname{idx}\mathbf{m} = 0$, $\mathbf{m}$ *is isomorphic to one of the tuples in* (6.6) *with* $k = 1, 2, 3, \dots$.

Proof. Remark 5.9 assures that $d_{max}(\mathbf{m}) = 0$ and $n = m_{j,1}n_j$. Then the proof of the final part of Lemma 6.2 shows the corollary. □

Lemma 6.4. *Let* $c_0, \dots, c_p$ *be* $p+1$ *distinct points in* $\mathbb{C} \cup \{\infty\}$. *Let* $n_0, n_1, \dots, n_p$ *be non-negative integers and let* $a_{j,\nu}$ *be complex numbers for* $j = 0, \dots, p$ *and* $\nu = 1, \dots, n_j$. *Put* $\tilde n := n_0 + \cdots + n_p$. *Then there exists a unique polynomial* $f(x)$ *of* *degree* $\tilde n - 1$ *such that*

$$\begin{aligned} f(x) &= a_{j,1} + a_{j,2}(x-c_j) + \cdots + a_{j,n_j}(x-c_j)^{n_j-1} \\ &\quad + o(|x-c_j|^{n_j-1}) \qquad (x \to c_j,\ c_j \neq \infty), \\ x^{1-\tilde n} f(x) &= a_{j,1} + a_{j,2}x^{-1} + a_{j,n_j}x^{1-n_j} + o(|x|^{1-n_j}) \\ &\quad (x \to \infty,\ c_j = \infty). \end{aligned} \tag{6.8}$$

Moreover the coefficients of $f(x)$ *are linear functions of the* $\tilde n$ *variables* $a_{j,\nu}$.

Proof. We may assume $c_p = \infty$ with allowing $n_p = 0$. Put $\tilde n_i = n_0 + \cdots + n_{i-1}$ and $\tilde n_0 = 0$. For $k = 0, \dots, \tilde n - 1$ we define

$$f_k(x) := \begin{cases} (x-c_i)^{k-\tilde n_i} \prod_{\nu=0}^{i-1}(x-c_\nu)^{n_\nu} & (\tilde n_i \le k < \tilde n_{i+1},\ 0 \le i < p), \\ x^{k-\tilde n_p} \prod_{\nu=0}^{n_{p-1}}(x-c_\nu)^{n_\nu} & (\tilde n_p \le k < \tilde n). \end{cases}$$

Since $\deg f_k(x) = k$, the polynomials $f_0(x), f_1(x), \dots, f_{\tilde n-1}(x)$ are linearly independent over $\mathbb{C}$. Put $f(x) = \sum_{k=0}^{\tilde n-1} u_k f_k(x)$ with $c_k \in \mathbb{C}$ and

$$v_k = \begin{cases} a_{i,k-\tilde n_i+1} & (\tilde n_i \le k < \tilde n_{i+1},\ 0 \le i < p), \\ a_{p,\tilde n-k} & (\tilde n_p \le k < \tilde n) \end{cases}$$

by (6.8). The correspondence which maps the column vectors $u := (u_k)_{k=0,\dots,\tilde n-1} \in \mathbb{C}^{\tilde n}$ to the column vectors $v := (v_k)_{k=0,\dots,\tilde n-1} \in \mathbb{C}^{\tilde n}$ is given by $v = Au$ with a square matrix A of size $\tilde n$. Then A is an upper triangular matrix of size $\tilde n$ with non-zero diagonal entries and therefore the lemma is clear. $\square$

6.2. Existence theorem

Definition 6.5 (top term). Let

$$P = a_n(x)\tfrac{d^n}{dx^n} + a_{n-1}(x)\tfrac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\tfrac{d}{dx} + a_0(x)$$

be a differential operator with polynomial coefficients. Suppose $a_n \neq 0$. If $a_n(x)$ is a polynomial of degree k with respect to x, we define $\operatorname{Top} P := a_{n,k}x^k\partial^n$ with the coefficient $a_{n,k}$ of the term x^k of $a_n(x)$. We put $\operatorname{Top} P = 0$ when $P = 0$.

Theorem 6.6. *Suppose* $\mathbf{m} \in \mathcal{P}_{p+1}^{(n)}$ *satisfies* (6.3). *Retain the notation in Definition 6.1.*

i) *We have* $N_1(\mathbf{m}) = p - 2$ *and*

$$\sum_{\nu=1}^{n-1} N_\nu(\mathbf{m}) = \operatorname{Pidx}\mathbf{m}. \tag{6.9}$$

ii) *Suppose* $p \ge 2$ *and* $N_\nu(\mathbf{m}) \ge 0$ *for* $\nu = 2, \dots, n-1$. *Put*

$$q_\nu^0 := \#\{i\,;\, \widetilde m_{0,i} \ge n-\nu,\ i \ge 0\}, \tag{6.10}$$

$$I_{\mathbf{m}} := \{(j,\nu) \in \mathbb{Z}^2\,;\, q_\nu^0 \le j < q_\nu^0 + N_\nu(\mathbf{m}) \text{ and } 1 \le \nu \le n-1\}. \tag{6.11}$$

Then there uniquely exists a Fuchsian differential operator P *of the normal form* (4.43) *which has the Riemann scheme* (4.15) *with* $c_0 = \infty$ *under the Fuchs relation* (4.16) *and satisfies*

$$\frac{1}{(\deg P - j - \nu)!}\frac{d^{\deg P - j - \nu} a_{n-\nu-1}}{dx^{\deg P - j - \nu}}(0) = g_{j,\nu} \qquad (\forall (j,\nu) \in I_{\mathbf{m}}). \tag{6.12}$$

Here $\big(g_{j,\nu}\big)_{(j,\nu)\in I_{\mathbf m}} \in \mathbb C^{\operatorname{Pidx}\mathbf m}$ *is arbitrarily given. Moreover the coefficients of* P *are polynomials of* x, $\lambda_{j,\nu}$ *and* $g_{j,\nu}$ *and satisfy*

(6.13)
$$x^{j+\nu}\operatorname{Top}\Big(\frac{\partial P}{\partial g_{j,\nu}}\Big)\partial^{\nu+1} = \operatorname{Top}P \quad\text{and}\quad \frac{\partial^2 P}{\partial g_{j,\nu}\partial g_{j',\nu'}} = 0 \quad ((j,\nu),\,(j',\nu')\in I_{\mathbf m}).$$

Fix the characteristic exponents $\lambda_{j,\nu}\in\mathbb C$ *satisfying the Fuchs relation. Then all the Fuchsian differential operators of the normal form with the Riemann scheme* (4.15) *are parametrized by* $(g_{j,\nu})\in\mathbb C^{\operatorname{Pidx}\mathbf m}$. *Hence the operators are unique if and only if* $\operatorname{Pidx}\mathbf m = 0$.

PROOF. i) Since $\widetilde m_{j,1} = n-n_j \le n-2$, $N_1(\mathbf m) = 2(p-1)+1-(p+1) = p-2$ and

$$\begin{aligned}
&\sum_{\nu=1}^{n-1}\#\{(j,i)\in\mathbb Z^2\,;\, i\ge 0,\ 0\le j\le p,\ \widetilde m_{j,i}\ge n-\nu\}\\
&\quad=\sum_{j=0}^{p}\Big(\sum_{\nu=0}^{n-1}\#\{i\in\mathbb Z_{\ge 0}\,;\,\widetilde m_{j,i}\ge n-\nu\}-1\Big)\\
&\quad=\sum_{j=0}^{p}\Big(\sum_{i=0}^{m_{j,1}}\widetilde m_{j,i}-1\Big)=\sum_{j=0}^{p}\Big(\sum_{i=0}^{m_{j,1}}\sum_{\nu=1}^{n_j}\max\{m_{j,\nu}-i,0\}-1\Big)\\
&\quad=\sum_{j=0}^{p}\Big(\sum_{\nu=1}^{n_j}\frac{m_{j,\nu}(m_{j,\nu}+1)}{2}-1\Big)\\
&\quad=\frac12\Big(\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}m_{j,\nu}^2+(p+1)(n-2)\Big),
\end{aligned}$$

$$\begin{aligned}
\sum_{\nu=1}^{n-1}N_\nu(\mathbf m) &= (p-1)\Big(\frac{n(n+1)}{2}-1\Big)+(n-1)-\frac12\Big(\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}m_{j,\nu}^2+(p+1)(n-2)\Big)\\
&=\frac12\Big((p-1)n^2+2-\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}m_{j,\nu}^2\Big)=\operatorname{Pidx}\mathbf m.
\end{aligned}$$

ii) Put

$$\begin{aligned}
P &= \sum_{\ell=0}^{pn}x^{pn-\ell}p^P_{0,\ell}(\vartheta)\\
&= \sum_{\ell=0}^{pn}(x-c_j)^\ell p^P_{j,\ell}\big((x-c_j)\partial\big) \qquad (1\le j\le p),\\
h_{j,\ell}(t) &:= \begin{cases}\prod_{\nu=1}^{n_0}\prod_{0\le i<m_{0,\nu}-\ell}\big(t+\lambda_{0,\nu}+i\big) & (j=0),\\ \prod_{\nu=1}^{n_j}\prod_{0\le i<m_{j,\nu}-\ell}\big(t-\lambda_{j,\nu}-i\big) & (1\le j\le p),\end{cases}\\
p^P_{j,\ell}(t) &= q^P_{j,\ell}(t)h_{j,\ell}(t)+r^P_{j,\ell}(t) \qquad (\deg r^P_{j,\ell}(t)<\deg h_{j,\ell}(t)).
\end{aligned}$$

Here $p^P_{j,\ell}(t)$, $q^P_{j,\ell}(t)$, $r^P_{j,\ell}(t)$ and $h_{j,\ell}(t)$ are polynomials of t and

$$\deg h_{j,\ell} = \sum_{\nu=1}^{n_j}\max\{m_{j,\nu}-\ell,0\}. \tag{6.14}$$

The condition that P of the form (4.43) have the Riemann scheme (4.15) if and only if $r^P_{j,\ell}=0$ for any j and ℓ. Note that $a_{n-k}(x)\in\mathbb C[x]$ should satisfy

$$\deg a_{n-k}(x)\le pn-k \quad\text{and}\quad a^{(\nu)}_{n-k}(c_j)=0 \quad (0\le\nu\le n-k-1,\ 1\le k\le n), \tag{6.15}$$

which is equivalent to the condition that P is of the Fuchsian type.

Put $P(k) := \bigl(\prod_{j=1}^{p}(x-c_j)^n\bigr)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-k}(x)\frac{d^{n-k}}{dx^{n-k}}$.

Assume that $a_{n-1}(x), \dots, a_{n-k+1}(x)$ have already defined so that $\deg r_{j,\ell}^{P(k-1)} < n-k+1$ and we will define $a_{n-k}(x)$ so that $\deg r_{j,\ell}^{P(k)} < n-k$.

When $k = 1$, we put

$$a_{n-1}(x) = -a_n(x)\sum_{j=1}^{p}(x-c_j)^{-1}\Bigl(\sum_{\nu=1}^{n_j}\sum_{i=0}^{m_{j,\nu}-1}(\lambda_{j,\nu}+i) - \frac{n(n-1)}{2}\Bigr)$$

and then we have $\deg r_{j,\ell}^{P(1)} < n-1$ for $j = 1, \dots, p$. Moreover we have $\deg r_{0,\ell}^{P(1)} < n-1$ because of the Fuchs relation (cf. (2.21)).

Suppose $k \ge 2$ and put

$$a_{n-k}(x) = \begin{cases} \sum_{\ell\ge0} c_{0,k,\ell}x^{pn-k-\ell}, & \\ \sum_{\ell\ge0} c_{j,k,\ell}(x-c_j)^{n-k+\ell} & (j = 1, \dots, p) \end{cases}$$

with $c_{i,j,\ell} \in \mathbb{C}$. Note that

$$\begin{aligned} a_{n-k}(x)\partial^{n-k} &= \sum_{\ell\ge0} c_{0,k,\ell}x^{(p-1)n-\ell}\prod_{i=0}^{n-k-1}(\vartheta - i) \\ &= \sum_{\ell\ge0} c_{j,k,\ell}(x-c_j)^{\ell}\prod_{i=0}^{n-k-1}\bigl((x-c_j)\partial - i\bigr). \end{aligned}$$

Then $\deg r_{j,\ell}^{P(k)} < n-k$ if and only if $\deg h_{j,\ell} \le n-k$ or

$$c_{j,k,\ell} = -\frac{1}{(n-k)!}\Bigl(\frac{d^{n-k}}{dt^{n-k}} r_{j,\ell}^{P(k-1)}(t)\Bigr)\Big|_{t=0}. \tag{6.16}$$

Namely, we impose the condition (6.16) for all (j,ℓ) satisfying

$$\widetilde{m}_{j,\ell} = \sum_{\nu=1}^{n_j}\max\{m_{j,\nu} - \ell, 0\} > n-k.$$

The number of the pairs (j,ℓ) satisfying this condition equals $(p-1)k+1-N_{k-1}(\mathbf{m})$. Together with the conditions $a_{n-k}^{(\nu)}(c_j) = 0$ for $j = 1, \dots, p$ and $\nu = 0, \dots, n-k-1$, the total number of conditions imposing to the polynomial $a_{n-k}(x)$ of degree $pn-k$ equals

$$p(n-k) + (p-1)k + 1 - N_{k-1}(\mathbf{m}) = (pn-k+1) - N_{k-1}(\mathbf{m}).$$

Hence Lemma 6.4 shows that $a_{n-k}(x)$ is uniquely defined by giving $c_{0,k,\ell}$ arbitrarily for $q_{k-1}^0 \le \ell < q_{k-1}^0 + N_{k-1}(\mathbf{m})$ because $q_{k-1}^0 = \#\{\ell \ge 0\,;\, \widetilde{m}_{0,\ell} > n-k\}$. Thus we have the theorem. □

Remark 6.7. The numbers $N_\nu(\mathbf{m})$ don't change if we replace a $(p+1)$-tuple $\mathbf{m}$ of partitions of n by the $(p+2)$-tuple of partitions of n defined by adding a trivial partition $n = n$ of n to $\mathbf{m}$.

Example 6.8. We will examine the number $N_\nu(\mathbf{m})$ in Theorem 6.6. In the case of Simpson's list (cf. §13.2) we have the following.

(H_n: hypergeometric family)

$$\begin{aligned} \mathbf{m} &= n-11, 1^n, 1^n \\ \widetilde{\mathbf{m}} &= n, n-2, n-3, \dots 1; n; n \end{aligned}$$

(EO_{2m}: even family) $\mathbf{m} = mm, mm-11, 1^{2m}$

$\widetilde{\mathbf{m}} = 2m, 2m-2, \dots, 2; 2m, 2m-3, \dots, 1; 2m$

(EO_{2m+1}: odd family) $\mathbf{m} = m+1m, mm1, 1^{2m+1}$

$\widetilde{\mathbf{m}} = 2m+1, 2m-1, \dots, 1; 2m+1, 2m-2, \dots, 2; 2m+1$

(X_6: extra case) $\mathbf{m} = 42, 222, 1^6$

$\widetilde{\mathbf{m}} = 6, 4, 2, 1; 6, 3; 6$

In these cases $p = 2$ and we have $N_\nu(\mathbf{m}) = 0$ for $\nu = 1, 2, \dots, n-1$ because

$$\begin{aligned}\widetilde{\mathbf{m}} &:= \{\widetilde{m}_{j,\nu}\,;\ \nu = 0, \dots, m_{j,1}-1,\ j = 0, \dots, p\}\\ &= \{n, n, n, n-2, n-3, n-4, \dots, 2, 1\}.\end{aligned} \tag{6.17}$$

See Proposition 6.17 ii) for the condition that $N_\nu(\mathbf{m}) \ge 0$ for $\nu = 1, \dots, \operatorname{ord}\mathbf{m} - 1$.

We give other examples:

$\mathbf{m}$	Pidx	$\widetilde{\mathbf{m}}$	$N_1, N_2, \dots, N_{\operatorname{ord}\mathbf{m}-1}$
$221, 221, 221$	0	$52, 52, 52$	$0, 1, -1, 0$
$21, 21, 21, 21\,(P_3)$	0	$31, 31, 31, 31$	$1, -1$
$22, 22, 22$	-3	$42, 42, 42$	$0, -2, -1$
$11, 11, 11, 11\,(\tilde{D}_4)$	1	$2, 2, 2, 2$	1
$111, 111, 111\,(\tilde{E}_6)$	1	$3, 3, 3$	$0, 1$
$22, 1111, 1111\,(\tilde{E}_7)$	1	$42, 4, 4$	$0, 0, 1$
$33, 222, 111111\,(\tilde{E}_8)$	1	$642, 63, 6$	$0, 0, 0, 0, 1$
$21, 21, 21, 111$	1	$31, 31, 31, 3$	$1, 0$
$222, 222, 222$	1	$63, 63, 63$	$0, 1, -1, 0, 1$
$11, 11, 11, 11, 11$	2	$2, 2, 2, 2, 2$	2
$55, 3331, 22222$	2	$10, 8, 6, 4, 2; 10, 6, 3; 10, 5$	$0, 0, 1, 0, 0, 0, 0, 0, 1$
$22, 22, 22, 211$	2	$42, 42, 42, 41$	$1, 0, 1$
$22, 22, 22, 22, 22$	5	$42, 42, 42, 42, 42$	$2, 0, 3$
$32111, 3221, 2222$	8	$831, 841, 84$	$0, 1, 2, 1, 1, 2, 1$

Note that if $\operatorname{Pidx}\mathbf{m} = 0$, in particular, if $\mathbf{m}$ is rigid, then $\mathbf{m}$ doesn't satisfy (6.4). The tuple $222, 222, 222$ of partitions is the second case in (6.6) with $k = 2$.

REMARK 6.9. Note that [**O6**, Proposition 8.1] proves that there exit only finite basic tuples of partitions with a fixed index of rigidity.

Those with index of rigidity 0 are of only 4 types, which are $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ given in the above (cf. Corollary 6.3, Kostov [**Ko**]). Namely, those are in the S_∞-orbit of

$$\{11, 11, 11, 11 \quad 111, 111, 111 \quad 22, 1111, 1111 \quad 33, 222, 111111\} \tag{6.18}$$

and the operator P in Theorem 6.6 with any one of this spectral type has one accessory parameter in its 0-th order term.

The equation corresponding to $11, 11, 11, 11$ is called Heun's equation (cf. [**SW, WW**]), which is given by the operator

$$\begin{aligned}P_{\alpha,\beta,\gamma,\delta,\lambda} = x(x-1)(x-c)\partial^2 &+ \big(\gamma(x-1)(x-c) + \delta x(x-c)\\ &+ (\alpha+\beta+1-\gamma-\delta)x(x-1)\big)\partial + \alpha\beta x - \lambda\end{aligned} \tag{6.19}$$

with the Riemann scheme

$$\begin{Bmatrix} x = 0 & 1 & c & \infty & \\ 0 & 0 & 0 & \alpha & ;\, x \\ 1-\gamma & 1-\delta & \gamma+\delta-\alpha-\beta & \beta & ;\, \lambda \end{Bmatrix}. \tag{6.20}$$

Here λ is an accessory parameter. Our operation cannot decrease the order of $P_{\alpha,\beta,\gamma,\delta,\lambda}$ but gives the following transformation.

$$\operatorname{Ad}(\partial^{1-\alpha})P_{\alpha,\beta,\gamma,\delta,\lambda} = P_{\alpha',\beta',\gamma',\delta',\lambda'},$$

$$\begin{cases} \alpha' = 2-\alpha,\ \beta' = \beta-\alpha+1,\ \gamma' = \gamma-\alpha+1,\ \delta' = \delta-\alpha+1, \\ \lambda' = \lambda + (1-\alpha)\big(\beta-\delta+1+(\gamma+\delta-\alpha)c\big). \end{cases} \tag{6.21}$$

Proposition 6.10. ([**O6**, Proposition 8.4]). *The basic tuples of partitions with index of rigidity -2 are in the S_∞-orbit of the set of the 13 tuples*

$$\begin{aligned}\{&11,11,11,11,11 \quad 21,21,111,111 \quad 31,22,22,1111 \quad 22,22,22,211\\ &211,1111,1111 \quad 221,221,11111 \quad 32,11111,11111 \quad 222,222,2211\\ &33,2211,111111 \quad 44,2222,22211 \quad 44,332,11111111 \quad 55,3331,22222\\ &66,444,2222211\}.\end{aligned}$$

Proof. Here we give the proof in [**O6**].

Assume that $\mathbf{m} \in \mathcal{P}_{p+1}$ is basic and monotone and $\operatorname{idx}\mathbf{m} = -2$. Note that (5.42) shows

$$0 \le \sum_{j=0}^{p}\sum_{\nu=2}^{n_j}(m_{j,1}-m_{j,\nu})\cdot m_{j,\nu} \le -\operatorname{idx}\mathbf{m} = 2.$$

Hence (5.42) implies $\sum_{j=0}^{p}\sum_{\nu=2}^{n_j}(m_{j,1}-m_{j,\nu})m_{j,\nu} = 0$ or 2 and we have only to examine the following 5 possibilities.

(A) $m_{0,1}\cdots m_{0,n_0} = 2\cdots 211$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le p$.
(B) $m_{0,1}\cdots m_{0,n_0} = 3\cdots 31$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le p$.
(C) $m_{0,1}\cdots m_{0,n_0} = 3\cdots 32$ and $m_{j,1} = m_{j,n_j}$ for $1 \le j \le p$.
(D) $m_{i,1}\cdots m_{i,n_0} = 2\cdots 21$ and $m_{j,1} = m_{j,n_j}$ for $0 \le i \le 1 < j \le p$.
(E) $m_{j,1} = m_{j,n_j}$ for $0 \le j \le p$ and $\operatorname{ord}\mathbf{m} = 2$.

Case (A). If $2\cdots 211$ is replaced by $2\cdots 22$, $\mathbf{m}$ is transformed into $\mathbf{m}'$ with $\operatorname{idx}\mathbf{m}' = 0$. If $\mathbf{m}'$ is indivisible, $\mathbf{m}'$ is basic and $\operatorname{idx}\mathbf{m}' = 0$ and therefore $\mathbf{m}$ is $211,1^4,1^4$ or $33,2211,1^6$. If $\mathbf{m}'$ is not indivisible, $\frac12\mathbf{m}'$ is basic and $\operatorname{idx}\frac12\mathbf{m}' = 0$ and hence $\mathbf{m}$ is one of the tuples in

$$\{211,22,22,22 \quad 2211,222,222 \quad 22211,2222,44 \quad 2222211,444,66\}.$$

Put $m = n_0 - 1$ and examine the identity

$$\sum_{j=0}^{p}\frac{m_{j,1}}{\operatorname{ord}\mathbf{m}} = p-1+(\operatorname{ord}\mathbf{m})^{-2}\Big(\operatorname{idx}\mathbf{m}+\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}(m_{j,1}-m_{j,\nu})m_{j,\nu}\Big)$$

Case (B). Note that $\operatorname{ord}\mathbf{m} = 3m+1$ and therefore $\frac{3}{3m+1}+\frac{1}{n_1}+\cdots+\frac{1}{n_p} = p-1$. Since $n_j \ge 2$, we have $\frac12 p - 1 \le \frac{3}{3m+1} < 1$ and $p \le 3$.

If $p=3$, we have $m=1$, $\operatorname{ord}\mathbf{m} = 4$, $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3} = \frac54$, $\{n_1,n_2,n_3\} = \{2,2,4\}$ and $\mathbf{m} = 31,22,22,1111$.

Assume $p=2$. Then $\frac{1}{n_1}+\frac{1}{n_2} = 1-\frac{3}{3m+1}$. If $\min\{n_1,n_2\} \ge 3$, $\frac{1}{n_1}+\frac{1}{n_2} \le \frac23$ and $m \le 2$. If $\min\{n_1,n_2\} = 2$, $\max\{n_1,n_2\} \ge 3$ and $\frac{3}{3m+1} \ge \frac16$ and $m \le 5$. Note that $\frac{1}{n_1}+\frac{1}{n_2} = \frac{13}{16}$, $\frac{10}{13}$, $\frac{7}{10}$, $\frac47$ and $\frac14$ according to $m = 5$, 4, 3, 2 and 1, respectively. Hence we have $m=3$, $\{n_1,n_2\} = \{2,5\}$ and $\mathbf{m} = 3331,55,22222$.

Case (C). We have $\frac{3}{3m+2}+\frac{1}{n_1}+\cdots+\frac{1}{n_p} = p-1$. Since $n_j \ge 2$, $\frac12 p-1 \le \frac{3}{3m+2} < 1$ and $p \le 3$. If $p=3$, then $m=1$, $\operatorname{ord}\mathbf{m} = 5$ and $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3} = \frac75$, which never occurs.

Thus we have $p=2$, $\frac{1}{n_1}+\frac{1}{n_2} = 1-\frac{3}{3m+2}$ and hence $m \le 5$ as in Case (B). Then $\frac{1}{n_1}+\frac{1}{n_2} = \frac{14}{17}$, $\frac{11}{14}$, $\frac{8}{11}$, $\frac58$ and $\frac25$ according to $m = 5$, 4, 3, 2 and 1, respectively.

Hence we have $m=1$ and $n_1=n_2=5$ and $\mathbf{m}=32,11111,11111$ or $m=2$ and $n_1=2$ and $n_2=8$ and $\mathbf{m}=332,44,11111111$.

Case (D). We have $\frac{2}{2m+1}+\frac{2}{2m+1}+\frac{1}{n_2}+\cdots+\frac{1}{n_p}=p-1$. Since $n_j\ge 3$ for $j\ge 2$, we have $p-1\le\frac{3}{2}\frac{4}{2m+1}=\frac{6}{2m+1}$ and $m\le 2$. If $m=1$, then $p=3$ and $\frac{1}{n_2}+\frac{1}{n_3}=2-\frac{4}{3}=\frac{2}{3}$ and we have $\mathbf{m}=21,21,111,111$. If $m=2$, then $p=2$, $\frac{1}{n_2}=1-\frac{4}{5}$ and $\mathbf{m}=221,221,11111$.

Case (E). Since $m_{j,1}=1$ and (5.42) means $-2=\sum_{j=0}^p 2m_{j,1}-4(p-1)$, we have $p=4$ and $\mathbf{m}=11,11,11,11,11$. $\square$

Remark 6.11. A generalization of Proposition6.10 is given in [**HiO**] which can be applied to equations with irregular singularities.

6.3. Divisible spectral types

Proposition 6.12. *Let* $\mathbf{m}$ *be any one of the partition of type* $\tilde D_4$*,* $\tilde E_6$*,* $\tilde E_7$ *or* $\tilde E_8$ *in Example 6.8 and put* $n=\operatorname{ord}\mathbf{m}$*. Then* $k\mathbf{m}$ *is realizable but it isn't irreducibly realizable for* $k=2,3,\ldots$*. Moreover we have the operator* P *of order* $k\operatorname{ord}\mathbf{m}$ *satisfying the properties in* Theorem 6.6 ii) *for the tuple* $k\mathbf{m}$*.*

Proof. Let $P(k,g)$ be the operator of the normal form with the Riemann scheme

$$\begin{Bmatrix} x=c_0=\infty & x=c_j\ (j=1,\ldots,p)\\ [\lambda_{0,1}-k(p-1)n+km_{0,1}]_{(m_{0,1})} & [\lambda_{j,1}+km_{j,1}]_{(m_{j,1})}\\ \vdots & \vdots\\ [\lambda_{0,n_1}-k(p-1)n+km_{0,1}]_{(m_{0,n_1})} & [\lambda_{j,n_j}+km_{j,n_j}]_{(m_{j,n_j})}\end{Bmatrix}$$

of type $\mathbf{m}$. Here $\mathbf{m}=\big(m_{j,\nu}\big)_{\substack{j=0,\ldots,p\\ \nu=1,\ldots,n_j}}$, $n=\operatorname{ord}\mathbf{m}$ and g is the accessory parameter contained in the coefficient of the 0-th order term of $P(k,g)$. Since $\operatorname{Pidx}\mathbf{m}=0$ means

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}m_{j,\nu}^2=(p-1)n^2=\sum_{\nu=0}^{n_0}(p-1)nm_{0,\nu},$$

the Fuchs relation (4.16) is valid for any k. Then it follows from Lemma 4.1 that the Riemann scheme of the operator $P_k(g_1,\ldots,g_k)=P(k-1,g_k)P(k-2,g_{k-1})\cdots P(0,g_1)$ equals

$$\begin{Bmatrix} x=c_0=\infty & x=c_j\ (j=1,\ldots,p)\\ [\lambda_{0,1}]_{(km_{0,1})} & [\lambda_{j,1}]_{(km_{j,1})}\\ \vdots & \vdots\\ [\lambda_{0,n_1}]_{(km_{0,n_1})} & [\lambda_{j,n_j}]_{(km_{j,n_j})}\end{Bmatrix} \tag{6.22}$$

and it contain an independent accessory parameters in the coefficient of νn-th order term of $P_k(g_1,\ldots,g_k)$ for $\nu=0,\ldots,k-1$ because for the proof of this statement we may assume $\lambda_{j,\nu}$ are generic under the Fuchs relation.

Note that

$$N_\nu(k\mathbf{m})=\begin{cases}1 & (\nu\equiv n-1\mod n),\\ -1 & (\nu\equiv 0\mod n),\\ 0 & (\nu\not\equiv 0,\ n-1\mod n)\end{cases}$$

for $\nu=1,\ldots,kn-1$ because

$$\widetilde{k\mathbf{m}}=\begin{cases}\{2i,2i,2i,2i\,;\,i=1,2,\ldots,k\} & \text{if }\mathbf{m}\text{ is of type }\tilde D_4,\\ \{ni,ni,ni,ni-2,ni-3,\ldots,ni-n+1\,;\,i=1,2,\ldots,k\} & \\ & \text{if }\mathbf{m}\text{ is of type }\tilde E_6,\ \tilde E_7\text{ or }\tilde E_8\end{cases}$$

under the notation (6.2) and (6.17). Then the operator $P_k(g_1,\dots,g_k)$ shows that when we inductively determine the coefficients of the operator with the Riemann scheme (6.22) as in the proof of Theorem 6.6, we have a new accessory parameter in the coefficient of the $\big((k-j)n\big)$-th order term and then the conditions for the coefficients of the $\big((k-j)n-1\big)$-th order term are overdetermined but they are automatically compatible for $j=1,\dots,k-1$.

Thus we can conclude that the operators of the normal form with the Riemann scheme (6.22) are $P_k(g_1,\dots,g_k)$, which are always reducible. $\square$

PROPOSITION 6.13. *Let k be a positive integer and let $\mathbf{m}$ be an indivisible $(p+1)$-tuple of partitions of n. Suppose $k\mathbf{m}$ is realizable and* $\operatorname{idx}\mathbf{m}<0$. *Then any Fuchsian differential equation with the Riemann scheme* (6.22) *is always irreducible if $\lambda_{j,\nu}$ is generic under the Fuchs relation*

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j} m_{j,\nu}\lambda_{j,\nu}=\operatorname{ord}\mathbf{m}-k\frac{\operatorname{idx}\mathbf{m}}{2}. \tag{6.23}$$

PROOF. Since $\operatorname{ord}k\mathbf{m}=k\operatorname{ord}\mathbf{m}$ and $\operatorname{idx}k\mathbf{m}=k^2\operatorname{idx}\mathbf{m}$, the above Fuchs relation follows from (4.32).

Suppose $Pu=0$ is reducible. Then Remark 4.17 ii) says that there exist $\mathbf{m}'$, $\mathbf{m}''\in\mathcal{P}$ such that $k\mathbf{m}=\mathbf{m}'+\mathbf{m}''$ and $0<\operatorname{ord}\mathbf{m}'<k\operatorname{ord}\mathbf{m}$ and $|\{\lambda_{\mathbf{m}'}\}|\in\{0,-1,-2,\dots\}$. Suppose $\lambda_{j,\nu}$ are generic under (6.23). Then the condition $|\{\lambda_{\mathbf{m}'}\}|\in\mathbb{Z}$ implies $\mathbf{m}'=\ell\mathbf{m}$ with a positive integer satisfying $\ell<k$ and

$$\begin{aligned}|\{\lambda_{\ell\mathbf{m}}\}|&=\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\ell m_{j,\nu}\lambda_{j,\nu}-\operatorname{ord}\ell\mathbf{m}+\ell^2\operatorname{idx}\mathbf{m}\\&=\ell\Big(\operatorname{ord}\mathbf{m}-k\frac{\operatorname{idx}\mathbf{m}}{2}\Big)-\ell\operatorname{ord}\mathbf{m}+\ell^2\operatorname{idx}\mathbf{m}\\&=\ell(\ell-k)\operatorname{idx}\mathbf{m}>0.\end{aligned}$$

Hence $|\{\lambda_{\mathbf{m}'}\}|>0$. $\square$

6.4. Universal model

Now we have a main result in Chapter 6 which assures the existence of Fuchsian differential operators with given spectral types.

THEOREM 6.14. *Fix a tuple* $\mathbf{m}=\big(m_{j,\nu}\big)_{\substack{0\le j\le p\\1\le\nu\le n_j}}\in\mathcal{P}_{p+1}^{(n)}$.

i) *Under the notation in* Definitions 4.10, 4.16 *and* 5.7, *the tuple* $\mathbf{m}$ *is realizable if and only if there exists a non-negative integer K such that $\partial_{max}^i\mathbf{m}$ are well-defined for $i=1,\dots,K$ and*

$$\begin{aligned}&\operatorname{ord}\mathbf{m}>\operatorname{ord}\partial_{max}\mathbf{m}>\operatorname{ord}\partial_{max}^2\mathbf{m}>\cdots>\operatorname{ord}\partial_{max}^K\mathbf{m},\\&d_{max}(\partial_{max}^K\mathbf{m})=2\operatorname{ord}\partial_{max}^K\mathbf{m}\quad\text{or}\quad d_{max}(\partial_{max}^K\mathbf{m})\le 0.\end{aligned}\tag{6.24}$$

ii) *Fix complex numbers $\lambda_{j,\nu}$. If there exists an irreducible Fuchsian operator with the Riemann scheme* (4.15) *such that it is locally non-degenerate* (cf. Definition 9.8), *then* $\mathbf{m}$ *is irreducibly realizable.*

Here we note that if P is irreducible and $\mathbf{m}$ is rigid, P is locally non-degenerate (cf. Definition 9.8).

Hereafter in this theorem we assume $\mathbf{m}$ is realizable.

iii) $\mathbf{m}$ *is irreducibly realizable if and only if* $\mathbf{m}$ *is indivisible or* $\operatorname{idx}\mathbf{m}<0$.

iv) *There exists a universal model $P_{\mathbf{m}}u=0$ associated with $\mathbf{m}$ which has the following property.*

Namely, $P_{\mathbf{m}}$ is the Fuchsian differential operator of the form

(6.25)
$$P_{\mathbf{m}} = \Big(\prod_{j=1}^{p}(x-c_j)^n\Big)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x),$$
$$a_j(x) \in \mathbb{C}[\lambda_{j,\nu}, g_1, \ldots, g_N], \quad \frac{\partial^2 a_j(x)}{\partial g_i \partial g_{i'}} = 0 \quad (1 \le i \le i' \le N,\ 0 \le j < n)$$

such that $P_{\mathbf{m}}$ has regular singularities at $p+1$ fixed points $x = c_0 = \infty, c_1, \ldots, c_p$ and the Riemann scheme of $P_{\mathbf{m}}$ equals (4.15) *for any $g_i \in \mathbb{C}$ and $\lambda_{j,\nu} \in \mathbb{C}$ under the Fuchs relation* (4.16)*. Moreover the coefficients $a_j(x)$ are polynomials of x, $\lambda_{j,\nu}$ and g_i with the degree at most $(p-1)n + j$ for $j = 0, \ldots, n$, respectively. Here g_i are called accessory parameters and we call $P_{\mathbf{m}}$ the* universal operator *of type $\mathbf{m}$.*

The non-negative integer N will be denoted by $\operatorname{Ridx}\mathbf{m}$ *and given by*

$$N = \operatorname{Ridx}\mathbf{m} := \begin{cases} 0 & (\operatorname{idx}\mathbf{m} > 0), \\ \gcd\mathbf{m} & (\operatorname{idx}\mathbf{m} = 0), \\ \operatorname{Pidx}\mathbf{m} & (\operatorname{idx}\mathbf{m} < 0). \end{cases} \tag{6.26}$$

Put $\overline{\mathbf{m}} = \big(\overline{m}_{j,\nu}\big)_{\substack{0\le j\le p \\ 1\le\nu\le n_j}} := \partial_{max}^K \mathbf{m}$ with the non-negative integer K given in i)*.*

When $\operatorname{idx}\mathbf{m} \le 0$*, we define*

$$q_\ell^0 := \#\{i\,;\, \sum_{\nu=1}^{\bar{n}_0} \max\{\overline{m}_{0,\nu} - i, 0\} \ge \operatorname{ord}\overline{\mathbf{m}} - \ell,\ i \ge 0\},$$
$$I_{\mathbf{m}} := \{(j,\nu) \in \mathbb{Z}^2\,;\, q_\nu^0 \le j \le q_\nu^0 + N_\nu - 1,\ 1 \le \nu \le \operatorname{ord}\overline{\mathbf{m}} - 1\}.$$

When $\operatorname{idx}\mathbf{m} > 0$*, we put $I_{\mathbf{m}} = \emptyset$.*

Then $\#I_{\mathbf{m}} = \operatorname{Ridx}\mathbf{m}$ and we can define I_i such that $I_{\mathbf{m}} = \{I_i\,;\, i = 1, \ldots, N\}$ and g_i satisfy (6.13) *by putting $g_{I_i} = g_i$ for $i = 1, \ldots, N$.*

v) *Retain the notation in* Definition 5.12*. If $\lambda_{j,\nu} \in \mathbb{C}$ satisfy*

$$\begin{cases} \sum_{j=0}^{p} \lambda(k)_{j,\ell(k)_j + \delta_{j,j_o}(\nu_o - \ell(k)_j)} \\ \notin \{0, -1, -2, -3, \ldots, m(k)_{j_o,\ell(k)_{j_o}} - m(k)_{j_o,\nu_o} - d(k) + 2\} \\ \quad \text{for any } k = 0, \ldots, K-1 \text{ and } (j_0, \nu_o) \text{ satisfying} \\ \quad m(k)_{j_o,\nu_o} \ge m(k)_{j_o,\ell(k)_{j_o}} - d(k) + 2, \end{cases} \tag{6.27}$$

any Fuchsian differential operator P of the normal form which has the Riemann scheme (4.15) *belongs to $P_{\mathbf{m}}$ with a suitable $(g_1, \ldots, g_N) \in \mathbb{C}^N$.*

$$\begin{cases} \text{If } \mathbf{m} \text{ is a scalar multiple of a fundamental tuple or simply reducible,} \\ (6.27) \text{ is always valid for any } \lambda_{j,\nu}. \end{cases} \tag{6.28}$$

$$\begin{cases} \text{Fix } \lambda_{j,\nu} \in \mathbb{C}. \text{ Suppose there is an irreducible Fuchsian differential} \\ \text{operator with the Riemann scheme (4.15) such that the operator is} \\ \text{locally non-degenerate or } K \le 1, \text{ then (6.27) is valid.} \end{cases} \tag{6.29}$$

Suppose $\mathbf{m}$ is monotone. Under the notation in §7.1*, the condition* (6.27) *is equivalent to*

$$\begin{aligned} &(\Lambda(\lambda)|\alpha) + 1 \notin \{0, -1, \ldots, 2 - (\alpha|\alpha_{\mathbf{m}})\} \\ &\qquad \text{for any } \alpha \in \Delta(\mathbf{m}) \text{ satisfying } (\alpha|\alpha_{\mathbf{m}}) > 1. \end{aligned} \tag{6.30}$$

Example 5.6 gives a Fuchsian differential operator with the rigid spectral type $21, 21, 21, 21$ which doesn't belong to the corresponding universal operator.

The fundamental tuple and the simply reducible tuple are defined as follows.

DEFINITION 6.15. i) (fundamental tuple) An irreducibly realizable tuple $\mathbf{m} \in \mathcal{P}$ is called *fundamental* if $\operatorname{ord} \mathbf{m} = 1$ or $d_{\max}(\mathbf{m}) \le 0$.

For an irreducibly realizable tuple $\mathbf{m} \in \mathcal{P}$, there exists a non-negative integer K such that $\partial_{max}^K \mathbf{m}$ is fundamental and satisfies (6.24). Then we call $\partial_{max}^K \mathbf{m}$ is a fundamental tuple corresponding to $\mathbf{m}$ and define $f\mathbf{m} := \partial_{max}^K \mathbf{m}$.

ii) (simply reducible tuple) A tuple $\mathbf{m}$ is *simply reducible* if there exists a positive integer K satisfying (6.24) and $\operatorname{ord} \partial_{max}^K \mathbf{m} = \operatorname{ord} \mathbf{m} - K$.

PROOF OF THEOREM 6.14. i) We have proved that $\mathbf{m}$ is realizable if the condition $d_{max}(\mathbf{m}) \le 0$ is valid. Note that the condition $d_{max}(\mathbf{m}) = 2 \operatorname{ord} \mathbf{m}$ is equivalent to the fact that $s\mathbf{m}$ is trivial. Hence Theorem 5.10 proves the claim.

iv) Now we use the notation in Definition 5.12. The existence of the universal operator is clear if $s\mathbf{m}$ is trivial. If $d_{max}(\mathbf{m}) \le 0$, Theorem 6.6 and Proposition 6.12 with Corollary 6.3 assure the existence of the universal operator $P_{\mathbf{m}}$ claimed in iii). Hence iii) is valid for the tuple $\mathbf{m}(K)$ and we have a universal operator P_K with the Riemann scheme $\{\lambda(K)_{\mathbf{m}(K)}\}$.

The universal operator P_k with the Riemann scheme $\{\lambda(k)_{\mathbf{m}(k)}\}$ are inductively obtained by applying $\partial_{\ell(k)}$ to the universal operator P_{k+1} with the Riemann scheme $\{\lambda(k+1)_{\mathbf{m}(k+1)}\}$ for $k = K-1, K-2, \dots, 0$. Since the claims in iii) such as (6.13) are kept by the operation $\partial_{\ell(k)}$, we have iv).

iii) Note that $\mathbf{m}$ is irreducibly realizable if $\mathbf{m}$ is indivisible (cf. Remark 4.17 ii)). Hence suppose $\mathbf{m}$ is not indivisible. Put $k = \gcd \mathbf{m}$ and $\mathbf{m} = k\mathbf{m}'$. Then $\operatorname{idx} \mathbf{m} = k^2 \operatorname{idx} \mathbf{m}'$.

If $\operatorname{idx} \mathbf{m} > 0$, then $\operatorname{idx} \mathbf{m} > 2$ and the inequality (5.19) in Lemma 5.3 implies that $\mathbf{m}$ is not irreducibly realizable. If $\operatorname{idx} \mathbf{m} < 0$, Proposition 6.13 assures that $\mathbf{m}$ is irreducibly realizable.

Suppose $\operatorname{idx} \mathbf{m} = 0$. Then the universal operator $P_{\mathbf{m}}$ has k accessory parameters. Using the argument in the first part of the proof of Proposition 6.12, we can construct a Fuchsian differential operator $\tilde{P}_{\mathbf{m}}$ with the Riemann scheme $\{\lambda_{\mathbf{m}}\}$. Since $\tilde{P}_{\mathbf{m}}$ is a product of k copies of the universal operator $P_{\overline{\mathbf{m}}}$ and it has k accessory parameters, the operator $P_{\mathbf{m}}$ coincides with the reducible operator $\tilde{P}_{\mathbf{m}}$ and hence $\mathbf{m}$ is not irreducibly realizable.

v) Fix $\lambda_{j,\nu} \in \mathbb{C}$. Let P be a Fuchsian differential operator with the Riemann scheme $\{\lambda_{\mathbf{m}}\}$. Suppose P is of the normal form.

Theorem 6.6 and Proposition 6.12 assure that P belongs to $P_{\mathbf{m}}$ if $K = 0$.

Theorem 5.2 proves that if $\partial_{max}^k P$ has the Riemann scheme $\{\lambda(k)_{\mathbf{m}(k)}\}$ and (6.27) is valid, then $\partial_{max}^{k+1} P = \partial_{\ell(k)} \partial_{max}^k P$ is well-defined and has the Riemann scheme $\{\lambda(k+1)_{\mathbf{m}(k+1)}\}$ for $k = 0, \dots, K-1$ and hence it follows from (5.27) that P belongs to the universal operator $P_{\mathbf{m}}$ because $\partial_{max}^K P$ belongs to the universal operator $P_{\mathbf{m}(K)}$.

If $\mathbf{m}$ is simply reducible, $d(k) = 1$ and therefore (6.27) is valid because $m(k)_{j,\nu} \le m(k)_{j,\ell(k)_\nu} < m(k)_{j,\ell(k)_\nu} - d(k) + 2$ for $j = 0, \dots, p$ and $\nu = 1, \dots, n_j$ and $k = 0, \dots, K-1$.

The equivalence of the conditions (6.27) and (6.30) follows from the argument in §7.1, Proposition 7.9 and Theorem 10.13.

ii) Suppose there exists an irreducible operator P with the Riemann scheme (4.15). Let $\mathbf{M} = (M_0, \dots, M_p)$ be the tuple of monodromy generators of the equation $Pu = 0$ and put $\mathbf{M}(0) = \mathbf{M}$. Let $\mathbf{M}(k+1)$ be the tuple of matrices applying the operations in §9.1 to $\mathbf{M}(k)$ corresponding to the operations $\partial_{\ell(k)}$ for $k = 0, 1, 2, \dots$.

Comparing the operations on $\mathbf{M}(k)$ and $\partial_{\ell(k)}$, we can conclude that there exists a non-negative integer K satisfying the claim in i). In fact Theorem 9.3 proves that $\mathbf{M}(k)$ are irreducible, which assures that the conditions (5.6) and (5.7) corresponding to the operations $\partial_{\ell(k)}$ are always valid (cf. Corollary 10.12). Therefore

$\mathbf{m}$ is realizable and moreover we can conclude that (6.29) implies (6.27). If $\operatorname{idx}\mathbf{m}$ is divisible and $\operatorname{idx}\mathbf{m} = 0$, then $P_{\mathbf{m}}$ is reducible for any fixed parameters $\lambda_{j,\nu}$ and g_i. Hence $\mathbf{m}$ is irreducibly realizable. □

Remark 6.16. i) The uniqueness of the universal operator in Theorem 6.14 is obvious. But it is not valid in the case of systems of Schlesinger canonical form (cf. Example 9.2).

ii) The assumption that $Pu = 0$ is locally non-degenerate seems to be not necessary in Theorem 6.14 ii) and (6.29). When $K = 1$, this is clear from the proof of the theorem. For example, the rigid irreducible operator with the spectral type $31, 31, 31, 31, 31$ belongs to the universal operator of type $211, 31, 31, 31, 31$.

6.5. Simply reducible spectral type

In this section we characterize the tuples of the simply reducible spectral type.

Proposition 6.17. i) *A realizable tuple* $\mathbf{m} \in \mathcal{P}^{(n)}$ *satisfying* $m_{0,\nu} = 1$ *for* $\nu = 1, \dots, n$ *is simply reducible if* $\mathbf{m}$ *is not fundamental.*

ii) *The simply reducible rigid tuple corresponds to the tuple in Simpson's list* (cf. §13.2) *or it is isomorphic to* $21111, 222, 33$.

iii) *Suppose* $\mathbf{m} \in \mathcal{P}_{p+1}$ *is not fundamental. Then* $\mathbf{m}$ *satisfies the condition* $N_\nu(\mathbf{m}) \geq 0$ *for* $\nu = 2, \dots, \operatorname{ord}\mathbf{m} - 1$ *in* Definition 6.1 *if and only if* $\mathbf{m}$ *is realizable and simply reducible.*

iv) *Let* $\mathbf{m} \in \mathcal{P}_{p+1}$ *be a realizable monotone tuple. Suppose* $\mathbf{m}$ *is not fundamental. Then under the notation in* §7.1, $\mathbf{m}$ *is simply reducible if and only if*

$$(\alpha|\alpha_{\mathbf{m}}) = 1 \quad (\forall \alpha \in \Delta(\mathbf{m})), \tag{6.31}$$

namely $[\Delta(\mathbf{m})] = 1^{\#\Delta(\mathbf{m})}$ (cf. Remark 7.11 ii)).

Proof. i) The claim is obvious from the definition.

ii) Let $\mathbf{m}'$ be a simply reducible rigid tuple. We have only to prove that $\mathbf{m} = \partial_{max}\mathbf{m}'$ is in the Simpson's list or $21111, 222, 33$ and $\operatorname{ord}\mathbf{m}' = \operatorname{ord}\mathbf{m} + 1$ and $d_{max}(\mathbf{m}) = 1$, then $\mathbf{m}'$ is in Simpson's list or $21111, 222, 33$. The condition $\operatorname{ord}\mathbf{m}' = \operatorname{ord}\mathbf{m} + 1$ implies $\mathbf{m} \in \mathcal{P}_3$. We may assume $\mathbf{m}$ is monotone and $\mathbf{m}' = \partial_{\ell_0,\ell_1,\ell_2}\mathbf{m}$. The condition $\operatorname{ord}\mathbf{m}' = \operatorname{ord}\mathbf{m} + 1$ also implies

$$(m_{0,1} - m_{0,\ell_0}) + (m_{1,1} - m_{1,\ell_0}) + (m_{2,1} - m_{2,\ell_0}) = 2.$$

Since $\partial_{max}\mathbf{m}' = \mathbf{m}$, we have $m_{j,\ell_j} \geq m_{j,1} - 1$ for $j = 0, 1, 2$. Hence there exists an integer k with $0 \leq k \leq 2$ such that $m_{j,\ell_j} = m_{j,1} - 1 + \delta_{j,k}$ for $j = 0, 1, 2$. Then the following claims are easy, which assures the proposition.

If $\mathbf{m} = 11, 11, 11$, $\mathbf{m}'$ is isomorphic to $1^3, 1^3, 21$.

If $\mathbf{m} = 1^3, 1^3, 21$, $\mathbf{m}'$ is isomorphic to $1^4, 1^4, 31$ or $1^4, 211, 22$.

If $\mathbf{m} = 1^n, 1^n, n-11$ with $n \geq 4$, $\mathbf{m}' = 1^{n+1}, 1^{n+1}, n1$.

If $\mathbf{m} = 1^{2n}, nn-11, nn$ with $n \geq 2$, $\mathbf{m}' = 1^{2n+1}, nn1, n+1n$.

If $\mathbf{m} = 1^5, 221, 32$, then $\mathbf{m}' = 1^6, 33, 321$ or $1^6, 222, 42$ or $21111, 222, 33$.

If $\mathbf{m} = 1^{2n+1}, n+1n, nn1$ with $n \geq 3$, $\mathbf{m}' = 1^{2n+2}, n+1n+1, n+1n1$.

If $\mathbf{m} = 1^6, 222, 42$ or $\mathbf{m} = 21111, 222, 33$, $\mathbf{m}'$ doesn't exists.

iii) Note that Theorem 6.6 assures that the condition $N_\nu(\mathbf{m}) \geq 0$ for $\nu = 1, \dots, \operatorname{ord}\mathbf{m} - 1$ implies that $\mathbf{m}$ is realizable.

We may assume $\mathbf{m} \in \mathcal{P}_{p+1}^{(n)}$ is standard. Put $d = m_{0,1} + \cdots + m_{p,1} - (p-1)n > 0$ and $\mathbf{m}' = \partial_{max}\mathbf{m}$. Then $m'_{j,\nu} = m_{j,\nu} - \delta_{\nu,1}d$ for $j = 0, \dots, p$ and $\nu \geq 1$. Under the notation in Definition 6.1 the operation ∂_{max} transforms the sets

$$\mathfrak{m}_j := \{\widetilde{m}_{j,k}\,;\, k = 0, 1, 2, \dots \text{ and } \widetilde{m}_{j,k} > 0\}$$

into

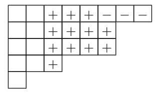
$$\mathfrak{m}'_j = \big\{\widetilde{m}_{j,k} - \min\{d, m_{j,1} - k\}\,;\, k = 0, \ldots, \max\{m_{j,1} - d, m_{j,2} - 1\}\big\},$$

respectively because $\widetilde{m}_{j,i} = \sum_\nu \max\{m_{j,\nu} - i, 0\}$. Therefore $N_\nu(\mathbf{m}') \le N_\nu(\mathbf{m})$ for $\nu = 1, \ldots, n-d-1 = \operatorname{ord}\mathbf{m}' - 1$. Here we note that

$$\sum_{\nu=1}^{n-1} N_\nu(\mathbf{m}) = \sum_{\nu=1}^{n-d-1} N_\nu(\mathbf{m}') = \operatorname{Pidx}\mathbf{m}.$$

Hence $N_\nu(\mathbf{m}) \ge 0$ for $\nu = 1, \ldots, n-1$ if and only if $N_\nu(\mathbf{m}') = N_\nu(\mathbf{m})$ for $\nu = 1, \ldots, (n-d) - 1$ and moreover $N_\nu(\mathbf{m}) = 0$ for $\nu = n-d, \ldots, n-1$. Note that the condition that $N_\nu(\mathbf{m}') = N_\nu(\mathbf{m})$ for $\nu = 1, \ldots, (n-d) - 1$ equals

$$m_{j,1} - d \ge m_{j,2} - 1 \quad \text{for} \quad j = 0, \ldots, p. \tag{6.32}$$



This is easy to see by using a Young diagram. For example, when $\{8, 6, 6, 3, 1\} = \{m_{0,1}, m_{0,2}, m_{0,3}, m_{0,4}, m_{0,5}\}$ is a partition of $n = 24$, the corresponding Young diagram is as above and then $\widetilde{m}_{0,2}$ equals 15, namely, the number of boxes with the sign $+$ or $-$. Moreover when $d = 3$, the boxes with the sign $-$ are deleted by ∂_{max} and the number $\widetilde{m}_{0,2}$ changes into 12. In this case $m_0 = \{24, 19, 15, 11, 8, 5, 2, 1\}$ and $m'_0 = \{21, 16, 12, 8, 5, 2\}$.

If $d \ge 2$, then $1 \in \mathfrak{m}_j$ for $j = 0, \ldots, p$ and therefore $N_{n-2}(\mathbf{m}) - N_{n-1}(\mathbf{m}) = 2$, which means $N_{n-1}(\mathbf{m}) \ne 0$ or $N_{n-2}(\mathbf{m}) \ne 0$. When $d = 1$, we have $N_\nu(\mathbf{m}) = N_\nu(\mathbf{m}')$ for $\nu = 1, \ldots, n-2$ and $N_{n-1}(\mathbf{m}) = 0$. Thus we have the claim.

iv) The claim follows from Proposition 7.9. □

Example 6.18. We show the simply reducible tuples with index 0 whose fundamental tuple is of type $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ (cf. Example 6.8).

$\tilde{D}_4$: $21,21,21,111$ $\;\;22,22,31,211$ $\;\;22,31,31,1111$

$\tilde{E}_6$: $211,211,1111$ $\;\;221,221,2111$ $\;\;221,311,11111$ $\;\;222,222,3111$ $\;\;222,321,2211$ $\;\;222,411,111111$ $\;\;322,331,2221$ $\;\;332,431,2222$ $\;\;333,441,3222$

$\tilde{E}_7$: $11111,2111,32$ $\;\;111111,2211,42$ $\;\;21111,2211,33$ $\;\;111111,3111,33$ $\;\;22111,2221,43$ $\;\;1111111,2221,52$ $\;\;22211,2222,53$ $\;\;11111111,2222,62$ $\;\;32111,2222,44$ $\;\;22211,3221,53$

$\tilde{E}_8$: $1111111,322,43$ $\;\;11111111,332,53$ $\;\;2111111,332,44$ $\;\;11111111,422,44$ $\;\;2211111,333,54$ $\;\;111111111,333,63$ $\;\;2221111,433,55$ $\;\;2222111,443,65$ $\;\;3222111,444,66$ $\;\;2222211,444,75$ $\;\;2222211,543,66$ $\;\;2222221,553,76$ $\;\;2222222,653,77$

In general, we have the following proposition.

Proposition 6.19. *There exist only a finite number of standard and simply reducible tuples with any fixed non-positive index of rigidity.*

Proof. First note that $\mathbf{m} \in \mathcal{P}_{p+1}$ if $d_{max}(\mathbf{m}) = 1$ and $\operatorname{ord}\mathbf{m} > 3$ and $\partial_{max}\mathbf{m} \in \mathcal{P}_{p+1}$. Since there exist only finite basic tuples with any fixed index of rigidity (cf. Remark 7.15), we have only to prove the non-existence of the infinite sequence

$$\mathbf{m}(0) \xleftarrow{\partial_{max}} \mathbf{m}(1) \xleftarrow{\partial_{max}} \cdots\cdots \xleftarrow{\partial_{max}} \mathbf{m}(k) \xleftarrow{\partial_{max}} \mathbf{m}(k+1) \xleftarrow{\partial_{max}} \cdots$$

such that $d_{max}(\mathbf{m}(k)) = 1$ for $k \ge 1$ and $\operatorname{idx}\mathbf{m}(0) \le 0$.

Put

$$\begin{aligned}\bar m(k)_j &= \max_\nu\{m(k)_{j,\nu}\},\\ a(k)_j &= \#\{\nu\,;\,m(k)_{j,\nu}=\bar m(k)_j\},\\ b(k)_j &= \begin{cases}\#\{\nu\,;\,m(k)_{j,\nu}=\bar m(k)_j-1\} & (\bar m(k)_j>1),\\ \infty & (\bar m(k)_j=1).\end{cases}\end{aligned}$$

The assumption $d_{max}(\mathbf{m}(k))=d_{max}(\mathbf{m}(k+1))=1$ implies that there exist indices $0\le j_k<j_k'$ such that

$$(a(k+1)_j,b(k+1)_j)=\begin{cases}(a(k)_j+1,b(k)_j-1) & (j=j_k \text{ or } j_k'),\\ (1,a(k)_j-1) & (j\ne j_k \text{ and } j_k')\end{cases}\tag{6.33}$$

and

$$\bar m(k)_0+\cdots+\bar m(k)_p=(p-1)\operatorname{ord}\mathbf{m}(k)+1\qquad(p\gg1)\tag{6.34}$$

for $k=1,2,\ldots$. Since $a(k+1)_j+b(k+1)_j\le a(k)_j+b(k)_j$, there exists a positive integer N such that $a(k+1)_j+b(k+1)_j=a(k)_j+b(k)_j$ for $k\ge N$, which means

$$b(k)_j\begin{cases}>0 & (j=j_k\text{ or } j_k'),\\ =0 & (j\ne j_k\text{ and } j_k').\end{cases}\tag{6.35}$$

Putting $(a_j,b_j)=(a(N)_j,b(N)_j)$, we may assume $b_0\ge b_1>b_2=b_3=\cdots=0$ and $a_2\ge a_3\ge\cdots$. Moreover we may assume $j'_{N+1}\le 3$, which means $a_j=1$ for $j\ge4$. Then the relations (6.33) and (6.35) for $k=N,N+1,N+2$ and $N+3$ prove that $\bigl((a_0,b_0),\cdots,(a_3,b_3)\bigr)$ is one of the followings:

$$((a_0,\infty),(a_1,\infty),(1,0),(1,0)),\tag{6.36}$$

$$((a_0,\infty),(1,1),(2,0),(1,0)),\tag{6.37}$$

$$((2,2),(1,1),(4,0),(1,0)),\ \ ((1,3),(3,1),(2,0),(1,0)),\tag{6.38}$$

$$((1,2),(2,1),(3,0),(1,0)),\tag{6.39}$$

$$((1,1),(1,1),(2,0),(2,0)).\tag{6.40}$$

In fact if $b_1>1$, $a_2=a_3=1$ and we have (6.36). Thus we may assume $b_1=1$. If $b_0=\infty$, $a_3=1$ and we have (6.37). If $b_0=b_1=1$, we have easily (6.40). Thus we may moreover assume $b_1=1<b_0<\infty$ and $a_3=1$. In this case the integers j_k'' satisfying $b(k)_{j_k''}=0$ and $0\le j_k''\le 2$ for $k\ge N$ are uniquely determined and we have easily (6.38) or (6.39).

Put $n=\operatorname{ord}\mathbf{m}(N)$. We may suppose $\mathbf{m}(N)$ is standard. Let p be an integer such that $m_{j,0}<n$ if and only if $j\le p$. Note that $p\ge2$. Then if $\mathbf{m}(N)$ satisfies (6.36) (resp. (6.37)), (6.34) implies $\mathbf{m}(N)=1^n,1^n,n-11$ (resp. $1^n,mm-11,mm$ or $1^n,m+1m,mm1$) and $\mathbf{m}(N)$ is rigid.

Suppose one of (6.38)–(6.40). Then it is easy to check that $\mathbf{m}(N)$ doesn't satisfy (6.34). For example, suppose (6.39). Then $3m_{0,1}-2\le n$, $3m_{1,1}-1\le n$ and $3m_{2,1}\le n$ and we have $m_{0,1}+m_{1,1}+m_{2,1}\le[\frac{n+2}{3}]+[\frac{n+1}{3}]+[\frac{n}{3}]=n$, which contradicts to (6.34). The relations $[\frac{n+2}{4}]+[\frac{n}{2}]+[\frac{n}{4}]\le n$ and $2[\frac{n+1}{2}]+2[\frac{n}{2}]=2n$ assure the same conclusion in the other cases. $\square$